TODAY

The exponential formula
+ applications

Announcement: HW 5 posted; due on Nov 22 (or earlier)

Last time: Egfs and combinatorial interpretation of multiplying egfs.
The Exponential formula

Have: A ‘deck’ sequence \( d_0, d_1, d_2, \ldots \)

\[
D(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!} \quad \text{<egf> \{d_n\} \_o}.
\]

Operation: Make ‘hands’ from decks.

\[
h(n,k) = \# \text{ of ways to partition } [n] \text{ into } k \text{ non-ordered subsets } \{S_1, S_2, \ldots, S_k\} \text{ and then building a deck on each } S_j \text{ independently}.
\]
Theorem: \[ \sum_{n \geq 0} h(n, k) \frac{X^n}{n!} = \frac{D(x)^k}{k!} \] 

That is, \( \{ h(n, k) \}_{n \geq 0} \) \( \text{egf} \) \( \frac{D(x)^k}{k!} \).

Proof: Consider \( P_k(x) = D(x)^k \) as an egf:

\[ P_k(x) = \sum_{n \geq 0} P(n, k) \frac{X^n}{n!} \]

From last lecture: \( P(n, k) \) has a combinatorial interpretation.
\[ P(n,k) = \text{# of ways to partition } \{n\} \text{ into } k \text{ ordered subsets } \{S_1, \ldots, S_k\} \text{ and building a deck on each } S_j \text{ independently.} \]

Main observation: Since all decks are of the same type, this is the same as the following.

1. Make an unordered partition \( \{S_1, \ldots, S_k\} \) of \( \{n\} \) into \( k \) sets.
2. Build a deck on each \( S_j \) independently.
3. Order the partition.
The number of ways to do \( \mathcal{O} \) is \( K_0! \).

Given an ordering of the partition, \( \mathcal{O} + \mathcal{O} \) can be done in \( h(n, k) \) ways by definition.

So \( p(n, k) = K_0! \cdot h(n, k) \).

Therefore, \( D^k(x) = P_k(x) = K_0! \sum_{n=0}^{\infty} \frac{h(n, k) x^n}{n!} \).

\[ \frac{D^k(x)}{K_0!} \leq \sum_{n=0}^{\infty} \frac{h(n, k) x^n}{n!} \geq 0 \]
A word on the 'main observation':

- It is important that we build the same deck on each subset of the partition, which is why we can do the unordered task first and order afterwards.

- If the decks were of different types (i.e., e.g., if $A(x), B(x)$) then we cannot build on unordered partitions first and order afterwards.

E.g.: Take $K=2$ and $A(x) \xrightarrow{\text{egf}} \{a_n\}$, $B(x) \xrightarrow{\text{egf}} \{b_n\}$ and $D(x) \xrightarrow{\text{egf}} \{d_n\}$.
\[ \left[ \frac{x^n}{n!} \right] A(x) \cdot B(x) = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \]

\[ = \sum_{\text{ordered partition } (S_1, S_2)} a_{|S_1|} b_{|S_2|} \quad (|S_1| + |S_2| = n) \]

\[ = \sum_{\text{unordered partitions } \{S_1, S_2\}} \left( a_{|S_1|} b_{|S_2|} + a_{|S_2|} b_{|S_1|} \right) \]

\[ \left[ \frac{x^n}{n!} \right] O^2(x) = \sum_{\text{unordered partitions } \{S_1, S_2\}} \left( d_{|S_1|} d_{|S_2|} + d_{|S_2|} d_{|S_1|} \right) \]

\[ \text{They are the same!} \]
The ‘grand’ exponential formula

\[ \sum_{h(n,k)} \frac{n^k}{n!} = \frac{D(x)^k}{k!} \quad \forall \]

Introduce variable \( y \). Multiply \( \sum \) by \( y^k \) and

sum over all \( k \geq 0 \).

\[ \sum_{h(n,k)} \frac{n^k}{n!} y^k = \sum_{k=0}^{\infty} \left[ yD(x) \right]^k \frac{y^k}{k!} = e^{yD(x)} \]

This gives the following.
Thm: Given deck sequence \(\{\mathbf{d}_n\}_{n=0}^\infty\), let
\[
H(x,y) = \sum_{n,k \geq 0} h(n,k) \frac{y^n}{k!},
\]
Then
\[
H(x,y) = e^{yD(x)}
\]

Corollary: Let \(h_n = \sum_{k \geq 0} h(n,k)\) = \# of hands of size \(n\) [with any number of decks].

If \(d_0 = 0\) then \(h_n < \infty\)
Then we have the following.
Given the sequence \( \{a_n \}_{n=0}^\infty \) with \( a_0 = 0 \),

\[
\sum_{n=0}^\infty h_n \frac{x^n}{n!} = e^{D(x)}
\]

**Examples**

1. Set partitions
2. Permutation in terms of cycles
3. Counting connected graphs
0) Set Partitions: Deck sequence \( d_0 = 0, \ d_n = 1 \) if \( n \neq 0 \).

\[
D(x) = \sum_{n=1}^\infty \frac{x^n}{n!} = e^{-1}.
\]

1. \( h(n, k) = \# \text{ of ways to partition } \{n\} \text{ into } k \text{ non-empty subsets} \]

\[
= S(n, k).
\]

2. \[
\begin{align*}
\sum_{n=0}^\infty \frac{x^n}{n!} y^k & \leq S(n, k) \frac{x^y}{y^k} \\
& = e^{x(e^{-1})}
\end{align*}
\]

3. \[ h_n = \text{Bell numbers} \rightarrow \sum_{n=0}^\infty \frac{x^n}{n!} = e^{x-1} \quad [\text{did this earlier}] \]
Stirling numbers of the first kind

Deck sequence \( d_0 = 0, \ d_n = (n-1)! = \# \text{ of cyclic permutations of length } n \).

- \( h(n,k) = \) partition \( \{n\} \) into non-empty ordered subsets \( \{S_1, S_2, \ldots, S_k\} \) and make a cyclic permutation involving each set \( S_j \)

\[
= (\text{cyc. perm. on } S_1) \ (\text{cyc. perm. on } S_2) \ldots (\text{cyc. perm. on } S_k)
\]

\[
= \left[ \binom{n}{k} \right] = \# \text{ of permutations of } \{n\} \text{ into } k \text{ cycles.}
\]
\[ D(x) = \leq \frac{(n-1)!}{n!} \frac{x^n}{n^x} = \leq \frac{x^n}{n^x} = \leq \int_0^x t^{n-1} \, dt \]

\[ = \int_0^x \left( \leq t^{n-1} \right) \, dt ; \quad |x| < 1 \]

\[ = \int_0^x \frac{1}{1-t} \, dt \]

\[ = - \log (1-x) \]

\[ H(x,y) = e^{yD(x)} = e^{-y \log (1-x)} = \frac{1}{(1-x)^y}. \]
\[ \begin{align*}
\sum_{n,k=0} \left[ \begin{array}{c} n \\ k \end{array} \right] y^k x^n & = \frac{1}{(1-x)^y} \quad \text{(use extended Bin. Thm.)} \\
& = \sum_{n,k=0} \left[ \begin{array}{c} n \\ k \end{array} \right] (-y)^n x^n \\
& = \sum_{n,k=0} (-1)^n (-y)^n \frac{x^n}{n!} \\
& = \sum_{n,k=0} (y+n-1)_n \frac{x^n}{n!} \\
\Rightarrow \sum_{k=0} \left[ \begin{array}{c} n \\ k \end{array} \right] y^k & = (y+n-1)_n = y (y+1) \cdots (y+n-1).
\end{align*} \]