TODAY

• Exponential generating functions
• Combinatorial theory: exponential formula
• Following chap 3.1–3.4 of text
  'generating functionology'
Exponential generating functions

- Given sequence of numbers \( \{a_n\}_{n \geq 0} \), its efg is
  \[
  A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}
  \]
  Notation: \( \{a_n\} \leftrightarrow A(x) \).

- Egfs can be used to solve many counting problems.
  Old: set partitions & permutation cycles
  New: labelled graphs
- It requires developing a centralized theory.
Combinatorial interpretation of multiplying e.g.f.s.

\[ A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!} \xrightarrow{\text{e.g.f.}} \{a_n\}_{n \geq 0} \]

\[ B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!} \xrightarrow{\text{e.g.f.}} \{b_n\}_{n \geq 0} \]

From before: \( A(x) \cdot B(x) = \sum_{n \geq 0} c_n x^n \), where

\[ c_n = \sum_{k=0}^{n} \frac{a_k b_{n-k}}{k! (n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \]

\[ \downarrow \text{combinatorial} \]
We get the following theorem.

Thm: If $A(x) \leftrightarrow \{a_n x^n \}_{n=0}^{\infty}$ and $B(x) \leftrightarrow \{b_n x^n \}_{n=0}^{\infty}$
then $A(x) \cdot B(x) \leftrightarrow \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right\}$.

**Combinatorial meaning**

Suppose $a_n = \#$ of objects of type $A$ having size $n$.
$b_n = \#$ of objects of type $B$ having size $n$.

- Do the experiment: **Partition** $\mathbb{N}$ into two sets $(S, T)$
  and build type $A$ object on $S$ (with size $|S|$) and type $B$
object on $T$ [with size $1T1$].

- **Ques:** How many ways are there of doing this experiment?

- **Ans:** Condition on $1S1$ and $1T1$. Say $1S1 = k$, so then $1T1 = n-k$ [they partition $n$].

- Given $1S1 = k$, $1T1 = n-k$: Build type A object on $S$ in $a_k$ ways. Then type B object on $T$ in $b_{n-k}$ ways. Everything is independent.

- So # ways given $1S1 = k$ & $1T1 = n-k$: $a_k b_{n-k}$. 
The total number of ways to do experiment is

\[ \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \]

Conditioning \( k=0 \) \( \uparrow \) choose \( S \) build objects

This provides a combinatorial interpretation of \( A(x), B(x) \).

And it generalizes....
Suppose $A_1(x), A_2(x), \ldots, A_K(x)$ are all egfs:

$$A_j(x) = \sum_{n \geq 0} a^{(j)}_n \frac{x^n}{n!},$$
given $a^{(j)}_n = \# \text{ of objects of type } j \text{ having size } n$.

Let $P(x) = A_1(x) \cdot A_2(x) \cdots A_K(x)$ and write

$$P(x) = \sum_{n \geq 0} P_n \frac{x^n}{n!} \xrightarrow{\text{egfs}} \{P_n\}.$$

Then $P_n$ counts the number of ways to do following experiment.
Partition \([n]\) into \(k\) ordered subsets \((S_1, \ldots, S_k)\).

Eg. \((\{1,2,3\}, \{3\}, \{4,5\}) \neq (\{3,3\}, \{1,2,3\}, \{4,5\})\)

are two different partitions of \([5]\) into 3 ordered sets.

- Build an object of type \(j\) on subset \(S_j\) \(1 \leq j \leq k\).

  This can be done in \(a_{S_j}\) ways and object has size \(|S_j|\).

- Put these objects together to make an ordered list of \(k\) objects, one of each type \(j\).
Counting $P_n$:

1. Conditioning on the sizes of the subsets $S_1, \ldots, S_k$.
   Suppose $|S_j|=S_j$. Then $(S_1, \ldots, S_k)$ is a composition of $n$:
   
   $S_1 + S_2 + \ldots + S_k = n$ with $S_j \geq 0$ integer.

2. Given composition $(S_1, \ldots, S_k)$, # of ordered partitions of $[n]$ with
   $|S_j|=S_j$ is the multinomial coefficient

   \[
   \binom{n}{S_1, S_2, \ldots, S_k}
   \]

   [Recall we solved this in Lecture 4].
Given ordered partition \((S_1, \ldots, S_k)\) of \(\mathbb{N}\), the number of ways to build an object is

\[
\prod_{j=1}^{k} a_{S_j}^{(i)} \quad \text{[By independence]}
\]

So,

\[
P_n = \sum_{(S_1, \ldots, S_k) \text{ compositions}} \binom{n}{S_1, \ldots, S_k} a_{S_1}^{(1)} \cdots a_{S_k}^{(k)}
\]

Choose ordered partition of given sizes

Build objects
By induction of the multiplication thm on slide 4, one can show the following.

**Egf multiplication Thm**: Let $A_1, \ldots, A_K$ be egfs with $A_j \leftrightarrow \{a_{jn} \}_{n \geq 0}$. Then

$$P(x) = A_1(x) \cdots A_K(x) \leftrightarrow \{p_{jn} \}_{n \geq 0}$$

with

$$P_n = \sum_{(S_1, \ldots, S_K) \in \mathbb{N}^K} \binom{n}{S_1, \ldots, S_K} a_1^{S_1} \cdots a_K^{S_K},$$

where $S_1 + \cdots + S_K = n$ and $S_j \geq 0$ are integers.
What is so good about this?

- To solve a comb. prob. of type , use egfs to write $p_n$ as a product of simpler egfs. Then read off coefficients.

Example: How many ways are there to divide a class of $n$ students into 3 groups labelled 1, 2, and 3?

Soln: Divide $\binom{n}{2}$ into an ordered partition $(S_1, S_2, S_3)$. 2 students

Build a group on students from each $S_j$. 3 groups
So let \( A_1, A_2, A_3 \) be the cdfs of

\[
A_j \left( \frac{e^{x}}{n} \right) \leq \# \text{ of ways to make a group of } n \text{ students from n} \]n

\[
= \sum_{j=0}^{n} 13^{n-0} = \frac{e^{x}}{n!} = e^{x}
\]

Want \( P_n \), where \( P(x) = \frac{e^{x}}{n!} \).

\[
P(x) = A_1(x) \cdot A_2(x) \cdot A_3(x) = e^{3x}
\]

So \( p_n = 3^n \) since \( e^{3x} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} \).
The Exponential formula [Building Hands from Decks]

- Have a sequence $\{d_n\}_{n=0}^\infty$.
  
  $d_n$ = # of decks of size $n$.

  Want to make a hand:

  - Partition $[n]$ into $K$ subsets $\{S_1, S_2, \ldots, S_k\}$.
    
    Not ordered.

  - Build a deck of size $S_j = |S_j|$ on set $S_j$.

  Weave them together to make a hand.
\( h(n,k) \): \# of hands of size \( n \) weaved with \( k \) decks (that is, \( \{u\} \) is partitioned into \( k \) subsets as above).

\[
\text{Thm: } \sum_{n>0} h(n,k) \frac{x^n}{n!} = \frac{D(x)^k}{k!}, \text{ where}
\]

\[
D(x) = \sum_{n>0} d(n) \frac{x^n}{n!} \xrightarrow{\text{egf}} \sum_{n>0} d(n) 3^n x^n.
\]

That is, \( \{ h(n,k) \}_{n>0} \xrightarrow{\text{egf}} \frac{D(x)^k}{k!} \).