**Mathematical Induction**

You have to prove a sequence of propositions $P(1), P(2), P(3), \ldots$

**Method**

① First verify the base case $P(1)$, or some first few $P(1), \ldots, P(k)$.

② Make the inductive hypothesis that $P(1), \ldots, P(n-1)$ are all true.

③ Use the inductive hypothesis to prove $P(n)$. 
Inductive arguments can have complicated forms.

- Can induct on multiple sequences of propositions at once.
  
  \[
  \begin{align*}
  & P(1), P(2), \ldots \\
  & Q(1), Q(2) \ldots \\
  \end{align*}
  \]

  \[
  \begin{align*}
  & P(1)+Q(1) \rightarrow Q(2) ; \\
  & P(1)+Q(2) \rightarrow P(2) ; \\
  & P(2)+Q(2) \rightarrow Q(3) ; \\
  \end{align*}
  \]

- **Transfinite induction**: Induction on an uncountable sequence of propositions.
  
  \[\Rightarrow\text{Requires the well-ordering principle, which is equivalent to the method of induction.}\]

  \[\Rightarrow\text{Well-ordering principle depends on the axiom of choice.}\]
Axiom of choice (a few words)

Statement: If $I$ is an arbitrary index set and $A_i, i \in I$ are non-empty sets then their product

$$\prod_{i \in I} A_i = \{ (a_i; i \in I) : a_i \in A_i \} \neq \emptyset.$$
Why the controversy? Because it implies certain 'paradoxical' propositions...

Thm [Banach-Tarski] A solid ball in $\mathbb{R}^3$ can be cut up into $\leq 10$ pieces and be rearranged into 2 identical solid balls as the first by using only translations, rotations and reflections of $\mathbb{R}^3$.

"Can turn a peanut into the sun."

Why has this not created a business enterprise?
- The pieces being cut up are non-measurable sets; they do not have well defined volume.
- Physical cutting mechanisms would give measurable sets.
Back to induction: Ex. 1.
Prove that every integer $n \geq 1$ has a binary decomposition

$$n = 2^{b_1} + 2^{b_2} + \ldots + 2^{b_k},$$
where $0 \leq b_1 < b_2 < \ldots < b_k$.

Proof: Base case: $n = 1 = 2^0$.

Inductive hyp: Every $n \leq m$ has a binary decomposition.

Inductive step: Take $m+1 = 2^{b_1} + \ldots + 2^{b_k}$.

If $m+1$ is a power of 2, then we are done. Otherwise, let $2^b$ be the largest num. s.t. $2^b < m+1$.

$$h = (m+1) - 2^b < m+1.$$
can write \( n = 2^{b_1} + \ldots + 2^{b_k} \) \( 0 \leq b_1 < \ldots < b_k \).

\[ \Rightarrow m+1 = 2^{b_1} + 2^{b_2} + \ldots + 2^{b_k} . \]

\( b \neq b_1, b_2, \ldots, b_k \) (claim)

If not, \( b = b_1 \) say, and so \( m+1 \geq 2^{b_1} + 2^{b_1} = 2^{b_1 + 1} \).

Contradiction since \( 2^b \) is largest s.t. \( 2^b \leq m+1 \).

Done by induction. \( \square \)
Another way to do induction

Proof by minimal counterexample (well-ordering principle).

Prove that every integer $n > 2$ has a prime factor.

Proof: Suppose not. Then there is a minimal $n$ such no prime divides it.

- $n$ is not a prime. So $\exists d \neq 1, n$ s.t. $d | n$.
- Then $2 \leq d < n \implies d$ has a prime divisor $p$.
- If $p | d$ and $d | n \implies p | n$. 

The inductive hypothesis is important.

Define a seq. \( a_n \) by \( a_1 = 2 \) and \( a_{n+1} = 2^{a_n} \).

\[
\begin{align*}
A_n &= \underbrace{2^2 \cdot \ldots \cdot 2}_{n \text{-term tower of 2s with height } n} \\
A_1 &= 4, \quad A_2 = 16, \quad A_3 = 2^{16} = 65536, \\
A_4 &= 2^{65536} > 65000.
\end{align*}
\]

Define \( b_n \) by \( b_1 = 100 \) and \( b_{n+1} = 100^{b_n} \).
Prove that $a_{n+3} \geq b_n$.

Use induction: $a_4 = 2^{14} > 100 = b_1$.

Inductive hyp.: $a_{n+3} \geq b_n$ for $1 \leq n \leq m$.

Inductive step: for $m+1$, $a_{m+4} = 2^{a_{m+3}}$

$\geq 2^{b_m} \ (\text{hyp.})$

$\geq b_{m+1} = 100^{b_m} \times$
Correct proof: New inductive hyp.

\[ a_{n+3} \geq 100 b_n \] (stronger hyp \( \Rightarrow a_{n+3} \geq b_n \)).

Base case: \( a_9 = 2 > 65000 \geq (100)^2 \).

Assume that hyp. holds for \( 1 \leq n \leq m \).

For \( m+1 \), \( a_{m+4} = \frac{16}{a_m} \geq 100 b_m \)

For \( m+1 \), \( a_{m+4} = 2 \geq 2 \)

\( \text{want} \geq 100 b_{m+1} \).
\[
\begin{align*}
\text{want} \quad (2^{100})^{b_m} & \geq (100)^{b_{m+1}} \\
\quad \implies 2 & \geq 100^{1 + \frac{1}{b_m}} \\
\text{since} \quad b_m \geq 1 & \implies 1 + \frac{1}{b_m} \leq 2 \\
\quad \implies (100)^{1 + \frac{1}{b_m}} & \leq 100^2 < 2^{100}.
\end{align*}
\]

Done by induction.