TODAY

The Catalan numbers

Announcement: Midterm exam info + practice problems on course webpage.

Covers: Chap. 3-7 of text
Lectures 3-20 except lec. 13 + supplementary notes

Last time: The Ballot Theorem
How many triangulations are there of an n-gon?

2-gon has 1 'triangulation' [line seg = degenerate Δ]

3-gon has 1 triangulation.

4-gon has 2 triangulations.
The 5-gon is a bit harder:

Let $C_n$ be the number of triangulations of an $(n+2)$-gon.

$c_0 = 1$, $c_1 = 1$, $c_2 = 2$, $c_3 = 5$. 
Think recursively

An \((n+2)\)-gon is \(\Delta\)ed recursively.

\[
L = (n+2) - (k-2) - \text{gon} \\
= (n-k+4) - \text{gon} \\
R = (k-1) - \text{gon}
\]

Condition of the third vertex, \(K\), of the \(\Delta\) containing edge 1—2. Note \(3 \leq k \leq n+2\).
• To complete the Δ-action, triangulate \( L \) and \( R \) independently.

\[
\text{# ways to Δ-ste L} = C_{n-k+2}
\]

\[
\text{# ways to Δ-ste R} = C_{k-3}
\]

# of ways to Δ-ste given that \( 1-2 \) is joined to vertex \( k \) is

\[
C_{k-3} \cdot C_{n-k+2}.
\]

By sum rule \( C_n = \sum_{k=3}^{n+2} C_{k-3} \cdot C_{n-k+2} \).
Thm: The number of triangulations of an \((n+2)\)-gon satisfies the recursion

\[
C_n = \sum_{k=0}^{n-1} C_k \cdot C_{n-1-k}, \quad C_0 = 1.
\]

\[
C_n = C_0 \cdot C_{n-1} + C_1 \cdot C_{n-2} + C_2 \cdot C_{n-3} + \ldots + C_{n-1} \cdot C_0
\]

(indices sum to \(n-1\)).
Back to the ballot problem.

Ballot problem $\Rightarrow$ Paths consisting of $U = (1, 1)$ and $D = (1, -1)$ steps s.t.

1. Path is non-negative
2. Goes from $(0, 0) \rightarrow (2n, 0)$
Last time: # such paths = \( P_n = \frac{1}{n+1} \binom{2n}{n} \).

Here is another way to ‘find’ \( P_n \).

- first time path hits \( y = 0 \) after time \( 0 \).
- Suppose the first time path hits 0 after time 0 is \( 2k \), \( 1 \leq k \leq n \).
This decomposes path into a
1) Non-negative path from $(2k,0)$ to $(2k,0)$
2) A non-negative path from $(0,0)$ to $(2k,0)$ that is strictly positive between times $[1,2k-1]$. 

# of $R$-paths $= P_{n-k}$.

# of $L$-paths $= P_{k-1}$.
\# of paths from \((0,0)\to(2n,0)\) = \\
\sum_{k=1}^{\infty} P_{k-1} \cdot P_{n-k} = \sum_{k=0}^{n-1} P_k \cdot P_{n-1-k},
\\
P_0 = 1.
\\
\bullet \quad P_n = C_n = \frac{1}{n+1} \binom{2n}{n}, \text{ by induction and the Ballot Theorem!}
The Catalan recursion

\[ C_n = \sum_{k=0}^{n-1} C_k \cdot C_{n-1-k}, \quad C_0 = 1 \]

The Catalan numbers: \[ C_n = \frac{1}{n+1} \binom{2n}{n} \].

What is the Catalan recursion?

It usually results from a divide and conquer approach to solving problems.
You have experienced the Catalan recursion before...

\[ N_1 = 42039 \quad N_2 = 8910 \]

\[ N_1 = 4 \cdot 10^0 + 3 \cdot 10^1 + 0 \cdot 10^2 + 1 \cdot 10^3 + 9 \cdot 10^4 \]

\[ N_2 = 1 \cdot 10^0 + 0 \cdot 10^1 + 1 \cdot 10^2 + 9 \cdot 10^3 + 8 \cdot 10^4 \]

\[ N_1 \cdot N_2 = (4.1) \cdot 10^0 + (4.0 + 1.3) \cdot 10^1 + (4.1 + 3.0 + 0.1) \cdot 10^2 + (4.9 + 3.2 + 0.0 + 1.1) \cdot 10^3 + \ldots \]
Making it formal:

Take two power series:

\[ A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n, \]

\[ B(x) = b_0 + b_1 x + b_2 x^2 + \cdots = \sum_{n=0}^{\infty} b_n x^n. \]

Write \[ A(x). B(x) = \sum_{n=0}^{\infty} c_n x^n \]
Claim: $m_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$.

$$= \sum_{k=0}^{n} a_k b_{n-k}.$$  \[ \text{called convolution.} \]

$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$  \[ \text{called convolution.} \]

$B(x) = b_0 + b_1 x + b_2 x^2 + \ldots$

$A(x) \cdot B(x) = (a_0 b_0) x^0 + (a_0 b_1 + b_0 a_1) x^1 + \ldots$
Now we can find formula for Cetlën numbers from the Cetlën recursion.

\[ C_n = \sum_{k=0}^{n-1} C_k \cdot C_{n-1-k}, \quad C_0 = 1 \]

Set \( n := n + 2 \) and write

\[ C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k}, \quad C_0 = 1. \]
Let \( C(x) = \sum_{n \geq 0} C_n x^n \).

- Multiply \( \Theta \) by \( x^n \) and sum over all \( n \geq 0 \).

LHS = \( \sum_{n \geq 0} C_{n+1} x^n = \frac{1}{x} \cdot \sum_{n \geq 0} C_{n+1} x^{n+1} \)

\[
\text{LHS} = \frac{C(x) - 1}{x}.
\]

\[
= \frac{1}{x} \sum_{n > 1} C_n x^n \quad [n \to n-1]
\]

\[
= \frac{1}{x} [C(x) - C(x^2)] = \frac{c(x) - 1}{x}.
\]
RHS: \[ \leq \left( \sum_{k=0}^{n} C_k C_{n-k} \right) X^n \]

\[ \leq 0 \quad \text{for} \quad n > 0 \]

\[ = C(x)^2 \quad \text{by our previous claim} \]

So \[ \frac{C(x) - 1}{X} = C(x)^2 \]

\[ \iff X \cdot C(x)^2 - C(x) + 1 = 0 \]