TODAY

The Ballot Theorem

Announcement: Midterm on Nov 1. Practice problems + info on course webpage by Monday.

Last time Fibonacci Identities
Election day. Two candidates: Ursula vs. David

\# voters = 2n

\# votes for Ursula = n

\# votes for David = n

Q: How many voting orders such that Ursula never trails David?
Example: \( n = 4 \)

Voting order: \( \text{UDUUDUDD} \)

<table>
<thead>
<tr>
<th>After vote #</th>
<th>1 2 3 4 5 6 7 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>U votes - D votes</td>
<td>1 0 1 2 1 2 1 0</td>
</tr>
</tbody>
</table>

By picture:

\( \text{U vote} = (1, 1) \)
\( \text{D vote} = (1, -1) \)
Picture is worth many words.

Any voting order gives a path starting from (0, 0) and consisting of:

- Up step $U = (1, 1)$ [vote for U]
- Down step $D = (1, -1)$ [vote for D].

$Y$-axis = $U$ votes - $D$ votes
$X$-axis = after vote num.
Dictionary: Voting $\leftrightarrow$ U/D paths

Suppose $P$ is a U/D path from $(0,0)$ to $(x,y)$

e.g. In the above a path from $(0,0)$ to $(10,1)$

\[
\begin{align*}
X &= \# \text{ total votes} = \# \text{ U steps} + \# \text{ D steps} \\
Y &= \# \text{U votes} - \# \text{D votes} = \# \text{ U steps} - \# \text{ D steps} \\
\Rightarrow \# \text{ U steps} &= \frac{X+Y}{2}, \quad \# \text{ D steps} = \frac{X-Y}{2}.
\end{align*}
\]
Back to our main question.

Count the complement.

Consider all voting orders with $U$ getting $n$ votes and $D$ getting $n$ votes.

$$\# \text{ voting orders} = \binom{2n}{n}.$$ 

Two kinds of voting orders.
- Good order: Ursula never trails David.
- Bad order: Ursula trails David at some time.

\[ | \text{Good orders} | = \binom{2n}{n} - | \text{Bad orders} | \]

\[ \text{want total} \]

Will count bad voting orders... by picture.
Bed voting order

An $n=4$ case

Any bad order is a path from $(0,6)$ to $(2n,0)$ that hits level $y=-1$. 
Take a bad order and consider

- First time it hits line $y = -1$
  (in example at time 5).
- Take the initial segment from time 0 to the time it hits $y = -1$ and reflect along line $y = -1$.

![Graph showing reflected bad order and reflected segment](image)
• After reflecting initial segment \( \rightarrow \) a path from \((0,-2)\) to \((2n,0)\) that hits \(y=-1\) at the same first time as the bad path.

**Reflection principle**

This is a bijection from bad voting orders \(\leftarrow\) paths from \((0,-2)\) to \((2n,0)\).

[Inverse: reflect again!]
\[ |\text{Bad orders}| = |\text{paths from } (0, -2) \to (2n, 0)| \]

\[ = |\text{paths from } (0, 0) \to (2n, 2)| \left[ \text{translate up by } 2 \right] \]

0. paths from \((0, 0)\to(2n, 2)\) \(\implies\) 
\[ \# U\text{ steps} = \frac{2n + 2}{2} = n + 1, \]
\[ \# D\text{ steps} = n - 1. \]

0. \(|\text{paths from } (0, 0) \to (2n, 2)| = \binom{\# \text{ total steps}}{\# U\text{ steps}} = \binom{2n}{n+1}.\]
\[ | \text{Good orders} | = \binom{2n}{n} - \binom{2n}{n+1} \]

\[ \quad \uparrow \quad \uparrow \]

\[ \quad \text{total} \quad \text{bad ones} \]

\[ = \frac{1}{n+1} \binom{2n}{n} . \]

Answer to Q: \( \frac{1}{n+1} \binom{2n}{n} \) voting orders.
Q: Ursula gets $u$ votes and David gets $d$ votes.

How many voting orders with Ursula never trailing David?

Answer: Paths from $(0, 0) \to (u+d, u-d)$ that never fall below $y=0$. 
As before: \[ |\text{Good}| = |\text{Total}| - |\text{Bad}| \]

\[ = \left( \frac{u+d}{u} \right) - |\text{Bad}| \]

Bad paths: As before, reflect segment of bad path from \((0,0)\) to \((t,1)\) about \(y=-1\).

Ex: reflect to hit \(y=-1\)
Bad paths $\leftrightarrow$ paths from $(0,-2) \rightarrow (u+d, u-d)$

$\leftarrow$ paths $\rightarrow$ $(0,0) \rightarrow (u+d, u-d+2)$

$|\text{Bad paths}| = \binom{u+d}{u+1}$.

$|\text{Good paths}| = \binom{u+d}{u} - \binom{u+d}{u+1} = \frac{u-d+1}{u+1} \cdot \binom{u+d}{d}$