TODAY

Some Fibonacci identities with combinatorial proofs

Announcement: HW 4 is posted, due Nov 8.
Midterm on Nov 1 in 35-225.

Last time: Fibonacci numbers $\leftrightarrow$ filings by $\square$ or $\square$
- Supplementary notes on course webpage.
We will prove the following.

\[ \sum_{i,j \geq 0} {n-i \choose j} {n-j \choose i} = f_{2n+1} . \]

\[ \sum_{k \geq 0} (k^n) f_{k-1} = f_{2n-1} . \]

\[ f_n^2 = f_{n+1} f_{n-1} + (-1)^n \quad [\text{Cassini's id.}] . \]
Identity 1: \[ \sum_{i,j \geq 0} (n-i)(n-j) = f_{2n+1} \]

RHS: # of tilings of \(1 \times (2n+1)\) board with \(S\) and \(D\).

LHS: • A tiling of \(1 \times (2n+1)\) board has an odd number of \(S\).
    • Consider the position of the median square.
    • Suppose \(i\) dominoes left of median square, \(j\) right.

\[ D = i + j \rightarrow 2(i+j) \text{ cells occupied by } D. \]
\[ S = 2n+1 - 2(i+j) = 2(n-i-j)+1 \]

- # tiles left of median \( S = n-j \) (i D)
  - " " right " " \( S = n-i \) (j D)
0. # ways to tile left of median \( S = \binom{n-j}{i} \)

- right \( \rightarrow \) \( S = \binom{n-i}{j} \)

# ways to tile with \( i D < \text{median} S \)

\[ j D \geq \text{median} S = \left( \binom{n-j}{i} \right) \cdot \left( \binom{n-i}{j} \right) \quad \text{[product rule]} \]

- Sum over all values of \( i, j \geq 0 \rightarrow \text{LHS} \)
Identity 2: \[ \sum_{k=0}^{n} \binom{n}{k} f_{k-1} = f_{2n-1} \]

RHS: \# of tilings of an \( 1 \times (2n-1) \) board.

CHS:  
1. There are at least \( \left\lfloor \frac{2n-1}{2} \right\rfloor = n \) tiles in any tiling of a \( 1 \times (2n-1) \) board.
2. Condition on \# of \( S \) in first \( n \) tiles. Suppose it is \( K \), \( 0 \leq K \leq n \).
The number of cells remaining to be tiled is:

\[(2n-1) - [k + 2(n-k)] = k-1\]

Complete the last \((k-1)\) cells in \(f_{k-1}\) ways.
Identity 3: \( f_n^2 = f_{n-1} \cdot f_{n+1} + (-1)^n \).

We will find almost bijection...

Set 1: Tilings of a pair of \( 1 \times n \) boards

\[
\begin{array}{cccc}
1 & 2 & \ldots & n \\
\end{array}
\]  
\text{top board}

\[
\begin{array}{cccc}
2 & \ldots & n & n+1 \\
\end{array}
\]  
\text{bottom board}

Set 2: Tilings of a \( 1 \times (n-1) \) board and a \( 1 \times (n+1) \) board.
\[ |\text{Set 1}| = f_n^2, \quad |\text{Set 2}| = f_{n-1} \cdot f_{n+2}. \]

- Consider overwriting.

Given a top and bottom board as in set 1 and set 2, a fault at cell i
means that both top and bottom board are breakable at cell i.

\[ \text{square} \]

\[ \begin{array}{c|c|c|c}
1 & 2 & i & 1 \\
\end{array} \]

\[ \begin{array}{c|c|c|c}
2 & i & 1 & 1 \\
\end{array} \]

square \[ \xleftarrow{\text{fault}} \]

\text{fault at i: Top and bottom board are both not covered by domino at cells i and i+1.}

Fault at 1 \[ \xleftarrow{\text{cell 1 of top board in a square.}} \]
Example of faults: $n = 8$

Faults: 1, 3, 5, 7, and 8.

Last fault: 8
Observe: Always a fault if at least one of the boards has a square.

**Bijection:** Set 1 with at least 1 fault $\rightarrow$
Set 2 $\ldots$ 1 $\ldots$

Given a tiling of Set 1 with a fault, take the fault, break both tilings (top & bottom) and swap the tails of the two tilings.
Tail swapping takes Set 1 tilings $\rightarrow$ Set 2 tilings.

Tail swapping twice $\rightarrow$ original tiling; bijection.

- Observe that after tail swapping the top tile has \( n+1 \) cells and bottom tile has \( n-1 \) cells.
As tail swapping is a bijection between
\[
\{ \text{set 1 w. a fault} \} \leftrightarrow \{ \text{set 2 w. a fault} \},
\]
\[
|\{ \text{set 1 w. a fault} \}| = |\{ \text{set 2 w. a fault} \}|
\]

Now consider 2 cases:

- n odd: Then set 1 always has a fault since top and bottom board must have a square.
Only 1 tiling of set 2 does not have a fault: all dominoes tiling.

\[
\begin{array}{c}
1:2 \\
2: \ldots \\
\end{array}
\] \} \text{No faults}

So for \( n \) odd,

\[
|\text{set 2 w. faults}| = |\text{set 1}| = f_n^2
\]

and

\[
|\text{set 2 w. faults}| = |\text{set 2}| - 1 = f_{n+1} \cdot f_{n-1} - 1
\]

Thus, \( f_n^2 = f_{n+1} \cdot f_{n-1} - 1 \) for \( n \) odd.
Case of $n$ even: Set 2 always has a fault because $n+1$ and $n-1$ are odd, so there is always a square in the tiling.

Set 2 has exactly 1 tiling w. no faults: all dominoes

\[ \begin{array}{cccccccc}
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
2 & 2 & 2 & 2 & \cdots & 2 & 2
\end{array} \]

\{ No faults. \}

Thus,

- $|\text{Set 1 w. faults}| = |\text{Set 1}| - 1 = f_n^2 - 1$
- $|\text{Set 2 w. faults}| = |\text{Set 2}| = f_{n+1} \cdot f_{n-1}$

We get $f_n^2 - 1 = f_{n+1} \cdot f_{n-1} \Rightarrow f_n^2 = f_{n+1} \cdot f_{n-1} + 1$. 