TODAY

- Fibonacci numbers: combinatorial view

Last time

- Solving linear recurrences

L3 supplementary handout on webpage.
\[ F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1. \]

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

Some history

- Considered Leonardo Bonacci of Pisa around 1200s.
  \[ \text{Fibonacci} \leftarrow \text{Filius Bonacci} \]
  \[ \text{Son of Bonacci} \rightarrow \]

- \( F_n \) = rabbit population in \( n \)-th generation
- Fibonacci did not use the recursive definition.
• DeMoivre (~1700) gave the recursive form

\[ F_{n+2} = F_{n+1} + F_n \]

• DeMoivre, Binet and Bernoulli independently found the solution

\[ F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \]

• Lucas (also introduced Tower of Hanoi prob.) attributed the sequence to Fibonacci in 1876.
A combinatorial interpretation.

Two kinds of tiles

\[ \square \rightarrow \text{square} \quad (1 \times 1) \]

\[ \sqcap \rightarrow \text{domino} \quad (1 \times 2) \]

Board of length \(1 \times n\):

\[ \text{cell 1} \quad \text{cell n} \]

Question: How many ways to tile using squares and dominos?
Ex:  \( n = 3 \) 
\[
\begin{align*}
    1 & \quad 2 & \quad 3 \\
\end{align*}
\]
3 squares
\[
\begin{align*}
    1 & \quad 2 & \quad 3 \\
\end{align*}
\]
1s, 1d \( \{ \) 3 ways
\[
\begin{align*}
    1 & \quad 2 & \quad 3 \\
\end{align*}
\]
1d, 1s \( \} \)

General approach: \( f_n = \# \text{ ways to tile } 1 \times n \) board with 1s and 1d.

Think recursively: To find \( f_n \), condition on whether first tile is 1s or 1d.
If 1st tile is S → tile remaining 1x(n-1) board in \( f_{n-1} \) ways.

If " " d → 1 " " 1x(n-2) board in \( f_{n-2} \) ways.

\[ f_n = f_{n-1} + f_{n-2}; \quad f_1 = 1, \ f_2 = 2 \]

By convention: \( f_0 = 1 \).

\[ f_n = F_{n+1} \quad \text{for} \quad n \geq 0. \]
Prove Fibonacci identities using the comb. view.
Ex: Prove $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.
(Running sum identity).

Proof: Q: How many ways are there to tile a $1 \times (n+1)$ board using $S$ and $D$ so there is at least 1 domino?

Express identity in terms of $f_n$: $f_0 + f_1 + \cdots + f_{n-1} = f_{n+1} - 1$. 
RHS = tile 1 x (n+1) board in fn+1 ways and subtract the all squares tiling \( \rightarrow \) fn+1 - 1 ways.

LHS = Condition on the location of the last domino in the tiling (\( \geq 1 \) domino).

If last domino on cells \( k \) and \( k+1 \) \( \rightarrow \) fn - 1 ways to do it.

Since \( 1 \leq k \leq n \) \( \rightarrow \) the RHS by the sum rule.
Identity 2 (Convolution identity)

\[ F_{m+n} = F_m \cdot F_{n+1} + F_{m-1} \cdot F_n. \]

Proof: 1) Convert into identity via \( F_n = f_{n-1} \)

\[ f_{m+n-1} = f_{m-1} \cdot f_n + f_{m-2} \cdot f_{n-2} \]

2) Apply \( m \to m+1 \): \( f_{m+n} = f_m \cdot f_n + f_{m-1} \cdot f_{n-1} \).

Q: How many ways to tile an \( 1 \times (m+n) \) board?
LHS: \( f_{m+n} \) ways by defn.

RHS:

Either the tiling is breakable at cell \( m \) or not breakable

\begin{align*}
\text{Breakable} & \Rightarrow \text{a tile ends at cell } m.
\end{align*}
- If breakable $\Rightarrow$ break into a $1 \times m$ board and a $1 \times n$ board and tile each independently.

  $\Rightarrow f_m \cdot f_n$ ways.

- If not breakable $\Rightarrow$

  Then tile each of these independently in $f_{m-1} \cdot f_{n-1}$ ways.
Sum rule \[ f_{m+n} = f_{m-1} \cdot f_{n-1} + f_m \cdot f_n \].
(Breaking a tiling is an important notion)

Fibonacci \leftrightarrow Binomial identity:

\[ f_n = \sum_{k=0}^{n} \binom{n}{k} \cdot \]

Proof: How many tilings of an \(1 \times n\) board?
To get RHS condition on the # of dominoes in the tiling.

- If there are $k$ dominoes in the tiling then
  - $k$ dominoes takes $2k$ cells
  - $n-2k$ cell taken by squares
  - In total, $(n-2k)+k = n-k$ tiles used.
- So we choose $k$ domino tiles from $(n-k)$ total tiles in $\binom{n-k}{k}$ ways. Sum over $k \geq 0 \rightarrow$ RHS.
Note that tiling is uniquely determined by choosing domino positions from the \((n-k)\) tiles.

e.g. \(n-k = 5\) tiles, \(k = 2\) dominoes

\(n-2k = 7 - 4 = 3\) squares

\(5 \times 3, \ 2 \times 2\); \(d\) in positions \((3,5)\) then

\[\text{tiling} = s, s, d, s, d \rightarrow \begin{array}{cccccc}
1 & 2 & \_ & \_ & 5 & 6 \end{array} \]