18.211 Assignment 4

Due: Nov 8, in class by 2:10 pm

**Reading and practice problems:** Read the supplementary notes on Fibonacci numbers and Catalan numbers from the course webpage. Do some of the problems listed on the handout on Fibonacci numbers. Do not hand in these exercises.

Hand in solutions to **any 7 of the following 10** problems:

1) Solve the recurrence $a_{n+3} = 5a_{n+2} - 8a_{n+1} + 4a_n$ with $a_0 = 0$ and $a_1 = a_2 = 1$.

2) Show that

$$ a_n = \left(\frac{1 + \sqrt{13}}{2}\right)^n + \left(\frac{1 - \sqrt{13}}{2}\right)^n $$

is an integer for every $n \geq 0$.

3) Call a set of positive integers fat if each of its elements is at least as large as its cardinality. For example, \{6, 10, 11, 20, 33, 34\} is fat but \{2, 100, 200\} is not. Let $f(n)$ be the number of fat subsets of $[n]$, counting the empty set as fat. Show that for $n \geq 2$,

$$ f(n + 1) = f(n) + f(n - 1). $$

4) Prove the following Fibonacci identity using a combinatorial argument.

$$ F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1. $$

5) Prove the sum of squares identity:

$$ F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}. $$

A non-combinatorial argument receives at most 70% of full credit.

6) Consider the recurrence $a_{n+2} = a_{n+1} - 2a_n + n$ with $a_0 = 1$ and $a_1 = 0$. Let $A(x) = \sum_{n \geq 0} a_n x^n$ be the generating function of the sequence $a_n$. Find the closed form of $A(x)$ as a rational function.

7) Use generating functions to find a closed form formula for $s(n) = \sum_{k=0}^{n} k(k-1)^2$.

8) Let $D_n$ be the number of dearrangements of $[n]$, that is, the number of permutations of $[n]$ with no fixed points. This question derives the generating function of the sequence $D_n$. 


I) Show that the $D_n$s satisfy
\[ n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}. \]

II) Let $D(x) = \sum_{n \geq 0} D_n \frac{x^n}{n!}$. Using part (I) show that
\[ D(x) = \frac{e^{-x}}{1-x}. \]

9) A Standard Young Tableaux of size $2 \times n$ is a $2 \times n$ array with the numbers $\{1, \ldots, 2n\}$ inserted into it such that the numbers increase along all rows and columns (going left to right for rows and top to bottom for columns). Prove that the number of such tableaux is $C_n = \frac{1}{n+1} \binom{2n}{n}$. Hint: Find a bijection with voting sequences of the Ballot Theorem.

10) Let $\gcd(a, b)$ denote the greatest common divisor of two integers $a$ and $b$. Prove that the Fibonacci numbers satisfy $\gcd(F_n, F_m) = F_{\gcd(n, m)}$. 