18.211 Assignment 2

Due: Oct 18, in class by 2:10 pm

Reading and practice problems: Read chapters 5, 6 and 7 from the textbook. Do exercises 6, 7, 11, 18, 29 from chapter 5, exercises 7, 17, 18, 26, 37(a) from chapter 6, and exercises 3, 9, 14, 19, 26 from chapter 7. Do not hand in these exercises.

Hand in solutions to any 7 of the following 10 problems:

1) Derive the following identities using the Binomial Theorem.
   
a) \( \sum_{k=0}^{n} k^2 \binom{n}{k} = (n+1)n2^{n-2} \).
   
b) \( \sum_{k=0}^{n} \frac{1}{k+2} \binom{n}{k} = \frac{n2^{n+1} + 1}{(n+2)(n+1)} \).

2) Find the number of integers in \([1700]\) that are relatively prime to 170. Show your work.

3) Consider the permutation of \([n]\) given by the arrangement \( n \ (n-1) \cdots 2 \ 1 \). Find the cycle decomposition of this permutation.

4) Provide a combinatorial proof of the following identity.
   
   \( S(n+1, m+1) = \sum_{k=0}^{n} \binom{n}{k} S(k, m) \).

5) The Binomial Theorem provides the relation \( \vec{v}(x) = P\vec{u}(x) \), where \( P \) is the infinite matrix with entries \( P(n, k) = \binom{n}{k} \) for \( n, k \geq 0 \), \( \vec{v}(x) \) is the infinite vector with \( n \)-th row \((1 + x)^n\) and \( \vec{u}(x) \) is the infinite vector with \( n \)-th row \( x^n \). Find the entries of inverse matrix \( P^{-1} \). Hint: Use the Binomial Theorem.

6) Suppose that a permutation \( \sigma \) of \([n]\) satisfies \( \sigma^7 = \text{id} \). Prove that the cycles of \( \sigma \) have length 1 or 7.

7) Use a combinatorial argument to prove that
   
   \( \left[ \begin{array}{c} n+1 \\ 2 \end{array} \right] = n! \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) \).
8) An index \( i \) is a \textit{record} of a permutation \( \sigma \) if \( \sigma(i) > \max\{\sigma(1), \ldots, \sigma(i-1)\} \). By convention, 1 is always a record. Prove that the number of permutations with exactly \( k \) records is \( \binom{n}{k} \).

Hint: Find a bijection.

9) Prove that for \( n, m \geq 0 \) and \( 0 \leq k \leq n \),

\[
\binom{n}{k} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n + m - j}{k - j}.
\]

10) Recall \( p(n) \) is the number of integer partitions of \( n \). Prove that

\[
\sum_{n \geq 0} p(n) x^n = \sum_{k \geq 0} \frac{x^{k^2}}{(1 - x)(1 - x^2) \cdots (1 - x^k)^2}.
\]