18.211 Assignment 2

Due: Oct 4, in class by 2:10 pm

**Reading and practice problems:** Read chapters 3 and 4 from the textbook. Do exercises 2, 4, 6, 8, 10, 30, 34, 35 from chapter 3 and exercises 3, 5, 7, 9, 13, 15, 37, 39 from chapter 4. Do not hand in these exercises.

Hand in solutions to any 7 of the following 10 problems:

1) Carefully write proofs of the following propositions. The proofs need not use combinatorial arguments but the reader should be able to follow your steps clearly.
   a) For \(1 \leq k \leq n - 1\), the binomial coefficients satisfy
      \[
      \binom{n}{k}^2 > \binom{n}{k - 1} \binom{n}{k + 1}.
      \]
   b) For \(0 \leq k \leq n\), the binomial coefficients \(\binom{n}{k}\) is maximized at \(k = \lfloor n/2 \rfloor\).

2) Determine the number of integer solutions to the equation
   \[x_1 + x_2 + x_3 + x_4 = 43,\]
   where (a) every \(x_i \geq 0\), and (b) every \(x_i \geq 3\).

3) Suppose you expand \((x + y + z)^{2017}\) into sums of terms of the form \(x^a y^b z^c\). How many of these terms have \(x\) appearing to at least the first power, i.e., \(a \geq 1\)? (Note that \(2017 x^{2016} y\) counts as one term rather than 2017 terms.)

4) This is a two part question about restricted arrangements.
   a) How many arrangements of MASSACHUSETTS have no adjacent S’s?
   b) How many arrangements of COMBINATORICS have the vowels in alphabetical order?

5) How many arrangements of RECURRENCE RELATION (ignoring the space) have the first R following the first E and the first C following the first N?

6) For \(n \geq 0\), consider the sequence
   \[a_n = \sum_{k \geq 0} \binom{n - k + 1}{k}.\]
Recall that \( \binom{n}{k} = 0 \) if \( k > n \). Verify that \( a_0 = 1 \) and \( a_1 = 2 \). Use a combinatorial argument to prove that

\[
a_n = a_{n-1} + a_{n-2}.
\]

7) For integers \( m, n \geq 1 \), find a simple formula for the number of non-decreasing functions \( f : [n] \rightarrow [m] \). For example, if \( n = 3 \) and \( m = 5 \) then a non-decreasing function is \( f(1) = 1, f(2) = 1, \) and \( f(3) = 2 \).

8) Provide a combinatorial proof of the identity

\[
n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n.
\]

9) Prove that

\[
\sum_{k \geq 1} \left( \binom{n}{\lfloor \log_3 k \rfloor} \right) = 2^{2n+1},
\]

where \( \binom{n}{k} = 0 \) if \( k > n \).

10) Suppose a positive integer \( n \) has \( k \) ones in its binary representation. Prove that the \( n \)-th row of Pascal’s triangle contains \( 2^k \) odd numbers. Hint: First show that \((1+x)^{2^j} \equiv 1 + x^{2^j} \pmod{2}\) as polynomials.