FALL 2022 LEARNING SEMINAR

Euler systems. Primary reference: [Gro91].

(1) Overview of Kolyvagin's result, Heegner points
- Review the Selmer and Tate–Shafarevich groups of elliptic curves (for example [Ski18, Section 2.1]).
- State the result (Proposition 2.1), and sketch the overall argument.
- Introduce the Heegner points and prove the Euler system relation (Proposition 3.7). Remark about the action of complex conjugation (Propositions 5.3, 5.4).
- Motivate the Kolyvagin derivatives D_l and the notion of Kolyvagin primes[1] of §3, and define the Kolyvagin classes $c(n)$.

(2) Kolyvagin system relations, global duality argument
- Describe the cohomology groups $H^1(K_\lambda, E[p])$ and the finite/singular isomorphism à la [MR04, Section 1.2] for λ a Kolyvagin prime[2]. Remark about the action of complex conjugation.
- Prove the Kolyvagin system relation (Proposition 6.2), writing it in the form of [MR04, Definition 3.1.3].
- Chebotarev argument in §9: prove that if $c^\pm \in H^1(K, E[p])^\pm$ are two classes, then there is a positive proportion of Kolyvagin primes for which $\mathrm{loc}_\lambda(c^\pm)$ is nonzero if $c^\pm$ is nonzero.
- Note that the image of the local Kummer maps are isotropic under the local pairing, and explain the global duality argument in §10.

(3) Euler system of cyclotomic units
- Discuss some of the history of class numbers of cyclotomic fields, if time permits.
- Define the Euler system of cyclotomic units for the case $F = \mathbb{Q}(\mu_p)^+$, as in [Rub00, Section III.2] and [Lan90, Appendix.1-4], using similar language as we used in the previous talks.
- Run through the Kolyvagin system argument modulo p^n to prove [Lan90, Appendix.Therem 4.1].

[1] This is the common name for primes l such that $\mathrm{Frob}(l) = \mathrm{Frob}(\infty)$ as in the paper.
[2] Note that the Frobenius of [MR04, Section 1.2], in our case, is the identity!

p-adic Hodge theory. Primary reference: [BC09].

(4) Hodge–Tate representations, general formalism
- §1.
- §2.1-2.2. Then define B_{HT} (Definition 2.4.7) and do Example 2.4.13, Example 2.3.3.
- §5 (except the comments regarding B_{dR}). Prove at least (1) and (2) of Theorem 5.2.1. Remark that D_{HT} inherits the graded structure of B_{HT} (as in Definition 2.4.12).
- Motivate the need for better period rings: (i) D_{HT} is not fully faithful (bottom of page 24), (ii) Theorem 2.4.6 (prove at least the first part) and the remark following it.

(5) De Rham representations
- §4. A lot of §4.2-4.4 already appeared in our previous learning seminar on p-adic Shtukas. Make this connection and feel free to skip proofs we already did[3].
- §6. Make sure to cover Example 6.3.9 and the remarks that follow.

(6) Crystalline and Semistable representations
- §7.3. Make sure to give Definition 7.3.4 and to mention that it is not an abelian category.
- Define the notion of weak admissibility (Definition 8.2.1, Lemma 8.1.13), and make the remark on top of page 109. Mention that this category is abelian (bottom of page 110).
- §9.1. Make sure to mention Theorem 9.1.10, Proposition 9.1.11 and Example 9.1.12.
- Mention that B_{st} exists and give, without proofs: (i) Theorem 9.3.4, and that it is in in fact an equivalence of categories, (ii) Theorem 9.3.5.

Bloch–Kato conjecture. Primary reference: [Bel09].

(7) Conjectures about Galois representations
- §1.1-1.2: Make sure to include Predictions 1.2 and 1.3. Mention the case in Exercise 1.7.
- §3: Make sure to include Conjecture 3.1 and the remarks following it. Make sure to state Conjecture 3.2 and how it looks in the case that V is polarized, as in equation (11).

(8) Bloch–Kato Selmer group and Bloch–Kato conjecture
- §2.1: In §2.1.1, mention Proposition 2.1 for G as in Exercise 2.1.c and 2.1.d. In 2.1.2, we already discussed the finite level Kummer maps, so give a brief reminder and construct κ as above Exercise 2.4. Cover the basic results in §2.1.3-§2.1.5: Propositions 2.2, 2.3, 2.5, 2.6, 2.7.

[3] For R as in the notes, the tilt of $(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ is $(R[1/p], R)$, and this gives $\theta\colon W(R) \to \mathcal{O}_{\mathbb{C}_p}$ which is surjective and with kernel primitive of degree 1 (see for example Lemma 2.12 of Vijay's notes of our previous learning seminar).

- §2.2: Give at least a sketch of Proposition 2.8, and prove Theorem 2.1. Comment on Propositions 2.9, 2.10. Comment on H^1_e, H^1_g without proofs, including Proposition 2.11 and the analogies in 2.2.3.
- §2.3: Deduce Theorem 2.2 from Proposition 2.7 and write it in the form of Remark 2.3. Comment on the examples of §2.3.2, §2.3.3.
- §4: State the conjecture, and do the examples in §4.1.2. Comment also on Prediction 4.1, Theorem 4.2 and Corollary 4.1. Comment on the examples in §4.3 if there is time.

Further topics.

(9) Introduction to Iwasawa theory [Was97, Chapter 13]
(10) Iwasawa theory of elliptic curves [Ski18]
(11) Beilinson's conjecture on special values of L-functions [Sch88]
(12) Introduction to Kudla's program [KR14, Li22a, Li22b]

References

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[Lan90] Serge Lang. *Cyclotomic fields I and II*, volume 121 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990. With an appendix by Karl Rubin.

[Li22a] Chao Li. From sum of two squares to arithmeti Siegel–Weil formulas, 2022. URL: http://www.math.columbia.edu/~chaoli/ASWSurvey.pdf.

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