FALL 2022 LEARNING SEMINAR: IWASAWA THEORY OF ELLIPTIC CURVES

MURILO CORATO ZANARELLA

November 28, 2022

Warning: These notes are meant to give a big picture overview of the subject. I will not try to spell out all the technical assumptions for the "big" theorems in this exposition, and many claims will only be approximately correct. For precise result, one should follow the references given.

1. General philosophy

Let $T \subseteq V$ be a lattice inside a geometric p-adic representation, and denote $W = V/T$. We consider the Bloch–Kato Selmer groups $H^1_f(F, ?)$ for $? \in \{T, V, W\}$. The group $H^1_f(F, W)$ contains interesting information besides just the dimension $\dim H^1_f(F, V)$. Namely, we define

$$Ш_f(W/F) := H^1_f(F, W)_{/\mathrm{div}}$$

where the subscript means the quotient by the maximal divisible submodule (which is the image of $H^1_f(F, V)$).

Example 1.1. If $T = \mathbb{Z}_p(1)$, then $Ш_f(W/F) = \mathrm{Cl}(F)[p^\infty]$. If $T = \mathbb{Z}_p$, then $Ш_f(W/F) = \mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Q}_p/\mathbb{Z}_p)$. If $T = T_pE$, then $Ш_f(W_pE/F) = Ш(E/F)[p^\infty]_{/\mathrm{div}}$, which is, of course, $Ш(E/F)[p^\infty]$ if this is finite.

Paraphrasing Kato, there are three phases of understanding of special values of L-functions. Here we think of V to be the p-adic realization of some motive.

(0) The Bloch–Kato conjecture predicts the order of vanishing $\mathrm{ord}_{s=0}L(s, V)$ to be $\dim H^1_f(F, V^*(1)) - \dim H^0(F, V^*(1))$. So let's assume this is 0.

(1) $L(0, V)$ is often algebraic except for certain *periods*. In some cases, Deligne and Beillison conjecture certain periods $\Omega_{V,r}$, such that $L(0, V) \in \Omega_{V,r} \cdot \overline{\mathbb{Q}}^\times$. We will denote $L(0, V)/\Omega_{V,r}$ by $L(0, V)_{alg}$.

(2) As we vary V in some suitable p-adic family, the values $L(0, V)_{alg}$ often vary p-adically as well.

(3) The value $L(0, V)_{alg}$ often have deep arithmetic significance.

Example 1.2. Last week we saw this for the family $\mathbb{Q}_p(k) \otimes \omega^k\chi$ for $\chi\colon \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \to \mathbb{Z}_p^\times$ as we vary $k \leq 0$. We have $L(0, \mathbb{Q}_p(k) \otimes \omega^k\chi) = L(k, \omega^k\chi)$, which is nonzero only if χ is odd. Then these values are exactly what are interpolated by $\mathscr{L}_{KL}^{\chi\omega^{-1}}$. As we discussed, this p-adic L-function is deeply related to the p-primary part of class groups of p-power cyclotomic fields.

This is still quite vague, so let's start to get more concrete. Let K_∞/K be a $\mathbb{Z}_p^d$-extension and denote $\Gamma := \mathrm{Gal}(K_\infty/K)$. Let $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$ be the Iwasawa algebra. It is isomorphic to $\mathbb{Z}_p[\![T_1, \ldots, T_d]\!]$.

For a suitable subset $\Xi \subseteq \mathrm{Hom}_{cont}(\Gamma, \overline{\mathbb{Q}_p}^\times)$ of characters, we will consider the p-adic representations $V(\chi)$ as $\chi \in \Xi$. Here $V(\chi)$ means twisting V by $G_K \twoheadrightarrow \Gamma \xrightarrow{\chi} \overline{\mathbb{Q}}_p^\times$ (after extending scalars to contain the image of χ). From the specialization morphisms $\chi\colon \Lambda \to \overline{\mathbb{Q}_p}^\times$, we have maps $H^1(K, \mathbb{T}_T) \to H^1(K, T(\chi))$, and we can define a Selmer group $\mathrm{Sel}_\Xi(T)$ to be the set of classes that specialize to $H^1_f(K,T(\chi))$ for all $\chi \in \Xi$. Similarly we can define $S_\Xi(T) \subseteq H^1(K,\mathbb{W}_T)$ and $X_\Xi(T) = S_\Xi(T)^\vee$. Of course, this will only actually capture the Selmer groups $H^1_f(K,V(\chi))$ if Ξ is chosen suitably.

Assume that almost all of $L(s,V^*(1)(\chi^{-1}))$ have the same order of vanishing r at $s=0$. Then we expect $\mathrm{Sel}_\Xi(T)$ and $X_\Xi(T)$ to have Λ-rank r. Furthermore, if $r=0$, then we can hope that $L(s,V^*(1)(\chi^{-1}))_{alg}$ vary p-adically. That is, that there exist an element $\mathscr{L}_{V,\Xi} \in \Lambda$ such that

$$\chi(\mathscr{L}_{V,\Xi}) = (*) \cdot L(0,V^*(1)(\chi^{-1}))_{alg}$$

up to some simple factors $(*)$. Finally, as we expect that $L(0,V^*(1)(\chi^{-1}))_{alg}$ is related to $Ш_f(W(\chi)/F)$, one can have the hopeful expectation that

$$\mathrm{Ch}(X_\Xi(T)) = (\mathscr{L}_{V,\Xi}).$$

There an ambiguity in this expectation, as the right hand side does not depend on the lattice T. However, different choices of T should only change the left side by a power of p, and we can hope that the choice of T determines a precise choice of period for $\mathscr{L}_{V,\Xi}$.

1.1. **Greenberg Selmer groups.** This is an exposition of the conjectures in [Gre89]. We consider the following condition for a p-adic place v of K.

Definition 1.3. A p-adic representation V of K_v is *ordinary* if there exists a $\mathbb{Q}_p[G_{K_v}]$-stable $\mathbb{Z}$-filtration $F^iV \subseteq V$ that is exhaustive and separated such that the action of inertia in $F^iV/F^{i+1}V$ is by ϵ^i_{cycl}. Denote $V^+ := F^1V$.

In particular, ordinary representations are de Rham with Hodge–Tate weight $-i$ of multiplicity $\dim F^iV/F^{i+1}V$.

Proposition 1.4. *If V is an ordinary K_v-representation, then*

$$H^1_g(K_v,V) = \ker(H^1(K_v,V) \to H^1(I_v,V/V^+)).$$

Proof. First note that $H^1_g(K_v,V/V^+) = H^1_{unr}(K_v,V/V^+)$ by dimension counting, as $(V/V^+)^{G_{K_v}}$ and $D^{\phi=1}_{crys}(V/V^+)$ are 0 because V/V^+ has only strictly positive Hodge–Tate weights. The second assertion follows since $\mathrm{Fil}^1 B^{\phi=1}_{crys} = 0$.

Now in Hao's talk we saw that $H^1(K_v,V\otimes B^+_{dR}) \to H^1(K_v,V\otimes B_{dR})$ is injective for V de Rham, and we also saw that $H^1(K_v,V^+\otimes B^+_{dR}) = 0$ as V^+ has strictly positive Hodge–Tate weights. Now the claim follows from the commutative diagram

$$\begin{array}{ccccc} & & H^1(K_v,V) & \xrightarrow{\alpha} & H^1(K_v,V/V^+) \\ & & \downarrow & & \downarrow \\ 0 = H^1(K_v,V^+\otimes B^+_{dR}) & \longrightarrow & H^1(K_v,V\otimes B^+_{dR}) & \hookrightarrow & H^1(K_v,(V/V^+)\otimes B^+_{dR}) \end{array}$$

as then

$$H^1_g(K_v,V) = \alpha^{-1}H^1_g(K_v,V/V^+) = \alpha^{-1}H^1_{unr}(K_v,V/V^+) = \ker\Big(H^1(K_v,V) \xrightarrow{\alpha} H^1(K_v,V/V^+) \to H^1(I_v,V/V^+)\Big). \qquad \square$$

Remark 1.5. For many cases of interest, we have that $H^1_f(K_v,V) = H^1_g(K_v,V)$. By dimension counting, this happens precisely if $D^{\phi=1}_{crys}(V^*(1)) = 0$. For example, this is true if V is pure of weight $w \neq -2$.

For a lattice $T \subseteq V$, we have the induced filtrations $F^iT \subseteq V^iT$, and $F^iW = F^iT \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

Proposition 1.6. *Assume that* $(F^0V/F^1V)^{G_{K_v}} = 0$. *Then* $H^1_g(K_v, W) = \mathrm{im}(H^1(K_v, W^+)_{\mathrm{div}} \to H^1(K_v, W))$. *We also have* $H^1_g(K_v, T) = \ker(H^1(K_v, T) \to H^1(K_v, T/T^+)_{/\mathrm{tor}})$.

Proof. The assumption guarantees that $H^1(K_v, V/V^+) \hookrightarrow H^1(I_v, V/V^+)$. Thus $H^1_g(K_v, V) = \mathrm{im}(H^1(K_v, V^+) \to H^1(K_v, V))$.

For the first claim, consider the commutative diagram

$$\begin{array}{ccc} H^1(K_v, V^+) & \longrightarrow & H^1(K_v, V) \\ \downarrow & & \downarrow \\ H^1(K_v, W^+) & \longrightarrow & H^1(K_v, W) \end{array}$$

So $H^1_g(K_v, W)$ is the image of the above composition. Since the image of the left map is $H^1(K_v, W^+)_{\mathrm{div}}$, the claim follows.

For the second claim, consider the commutative diagram

$$\begin{array}{ccc} H^1(K_v, T) & \longrightarrow & H^1(K_v, V) \\ \downarrow & & \downarrow \\ H^1(K_v, T/T^+) & \longrightarrow & H^1(K_v, V/V^+) \end{array}$$

So $H^1_g(K_v, W)$ is the kernel of the above composition. Since the bottom map has kernel $H^1(K_v, T/T^+)_{\mathrm{tor}}$, the claim follows. $\square$

Take $K = \mathbb{Q}$ and $\mathbb{Q}_\infty$ the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

Then for a $G_{\mathbb{Q}}$-stable lattice $T \subseteq V$, we have the induced filtration F^iT, and Greenberg defines the following Selmer group.[1]

Definition 1.7. $S_{\mathrm{Gr}}(\mathbb{Q}_\infty, W) \subseteq H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathbb{W}_T)$ defined by the local conditions: unramified at $v \nmid p$, and

$$H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathbb{W}_T) := \ker\left(H^1(\mathbb{Q}_p, \mathbb{W}_T) \to H^1(I_p, \mathbb{W}_{T/F^+T})\right).$$

This Selmer group correspond to the subset $\Xi_{\mathrm{Gr}} \subseteq \mathrm{Hom}_{cont}(\Gamma, \mathbb{Z}_p^\times)$ of finite order characters. If we look at $L(s, V(\chi))$ for $\chi \in \Xi_{\mathrm{Gr}}$, then their Archimedean factors are all the same[2], and they have a pole at 0 of order

$$r_V := \sum_{0 \le k < w/2} m_k(V) + (a^+(V) - m_{w/2}(V)),$$

where we let $a^+(V) = m_{w/2}(V) = 0$ if w is odd. So we expect that $L(s, V(\chi))$ have a zero of order exactly r_V at 0 for all but finitely many χ. So Greenberg conjectures

Conjecture 1.8. *For* $T \subseteq V$ *an ordinary* p*-adic representation,* $X_{\mathrm{Gr}}(\mathbb{Q}_\infty, T)$ *is a finitely generated* Λ*-module of rank* $r_{V^*(1)}$.

The case that $r_V = r_{V^*(1)} = 0$ is exactly the *critical* case considered by Deligne, where the special values are supposed to be algebraic up to a precise period. In the critical case and if V is ordinary, Coates and Perrin-Riou conjecture a precise p-adic interpolation property of $L(0, V(\chi))$. So there is an explicit conjectured p-adic L-function $\mathscr{L}_V \in \mathrm{Frac}(\Lambda)$.

Then Greenberg also conjectures

Conjecture 1.9. *For* $T \subseteq V$ *an ordinary* p*-adic representation with* $r_V = r_{V^*(1)} = 0$, *the characteristic ideal* $\mathrm{Ch}(X_{\mathrm{Gr}}(\mathbb{Q}_\infty, T^*(1)))$ *is the numerator of* $\mathscr{L}_V$ *as ideals in* $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

[1] This is non-standard notation.
[2] As twisting by finite order characters does not change the Hodge–Tate weights

Here the ambiguity of powers of p come from an ambiguity in the definition of $\mathscr{L}_V$ and also on the choice of lattice T. There should also be a natural way to "normalize" $\mathscr{L}$ with respect to T to get the equality in Λ.

Example 1.10. Take $T = \mathbb{Z}_p(k)$. Then for $\Sigma = \{p, \infty\}$,

$$S_{\mathrm{Gr}}(\mathbb{Q}_\infty, \mathbb{Z}_p(k)) = \begin{cases} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathbb{W}_{\mathbb{Z}_p(k)}) & \text{if } k \geq 1, \\ H^1_{unr}(\mathbb{Q}, \mathbb{W}_{\mathbb{Z}_p(k)}) & \text{if } k \leq 0. \end{cases}$$

So if $X_\infty = \varprojlim_n \mathrm{Cl}(\mathbb{Q}(\mu_{p^n})[p^\infty]$ denotes the $\Lambda[\Delta]$-module of last time, we have $X_{\mathrm{Gr}}(\mathbb{Q}_\infty, \mathbb{Z}_p(k)) = X_\infty^{\omega^k} \otimes \epsilon_{cycl}^k$ if $k \leq 0$, and

$$0 \to \mathrm{Hom}\left((\varinjlim_n \mathrm{Cl}(\mathbb{Q}(\mu_{p^n})[p^\infty])^{\omega^{1-k}},\ \mathbb{Q}_p/\mathbb{Z}_p \right) \otimes \epsilon_{cycl}^{k-1} \to X_{\mathrm{Gr}}(\mathbb{Q}_\infty, \mathbb{Z}_p(k)) \to \mathrm{Hom}\left((\varinjlim_n (\mathcal{O}^{\times,p}_{\mathbb{Q}(\mu_{p^n})})^{\omega^{1-k}},\ \mathbb{Q}_p/\mathbb{Z}_p \right) \otimes \epsilon_{cycl}^{k-1} \to 0$$

if $k \geq 1$. So indeed, we can see that $X_{\mathrm{Gr}}(\mathbb{Q}_\infty, \mathbb{Z}_p(k))$ has rank 1 iff $k \geq 1$ is odd, corresponding to the trivial zero at $1-k$ for even nontrivial Dirichlet characters. The critical cases are if $k \geq 1$ is even or $k \leq 0$ is odd. So the above conjecture recovers the Iwasawa main conjecture. Note that for $k \leq 0$ even (which is non-critical), the characteristic ideal is not a p-adic L-function.

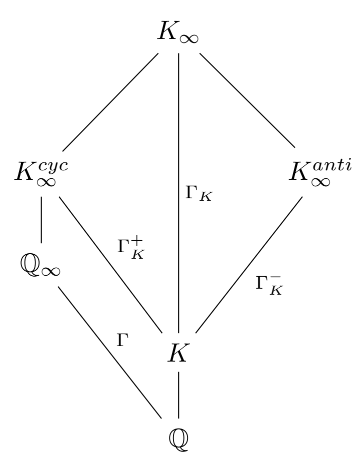
2. Greenberg Selmer groups of elliptic curves

Let $E/\mathbb{Q}$ be an elliptic curve, and $p \geq 5$. We consider $T_pE \subseteq V_pE$. Recall that V_pE is polarized of motivic weight -1, and has Hodge–Tate weights $0, 1$. For simplicity, we will also assume

(irred) $\qquad E[p]$ is an irreducible $G_{\mathbb{Q}}$-module.

Let K be a quadratic imaginary field. For simplicity, we will assume that $(N_E, D_K) = 1$, and that $p \nmid D_K$.

We will consider the extensions



and choose topological generators $\gamma, \gamma^+, \gamma^-$ of $\Gamma, \Gamma_K^+, \Gamma_K^-$. So if $T^? = \gamma^? - 1$ for $? \in \{\emptyset, +, -\}$, we have $\Lambda = \mathbb{Z}_p[\![T]\!]$, $\Lambda_K^+ = \mathbb{Z}_p[\![T^+]\!]$, $\Lambda_K^- = \mathbb{Z}_p[\![T^-]\!]$ and $\Lambda_K = \mathbb{Z}_p[\![T^+, T^-]\!]$ the corresponding Iwasawa algebras.

For what follows, let F_∞/F be one of the four extensions above.

2.1. Greenberg main conjectures.

V_pE is an ordinary representation at a place p exactly when one of the following:

(1) E has good reduction at p and $p \nmid a_p(E)$. That is, E has good non-supersingular reduction.

(2) E has multiplicative reduction.

In the first case, the reduction $\widetilde{E}/\mathbb{F}_p$ has $T_p\widetilde{E} \simeq \mathbb{Z}_p$, and the surjection $T_pE \twoheadrightarrow T_p\widetilde{E}$ give us the filtration. In the second case, the surjection comes from Tate's parametrization $E(\overline{\mathbb{Q}_p}) \simeq \overline{\mathbb{Q}_p}^{\times}/q^{\mathbb{Z}} \xrightarrow{\text{val}} \mathbb{Q}/e\mathbb{Z}$. There is also a lot of work that has been done to do Iwasawa theory in the case of supersingular reduction, but we will not consider this here for simplicity.[3]

In both cases, V/V^+ is unramified, and Frobenius act by multiplication by α_p. In the first case, α_p is the unit root of $x^2 - a_px + p$, and in the second case it is a_p.

Then we can define the Greenberg local condition at p as before. As for the places not above p, we have:

Proposition 2.1. *Let $v \nmid p$ be a place of F.*

(1) $H^1_{unr}(F_v, \mathbb{W}_T)$ has finite exponent as an abelian group.
(2) If E has good reduction at v, then $H^1_{unr}(F_v, \mathbb{W}_T) = 0$.
(3) If v only has finitely many primes above it in F_∞/F, then $H^1_{unr}(F_v, \mathbb{W}_T) = 0$ as well.

Proof. The third point has the same proof as in the last talk (where we considered the cyclotomic extension of $\mathbb{Q}$).

For L/F_v a finite extension, let $\mathcal{E}$ be the Néron model of E over $\mathcal{O}_L$. Let $\mathcal{E}^0$ be the open subgroup scheme of $\mathcal{E}$ whose generic fiber is E and special fiber is the identity component of the special fiber $\mathcal{E}_0$ of $\mathcal{E}$. Then we have an exact sequence

$$0 \to \mathcal{E}^0(L^{unr}) \to \mathcal{E}(L^{unr}) \to \pi_0(\mathcal{E}_0) \to 0$$

and $\mathcal{E}(L^{unr}) = E(L^{unr})$. But by Lang's theorem, $H^1(k(L), \mathcal{E}^0(L^{unr})) = 0$. Also, $H^2(k(L), \mathcal{E}^0(L^{unr})) = 0$ because $G_{k(L)}$ has cohomological dimension 1. Thus $H^1_{unr}(L, W) \simeq H^1(k(L), \pi_0(\mathcal{E}_0))[p^\infty]$, which has size $H^0(k(L), \pi_0(\mathcal{E}_0))[p^\infty]$ since $\pi_0(\mathcal{E}_0)$ is finite. In particular, $H^1_{unr}(L, W) = 0$ if E has good reduction over L, and in general $\#\pi_0(\mathcal{E}_0)$ kills $H^1_{unr}(L, W)$ independently of L. Now the first two claims follow from $H^1_{unr}(F_v, \mathbb{W}_T) = \varinjlim_n \bigoplus_{w|v} H^1_{unr}(F_{n,w}, W)$. □

We note that the third condition can only happen if F_∞/F is a $\mathbb{Z}_p$-extension. For $\mathbb{Q}_\infty/\mathbb{Q}$ and K^{cyc}_∞/K, this happens for any $v \nmid p$. For K^{anti}_∞/K, we have the following splitting behaviour for $v \nmid p$: i) if v is split in K, then it is totally inert in K^{anti}_∞/K, ii) if v is inert in K, then it is totally split in K^{anti}_∞/K.

Given this, we may define the following Iwasawa theoretic Selmer groups:

Definition 2.2. Let $S(F_\infty, E) \subseteq H^1(F_\Sigma/F, \mathbb{W}_T)$ be the Selmer group defined by the unramified local conditions for $v \nmid p$, and Greenberg at $v \mid p$. Let also $S^0(F_\infty, E)$ be the Selmer group defined by the trivial local conditions for $v \nmid p$, and Greenberg at $v \mid p$.[4]

Proposition 2.3. *$S^0(F_\infty, E)$ is identified with the direct limit $\varinjlim_{F \subseteq F' \subseteq F_\infty} H^1_f(F', W_pE)$.*

Proof. We have $H^1_f(F_v, W) = 0$ for $v \nmid p$. So it suffices to see that $\varinjlim_n \bigoplus_{w|v} H^1_f(F_{n,w}, W) = H^1_{\mathrm{Gr}}(F_v, \mathbb{W}_T)$. This follows from Shapiro's lemma, Proposition 1.6, the fact that $H^1_?(L, V)$ for $? \in \{e, f, g\}$ are the same for any p-adic field L, as $V \simeq V^*(1)$ are pure of weight -1, and the fact that if $v \mid p$, $\varinjlim_n \bigoplus_{w|v} H^1(F_{n,w}, W^+)_{/\mathrm{div}} = 0$.

For the last claim, note that for a p-adic field L, we have $H^1(L, W^+)_{/\mathrm{div}} \hookrightarrow H^2(L, T^+)_{\mathrm{tor}}$ and $H^2(L, T^+)_{\mathrm{tor}}$ is dual to $H^0(L, W/W^+)_{/\mathrm{div}}$. Thus $\varinjlim_n \bigoplus_{w|v} H^1(F_{n,w}, W/W^+)_{/\mathrm{div}}$ injects into the dual of $\varprojlim_n \bigoplus_{w|v} ((W/W^+)^{G_{F_{n,w}}}_{/\mathrm{div}}$. But for n sufficiently large, all primes of F_n above p are totally ramified along F_∞. So we are looking at $\varprojlim_n ((W/W^+)^{G_{L_n}}_{/\mathrm{div}}$ for a

[3]See Skinner's notes [Ski18] for some references.
[4]For comparison with [Ski18], $S^0(F_\infty, E)$ corresponds to $S(E/F_\infty)$.

totally ramified $\mathbb{Z}_p^d$-extension L_∞/L of p-adic fields. But W/W^+ is unramified, and the restriction maps $(W/W^+)^{G_{L_n}} \to (W/W^+)^{G_{L_{n+1}}}$ are identities, and thus the inverse limit is 0. □

For $\mathbb{Q}_\infty/\mathbb{Q}$ and K_∞/K, one expects the L-values $L(1,E,\chi) = L(0, V\otimes\chi)$ to be nonzero most of the time for finite order characters. Indeed, we have p-adic L-functions $\mathscr{L}_{\mathbb{Q}_\infty,E} \in \Lambda$ and $\mathscr{L}_{K_\infty,E} \in \Lambda_K$. For example, for a finite order character $\chi\colon \Gamma \to \overline{\mathbb{Q}_p}^\times$ of conductor p^t, we have

$$\chi(\mathscr{L}_{\mathbb{Q}_\infty,E}) = e_p(\chi)\frac{L(1,E,\chi^{-1})}{\Omega_E}, \quad e_p(\chi) = \begin{cases} \alpha_p^{-t}\frac{p^t}{G(\chi^{-1})} & \text{if } t>0, \\ \left(1-\frac{1}{\alpha_p}\right)^{2-\nu_p(N_E)} & \text{if } t=0. \end{cases}$$

$\mathscr{L}_{\mathbb{Q}_\infty,E}$ was first constructed by Amice–Vêlu and Vishik, see [MTT86]. $\mathscr{L}_{K_\infty,E}$ was constructed by Perrin–Riou [PR88].

We have the Iwasawa main Conjectures[5]

Conjecture 2.4 (Cyclotomic main conjecture). *$X(\mathbb{Q}_\infty,E)$ is Λ-torsion, and its characteristic ideal is $(\mathscr{L}_{\mathbb{Q}_\infty,E})$.*

Conjecture 2.5 (Two-variable main conjecture). *$X^0(K_\infty,E)$ is Λ_K-torsion, and its characteristic ideal is $(\mathscr{L}_{K_\infty,E})$.*

Now we restrict to the case of good ordinary reduction. Note that if $\mathbb{Q}\subseteq F\subseteq \mathbb{Q}_\infty$, then by inflation restriction

$$H^1(F/\mathbb{Q}, W^{G_F}) \to H^1(G_{\mathbb{Q},\Sigma}, W) \to H^1(G_{F,\Sigma}, W)$$

and since $W^{G_F} = E(F)[p^\infty]$ is finite and $F/\mathbb{Q}$ is cyclic, $\#H^1(F/\mathbb{Q},W^{G_F}) = \#\hat{H}^0(F/\mathbb{Q},W^{G_F})$ and $\hat{H}^0(F/\mathbb{Q},W^{G_F}) = W^{G_\mathbb{Q}}/\mathrm{Tr}_{F/\mathbb{Q}}W^{G_F} = E(\mathbb{Q})[p^\infty]/\mathrm{Tr}_{F/\mathbb{Q}}E(F)[p^\infty]$. In particular, since we are assuming (irred), we have $H^1_f(\mathbb{Q},W) \hookrightarrow S(\mathbb{Q}_\infty,E)[T]$. Analyzing it further, one can prove

Proposition 2.6 ([Gre99]). *If $X(\mathbb{Q}_\infty,E)$ is Λ-torsion and E has good ordinary reduction at p, then there is an exact sequence*

$$0 \to H^1_f(\mathbb{Q},W) \to S(\mathbb{Q}_\infty,E)[T] \to \prod_{l\in\Sigma} K_l$$

where $K_l = \ker(H^1_{/f}(\mathbb{Q}_l,W) \to H^1_{/\mathrm{Gr}}(\mathbb{Q}_l,\mathbb{W}_T))$. We have

$$\#K_l = \begin{cases} |c_l(E/\mathbb{Q})|_p^{-1} & \text{if } l\neq p, \\ \#(\mathbb{Z}_p/(1-\alpha_p))^2 & \text{if } l=p. \end{cases}$$

Furthermore, if $H^1_f(\mathbb{Q},W)$ is finite, then the above exact sequence is also exact on the right.

This implies that

Corollary 2.7. *Assume $X(\mathbb{Q}_\infty,E)$ is Λ-torsion. Then we have*

$$r(E/\mathbb{Q}) = 0 \text{ and } Ш(E/\mathbb{Q})[p^\infty] \text{ is finite} \iff T\nmid \mathrm{Ch}(X(\mathbb{Q}_\infty,E)).$$

Proof. The above implies that

$$H^1_f(\mathbb{Q},W) \text{ is finite} \iff X^0(\mathbb{Q}_\infty,E)/TX(\mathbb{Q}_\infty,E) \text{ is finite.}$$

But the right hand side is a finite quantity times $\Lambda/(T,\mathrm{Ch}(X(\mathbb{Q}_\infty,E)))$. This is finite if and only if $T\nmid \mathrm{Ch}(X(\mathbb{Q}_\infty,E))$. □

[5]If (irred) does not hold, then the equality of characteristic ideals must be modified by a factor of p

Together with the main conjecture, this would imply the rank 0 case of Bloch–Kato for E:

$$r(E/\mathbb{Q}) = 0 \text{ and } Ш(E/\mathbb{Q})[p^\infty] \text{ is finite } \iff L(1,E) \neq 0.$$

But in the rank 0 case, we can do even better, since

Proposition 2.8 ([Gre99, Proposition 4.8]). *$X(\mathbb{Q}_\infty, E)$ has no nonzero pseudo-null submodules.*

So the main conjecture would imply that: if $L(1,E) \neq 0$, then

$$\#\mathbb{Z}_p/\mathrm{triv}(\mathscr{L}_{\mathbb{Q}_\infty,E}) = \#H^1_f(\mathbb{Q}, W) \cdot \prod_{l\in\Sigma} \#K_l,$$

that is, that

$$\#\mathbb{Z}_p/\left(\alpha_p^{-2}(\alpha_p-1)^2\frac{L(1,E)}{\Omega_E}\right) = |Ш(E/\mathbb{Q})|_p^{-1} \cdot \prod_{l|N_E} |c_l(E/\mathbb{Q})|_p^{-1} \cdot \#\mathbb{Z}_p/(1-\alpha_p)^2,$$

which is simply

$$\left|\frac{L(1,E)}{\Omega_E}\right|_p^{-1} = \left|Ш(E/\mathbb{Q}) \cdot \prod_{l|N_E} c_l(E/\mathbb{Q})\right|_p^{-1}.$$

This is the p-part of the BSD formula. See [SU14, Theorem 2] for precise results on this.

2.2. Anticyclotomic extension.

The situation over the anticyclotomic extension is more delicate. Write $N_E = N^+N^-$ where primes in N^+ are split in K and primes in N^- are inert in K. We will assume that N^- is square-free. Then a local root number computation shows that for any $\chi\colon \Gamma_K^- \to \mathbb{Z}_p^\times$ of finite order,

$$\epsilon(E,\chi) = (-1)^{\nu(N^-)+1}.$$

In particular, we can only expect $L(E,\chi,1)$ to be nonzero for almost all χ when N^- is a product of an *odd* number of primes.

2.2.1. Case $\epsilon = 1$.

In the case $\epsilon = 1$, we have a Jacquet–Langlands transfer of f_E to a definite quaternion algebra B of discriminant $N^-\infty$. Up to a certain normalization, this is a modular form ϕ of level N^+ of B. A formula of Gross, and generalized by Shou-Wu Zhang give us that for $\chi\colon \Gamma_K^- \to \mathbb{Z}_p^\times$ of finite order,

$$\frac{L(E/K,\chi^{-1},1)}{\Omega_E} = \frac{4\eta_{E,N^+,N^-}}{w_K^2\sqrt{-D_K}}|\phi(P_\chi)|^2$$

where P_χ are certain CM cycles, and $\eta_{E,N^+,N^-} \in \mathbb{Z}_p$ is a factor related to the normalization of the Jacquet–Langlands transfer.

The periods $\phi(P_\chi)$ can be p-adically interpolated[6] as in [BD05, Definition 1.6] into a $\mathscr{L}_\phi \in \Lambda_K^-$.

Conjecture 2.9 (Anticyclotomic main conjecture for $\epsilon = 1$). *$X^0(K_\infty^{anti}, E)$ is Λ_K^--torsion, and its characteristic ideal is $(\mathscr{L}_\phi)^2$ in $\Lambda_K^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Remark 2.10. The factor η_{E,N^+,N^-} is related to the product of Tamagawa factors at primes of N^-, but is not always exactly that, see for example [RT97].

[6] This requires p to be ordinary.

2.2.2. **Case $\epsilon = -1$.** In this case, we expect $X(K_\infty^{anti}, E)$ to have rank 1, and we would hope to interpolate $L'(E/K, \chi^{-1}, 1)$. However, we do not do this directly. For this discussion, we need that N is square-free if $N^- \neq 1$.

A root number computation shows that if $\chi\colon \Gamma_K^- \to \overline{\mathbb{Q}_p}^\times$ is associated to an unramified algebraic Hecke character of infinity type $(n, -n)$ for $n \geq 1$ and $n \equiv 0 \mod (p-1)$. Then the root number of $L(E, \chi^{-1}, s)$ is forced to be 1. So consider $\Xi_{BDP} \subseteq \mathrm{Hom}_{cont}(\Gamma_K^-, \overline{\mathbb{Q}_p}^\times)$ the subset of such characters. Then again a Waldspurger-type formula says that

$$\frac{L(E/K, \chi^{-1}, 1)}{\Omega_\infty^{4n}} = \eta_{E,N^+,N^-} \cdot (*) \cdot (L(E/K, \chi^{-1}, 1)_{alg})^2$$

where Ω_∞ is a complex period, and $L(E/K, \chi^{-1}, 1)_{alg}$ is the result of applying certain powers of the Mass–Shimura operator to the Jacquet–Langlands transfer of f_E, and then evaluate this at a CM divisor determined by χ. Then one can p-adically interpolate, for $\chi \in \Xi_{BDP}$, the quantity

$$e_p(\chi) \frac{L(E/K, \chi^{-1}, 1)}{\Omega_p^{2n}}$$

where Ω_p is a certain p-adic period and

$$e_p(\chi) = \begin{cases} L(E/K_{\overline{v}}, \chi^{-1}, 1)^{-1} & \text{if } p = v\overline{v} \text{ in } K \text{ where } v \text{ corresponds to } \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}, \\ 1 & \text{otherwise.} \end{cases}$$

This corresponds to an element $\mathscr{L}_{BDP} \in (\Lambda_K^-)^{ur} := \mathbb{Z}_p^{ur}[\![\Gamma_K^-]\!]$. In the case p is split, this was done [BDP13] in the case $N^- = 1$, [HB15] for general N^- (and [LZZ18] over totally real fields). In the case p is non-split, this was done by [AI19].[7]

Finite order characters $\chi\colon \Gamma_K^- \to \overline{\mathbb{Z}_p}^\times$ are now *outside* the interpolation range, but one can prove a p-adic Gross–Zagier formula. In this sense, $\mathscr{L}_{BDP}$ is still capturing the information of $L'(E/K, 1)$ via Gross–Zagier, and more generally Yuan–Zhang–Zhang [YZZ13].

Theorem 2.11 (BDP formula, [HB15, Proposition 8.13])**.** *We have*

$$\mathrm{triv}(\mathscr{L}_{BDP}) = e_p(1) \cdot \log_{\omega_E} y_K^{N^+,N^-}$$

where $y_K^{N^+,N^-}$ is a certain generalized Heegner point on $E(K)$, and $\log_{E(K_v)}$ is the formal group logarithm. There is a similar formula for other finite order characters.

Remark 2.12. In the case $N^- = 1$, the above y_K is the usual Heegner point in $E(K)$, and the above logarithm can be identified with the logarithm on the formal group associated to E.

Now assume p is split. The characters $\chi \in \Xi_{BDP}$ have Hodge–Tate weights < -1 at v and > 1 at $\overline{v}$. So $H^1_f(K_{\overline{v}}, V(\chi)) = 0$, while $H^1_f(K_v, V(\chi)) = H^1(K_v, V(\chi))$. Now consider

Definition 2.13. Let $S_{?_1,?_2}(K_\infty^?, E)$ for $?_1, ?_2 \in \{\mathrm{Gr}, \emptyset, 0\}$ denote the Iwasawa theoretic Selmer groups where the local condition at v is given by $?_1$, and at $\overline{v}$ by $?_2$. Here $\emptyset$ means no condition, and 0 means the strict condition. We also consider $S^0_{?_1,?_2}(K_\infty, E)$ having strict local condition for $w \nmid p$.

Then we expect

[7] [Kri21] also has a construction of a p-adic L-function in the non-split case, but it lacks an interpolation formula as above, so at the moment we cannot compare them.

Conjecture 2.14 (BDP anticyclotomic main conjecture)**.** *$X^0_{\emptyset,0}(K^{anti}_\infty, E)$ is Λ^-_K-torsion, and its characteristic ideal is given by $(\mathscr{L}_{BDP})^2$ in $(\Lambda^-_K)^{ur} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

It is also worth pointing out that there is a two-variable version of this. Such p-adic L-function $\mathscr{L}_{K_\infty,BDP} \in (\Lambda_K)^{ur}$ interpolates special values for $\chi\colon \Gamma_K \to \overline{\mathbb{Q}_p}^\times$ associated to unramified Hecke characters of infinity type (n,m) where $n \geq 1, m \leq -1$, and $n,m \equiv 0 \mod (p-1)$. We expect

Conjecture 2.15 (BDP two-variable main conjecture)**.** *$X_{\emptyset,0}(K_\infty, E)$ is Λ_K-torsion, and its characteristic ideal is given by $\mathscr{L}_{K_\infty,BDP}$ in $(\Lambda_K)^{ur} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Similarly as before, a control theorem assuming the BDP anticyclotomic main conjecture would give the order of $H^1_{\emptyset,0}(K,W)$, in terms of $\log y_K$. With some work, this gives $E(K)/\mathbb{Z}\cdot y_K^{n^+,N^-}$, in terms of $\text{Ш}(E/K)[p^\infty]$, which by Gross–Zagier or Yuan–Zhang–Zhang gives the p part of the BSD formula in rank 1. See [JSW17] for precise results on this.

3. Relation with Euler systems

3.1. **Perrin–Riou regulator maps.** From the exact sequence $0 \to \mathbb{Q}_p \to B^{\phi=1}_{crys} \to B_{dR}/B^+_{dR} \to 0$, we get for a de Rham V and a padic field F that

$$0 \to V^{G_F} \to D^{\phi=1}_{crys}(V) \to D_{dR}(V)/D^+_{dR}(V) \xrightarrow{\exp_V} H^1_e(F,V) \to 0.$$

Now assume that $D^{\phi=1}_{crys}(V) = 0$. This also implies that $H^1_e(F,V) = H^1_f(F,V)$. Then the inverse of the above map is the *Bloch–Kato logarithm*

$$\log_V\colon H^1_f(F,V) \xrightarrow{\sim} \frac{D_{dR}(V)}{D^+_{dR}(V)}.$$

Moreover, if also $D^{\phi=1}_{crys}(V^*(1)) = 0$, then by dualizing the map $\exp_{V^*(1)}$ we obtain

$$\exp^*_V\colon H^1_{/f}(F,V) \xrightarrow{\sim} D^+_{dR}(V).$$

If F_∞/F is a Lubin–Tate extension, V is crystalline and has non-negative Hodge–Tate weights, then Perrin–Riou and others[8] proved that $H^1_{Iw}(F,T)/V^{G_{F_\infty}}$ is a torsion-free Λ-module of rank $\dim_{\mathbb{Q}_p} V$, and constructed a *regulator* map

$$\mathcal{L}_V\colon H^1_{Iw}(F,T) \to \mathscr{H}(\Gamma) \otimes D_{cris}(V)$$

where $\mathscr{H}(\Gamma)$ is a certain algebra of distributions, with $\Lambda \subseteq \mathscr{H}(\Gamma)$. This regulator map was defined to interpolate Bloch–Kato logarithms when specializing to $V(k)$ for $k \gg 0$ as in [PR94, Théorème], but it also interpolates Bloch–Kato dual exponentials when specializing to $V(k)$ for $k \ll 0$, as proven by Colmez.[9]

Often, one can choose suitable $\eta \in D_{cris}(V^*(1))$ so that the composition of the above with $\alpha \otimes \beta \mapsto \alpha \cdot \langle \beta, \eta\rangle$ lies in Λ. In the case of an ordinary elliptic curve $V = V_pE$ over $\mathbb{Q}_p$, V^+ and V/V^+ are of dimension 1, and in in many cases we can normalize the regulator map to obtain injections with finite cokernel

$$\mathrm{Log}\colon H^1_f(\mathbb{Q}_p, \mathbb{T}_T) \otimes_\Lambda \Lambda^{ur} \hookrightarrow \Lambda^{ur}, \quad \mathrm{Col}\colon H^1_{/f}(\mathbb{Q}_p, \mathbb{T}_T) \hookrightarrow \Lambda.$$

[8] See for example [LLZ11].
[9] See for example [Ber03].

In settings where we have Euler systems, they often afford global cohomology classes in $\mathrm{Sel}_?(F_\infty, \mathbb{T}_T)$, whose localizations are related to p-adic L-functions via these regulator maps. See also [BCD+14] for a good discussion about some cases of this. We will see some examples in what follows.

3.2. **Euler systems.** We will denote by $\mathrm{Sel}(F_\infty, E) = H^1_f(F, \mathbb{T}_T)$, with the modifications similarly to $S(F_\infty, E)$. In all this discussion, we assume that p splits in K and that p has ordinary good reduction.[10]

3.2.1. **Cyclotomic main conjecture.** In the case of $\mathbb{Q}_\infty/\mathbb{Q}$, Kato [Kat04] produced an Euler system which affords us a free rank 1 Λ-module

$$Z_{Kato} \subseteq \mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E).$$

Moreover, a deep explicit reciprocity law proven by Kato says that

Theorem 3.1 (Reciprocity law)**.** *Under the Coleman map* $\mathrm{Col}\colon H^1_{/f}(\mathbb{Q}_p, \mathbb{T}_T) \hookrightarrow \Lambda$, $\mathrm{loc}_p(Z_{Kato})$ *is sent to* $\mathscr{L}_{\mathbb{Q}_\infty,E} \cdot \Lambda$.

It is known that $\mathscr{L}_{\mathbb{Q}_\infty,E}$ is non-zero. This is how we know that Z_{Kato} is non zero. It also implies that $\mathrm{Sel}(\mathbb{Q}_\infty, E) \cap Z_{Kato} = 0$. By global duality,

$$0 \to \mathrm{Sel}(\mathbb{Q}_\infty, E) \to \mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E) \to H^1_{/f}(\mathbb{Q}_p, T) \to X(\mathbb{Q}_\infty, E) \to X_0(\mathbb{Q}_\infty, E) \to 0,$$

and we can divide by Z_{Kato}

$$0 \to \mathrm{Sel}(\mathbb{Q}_\infty, E) \to \frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)}{Z_{Kato}} \to \frac{H^1_{/f}(\mathbb{Q}_p, T)}{\mathrm{loc}_p(Z_{Kato})} \to X(\mathbb{Q}_\infty, E) \to X_0(\mathbb{Q}_\infty, E) \to 0.$$

Using this, one can prove that the cyclotomic main conjecture is equivalent to:

Conjecture 3.2 (Cyclotomic main conjecture without L-functions)**.** $\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)$ *is a rank* 1 *torsion-free* Λ*-module, and* $\mathrm{Ch}\left(\frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty,E)}{Z_{Kato}}\right) = \mathrm{Ch}(X_0(\mathbb{Q}_\infty, E))$.

Kato proved that $X_0(\mathbb{Q}_\infty, E)$ is Λ-torsion, $\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)$ is rank 1 torsion-free and the "Euler system divisibility"

$$\mathrm{Ch}(X_0(\mathbb{Q}_\infty, E)) \text{ divides } \mathrm{Ch}\left(\frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)}{Z_{Kato}}\right)$$

using his Euler system.

Proof of equivalence. Using that $\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)$ is a rank 1 torsion-free Λ-module, we have that $\frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty,E)}{Z_{Kato}}$, and hence $\mathrm{Sel}(\mathbb{Q}_\infty, E)$, are Λ-torsion. But $\mathrm{Sel}(\mathbb{Q}_\infty, E) \subseteq \mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)$ and the latter is torsion-free, so this means that $\mathrm{Sel}(\mathbb{Q}_\infty, E)$ is zero. From the exact sequence above, we would thus conclude that $X(\mathbb{Q}_\infty, E)$ is Λ-torsion.

Hence from Kato's result we obtain the exact sequence of torsion Λ-modules

$$0 \to \frac{\mathrm{Sel}_\emptyset(\mathbb{Q}_\infty, E)}{Z_{Kato}} \to \frac{H^1_{/f}(\mathbb{Q}_p, T)}{\mathrm{loc}_p(Z_{Kato})} \to X(\mathbb{Q}_\infty, E) \to X_0(\mathbb{Q}_\infty, E) \to 0.$$

Now the equivalence of equalities of characteristic ideals follows from the reciprocity law. □

More precisely, the above proof shows that Kato's divisibility translate to the divisibility

$$(\mathscr{L}_{\mathbb{Q}_\infty,E}) \text{ divides } \mathrm{Ch}(X(\mathbb{Q}_\infty, E)).$$

[10] There has been a lot of progress on extending these to non-split p or supersingular reduction.

Remark 3.3. Skinner–Urban [SU14] adapted the techniques of Ribet and Mazur–Wiles in the context of $GU(2,2)$ to prove the opposite divisibility in the two-variable main conjecture under some technical assumptions (crucially, one of them is that $\epsilon = 1$)

$$\mathrm{Ch}(X(K_\infty, E)) \text{ divides } (\mathscr{L}_{K_\infty,E}).$$

By specializing to the cyclotomic variable, this amounts to

$$\mathrm{Ch}(X(\mathbb{Q}_\infty, E)) \cdot \mathrm{Ch}(X(\mathbb{Q}_\infty, E^K)) \text{ divides } (\mathscr{L}_{\mathbb{Q}_\infty,E}) \cdot (\mathscr{L}_{\mathbb{Q}_\infty,E^K}).$$

So in combination with Kato's result, this proves the full cyclotomic main conjecture in some cases.

3.2.2. **Anticyclotomic main conjecture.** Let's assume that $N^- = 1$ for simplicity. Then we have the Euler system of Heegner points. They are (essentially) norm compatible in the anticyclotomic tower. So we get a free rank 1 Λ-module

$$Z_{Heeg} \subseteq \mathrm{Sel}(K_\infty^{anti}, E).$$

Even before the work of BDP, Perrin–Riou made the following conjecture

Conjecture 3.4 (Perrin–Riou's main conjecture)**.** *$X(K_\infty^{anti}, E)$ is a rank 1 Λ-module. There is a pseudo-isomorphism $X(K_\infty^{anti}, E) \sim \Lambda \oplus N \oplus N$ with $\mathrm{Ch}(N) = \mathrm{Ch}\left(\frac{\mathrm{Sel}(K_\infty^{anti},E)}{Z_{Heeg}}\right)$.*

There are analogues of this conjecture in the case $N^- \neq 1$ by using generalized Heegner points.

This conjecture can be show to be equivalent to the BDP main conjecture by a similar (in principle) but more complicated analysis as above. See [Cas17, Appendix A] for details. The crucial point is that

Theorem 3.5 (Reciprocity law, [Cas17, Theorem A.1])**.** *Under the big logarithm map $\mathrm{Log}\colon H^1_f(\mathbb{Q}_p, \mathbb{T}_T) \otimes_{\Lambda_K^-} (\Lambda_K^-)^{ur} \hookrightarrow (\Lambda_K^-)^{ur}$, we have $\mathrm{Log}(\mathrm{loc}_p(Z_{Heeg})) = \mathscr{L}_{BDP} \cdot (\Lambda_K^-)^{ur}$.*

As before, the Euler system nature of Heegner points allows one to prove the rank part and the "Euler system divisibility" (see [How04])

$$\mathrm{Ch}(N) \text{ divides } \mathrm{Ch}\left(\frac{\mathrm{Sel}(K_\infty^{anti}, E)}{Z_{Heeg}}\right).$$

Remark 3.6. Xin Wan [Wan20] adapted the argument of Skinner–Urban to $GU(3,1)$ to prove the opposite divisibility in the two-variable main conjecture under some technical assumptions for the case $\epsilon = -1$. As before, this affords a proof of the full anticyclotomic main conjecture in some cases.

3.2.3. **Two variable main conjectures.** Lei–Loeffler–Zerbes [LLZ14] have constructed a free submodule

$$Z_{LLZ} \subseteq \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K_\infty, E)$$

with *two* reciprocity laws, which have (essentially) been proven in [LLZ14] and [KLZ17]: under the maps $\mathrm{Col}\colon H^1_{/f}(K_{\overline{v}}, \mathbb{T}_T) \xrightarrow{\sim} \Lambda_K$ and $\mathrm{Log}\colon H^1_f(K_v, \mathbb{T}_T) \otimes_{\Lambda_K} \Lambda_K^{ur} \xrightarrow{\sim} \Lambda_K^{ur}$, we have

$$\mathrm{Col}(\mathrm{loc}_{\overline{v}}(Z_{LLZ})) = \mathscr{L}_{K_\infty,E} \cdot \Lambda_K, \quad \mathrm{Log}(\mathrm{loc}_v(Z_{LLZ})) = \mathscr{L}_{K_\infty,BDP} \cdot \Lambda_K^{ur}.$$

Conjecture 3.7 (Two variable main conjecture without L-functions)**.** *$\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K_\infty, E)$ is a torsion free rank 1 Λ_K-modules, Z_{LLZ} is nonzero and $\mathrm{Ch}(X_{\mathrm{Gr},0}(K_\infty, E)) = \mathrm{Ch}\left(\frac{\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K_\infty,E)}{Z_{LLZ}}\right)$.*

By arguments similar as above, given the reciprocity laws, this main conjecture is related to both of the two variable main conjectures: that $\mathrm{Ch}(X(K_\infty, E)) = (\mathscr{L}_{K_\infty,E})$ and that $\mathrm{Ch}(X_{\emptyset,0}(K_\infty, E)) = (\mathscr{L}_{K_\infty,BDP})$.

References

[AI19] Fabrizio Andreatta and Adrian Iovita. Katz type p-adic l-functions for primes p non-split in the cm field. *arXiv: Number Theory*, 2019.

[BCD+14] Massimo Bertolini, Francesc Castella, Henri Darmon, Samit Dasgupta, Kartik Prasanna, and Victor Rotger. p-adic L-functions and Euler systems: a tale in two trilogies. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 52–101. Cambridge Univ. Press, Cambridge, 2014.

[BD05] M. Bertolini and H. Darmon. Iwasawa's main conjecture for elliptic curves over anticyclotomic $\mathbb{Z}_p$-extensions. *Ann. of Math. (2)*, 162(1):1–64, 2005.

[BDP13] Massimo Bertolini, Henri Darmon, and Kartik Prasanna. Generalized Heegner cycles and p-adic Rankin L-series. *Duke Math. J.*, 162(6):1033–1148, 2013. With an appendix by Brian Conrad.

[Ber03] Laurent Berger. Bloch and Kato's exponential map: three explicit formulas. Number Extra Vol., pages 99–129. 2003. Kazuya Kato's fiftieth birthday.

[Cas17] Francesc Castella. p-adic heights of Heegner points and Beilinson-Flach classes. *J. Lond. Math. Soc. (2)*, 96(1):156–180, 2017.

[Gre89] Ralph Greenberg. Iwasawa theory for p-adic representations. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 97–137. Academic Press, Boston, MA, 1989.

[Gre99] Ralph Greenberg. Iwasawa theory for elliptic curves. In *Arithmetic theory of elliptic curves (Cetraro, 1997)*, volume 1716 of *Lecture Notes in Math.*, pages 51–144. Springer, Berlin, 1999.

[HB15] Ernest Hunter Brooks. Shimura curves and special values of p-adic L-functions. *Int. Math. Res. Not. IMRN*, (12):4177–4241, 2015.

[How04] Benjamin Howard. The Heegner point Kolyvagin system. *Compos. Math.*, 140(6):1439–1472, 2004.

[JSW17] Dimitar Jetchev, Christopher Skinner, and Xin Wan. The Birch and Swinnerton-Dyer formula for elliptic curves of analytic rank one. *Camb. J. Math.*, 5(3):369–434, 2017.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. Number 295, pages ix, 117–290. 2004. Cohomologies p-adiques et applications arithmétiques. III.

[KLZ17] Guido Kings, David Loeffler, and Sarah Livia Zerbes. Rankin-Eisenstein classes and explicit reciprocity laws. *Camb. J. Math.*, 5(1):1–122, 2017.

[Kri21] Daniel J. Kriz. *Supersingular p-adic L-functions, Maass-Shimura operators and Waldspurger formulas*, volume 212 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2021.

[LLZ11] Antonio Lei, David Loeffler, and Sarah Livia Zerbes. Coleman maps and the p-adic regulator. *Algebra Number Theory*, 5(8):1095–1131, 2011.

[LLZ14] Antonio Lei, David Loeffler, and Sarah Livia Zerbes. Euler systems for Rankin-Selberg convolutions of modular forms. *Ann. of Math. (2)*, 180(2):653–771, 2014.

[LZZ18] Yifeng Liu, Shouwu Zhang, and Wei Zhang. A p-adic Waldspurger formula. *Duke Math. J.*, 167(4):743–833, 2018.

[MTT86] B. Mazur, J. Tate, and J. Teitelbaum. On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent. Math.*, 84(1):1–48, 1986.

[PR88] Bernadette Perrin-Riou. Fonctions l p-adiques associees a une forme modulaire et a un corps quadratique imaginaire. *Journal of the London Mathematical Society*, s2-38(1):1–32, 1988.

[PR94] Bernadette Perrin-Riou. Théorie d'iwasawa des répresentations p-adiques sur un corps local (avec un appendice de j.-m. fontaine). *Inventiones mathematicae*, 115(1):81–150, 1994.

[RT97] Kenneth A. Ribet and Shuzo Takahashi. Parametrizations of elliptic curves by Shimura curves and by classical modular curves. volume 94, pages 11110–11114. 1997. Elliptic curves and modular forms (Washington, DC, 1996).

[Ski18] Christopher Skinner. Arizona Winter School 2018, Lecture Notes, 2018. URL: https://swc-math.github.io/aws/2018/2018SkinnerNotes.pdf.

[SU14] Christopher Skinner and Eric Urban. The Iwasawa main conjectures for GL_2. *Invent. Math.*, 195(1):1–277, 2014.

[Wan20] Xin Wan. Iwasawa main conjecture for Rankin-Selberg p-adic L-functions. *Algebra Number Theory*, 14(2):383–483, 2020.

[YZZ13] Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang. *The Gross-Zagier formula on Shimura curves*, volume 184 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2013.