0.1 p-adic Hodge Theory

Notations (0.1.0.1).

- Let K be a p-adic field with residue field k,
- $K_0 = W(k)[\frac{1}{p}]$ its maximal unramified subextension.

References are [Theory of p-adic Representations, J. M. Fontaine and Yi Ouyang]. [Cmi Summer School Notes On p-adic Hodge Theory, O. Brinon and B. Conrad]. [Notes on p-adic Hodge Theory, S. Hong], [An Excursion Into p-adic Hodge Theory: From Foundations To Recent Trends, F. Andreatta, R. Brasca, O. Brinon, X. Caruso, B. Chiarellotto, G. Freixas i Montplet, S. Hattori, N. Mazzari, S. Panozzo, M. Seveso, G. Yamashita]. [p-adic Hodge Theory For Rigid-Analytic Varieties, Scholze]

1 C_{dR}-Theorem

Thm. (0.1.1.1) [C_{dR}-Theorem, Faltings/Tsuji]. If $X \in \mathcal{S}\mathrm{ch}^{\mathrm{sm},proper}/K$, then for any $r \in \mathbb{N}$, there exists a canonical isomorphism

$$\gamma_{\mathrm{dR}}(X) : B_{\mathrm{dR}} \otimes_K H^r_{\mathrm{dR}}(X/K) \cong B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathrm{H}^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p).$$

which identifies filtrations and Gal_K-actions on both sides. Moreover γ_{dR} is functorial in X.

Proof: Cf.[Faltings, p-adic Hodge Theory]. or [p-adic Hodge for Rigid Analytic Varieties, Scholze], [BMS18]P104. □

Prop. (0.1.1.2)[deRham Comparison for Étale Cohomologies]. By taking the gradation of(0.1.1.1), by(0.1.7.4), there is a Hodge-like decomposition

$$\mathbb{C}_K \otimes_{\mathbb{Q}_p} \mathrm{H}^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{a+b=r} \mathbb{C}_K(-a) \otimes_K H^b(X, \Omega^a_X).$$

Example (0.1.1.3) [Elliptic Curve Case, Tensoring $\mathbb{C}_K$ Lost Informations]. For $E \in \mathcal{E}\mathrm{ll}/K$ with multiplicative reduction and $j(E) > 1$, by**??** and**??**,

$$E(\overline{K}) \cong \overline{K}^\times / q^{\mathbb{Z}}$$

as Gal_K-representations for some $q \in K^\times$. Thus $T_p(E) \cong q^{\mathbb{Q}_p/\mathbb{Z}_p}$ and there exists an exact sequence

$$0 \to \mathbb{Z}_p(1) \to T_p(E) \to \mathbb{Z}_p \to 0.$$

Then this sequence doesn't split when tensoring $\overline{K}$, but split when tensoring $\mathbb{C}_K$, by(0.1.1.2).

Proof: Suppose it splits ofter tensoring $\overline{K}$, then it splits after tensoring some finite extension K'. Then by projection of K' onto $\mathbb{Q}$, we see

$$0 \to \mathbb{Q}_p(1) \to V_p(E) \to \mathbb{Q}_p \to 0$$

is splitting as a $\mathrm{Gal}_{K'}$-representations. But this is not true, as any system of roots of p □

The Hodge-Tate representation comes from Falting's theorem on the Hodge-Tate decomposition on the etale cohomology of smooth proper varieties over a p-adic field K.

But the Hodge-Tate decomposition is too weak, to strengthen it, we want to add a filtration. The idea is to prolong $H^q(X, \Omega^p_X)$ to the filtration on the $H^*_{\mathrm{dR}}(X/K)$, which has a filtration with $\mathrm{gr}^j = H^{n-j}(X, \Omega^j_X)$. And we simulate this information on the ring-theoretic level.

2 Fontaine-Wintenberger Theory of Norm Fields

Cf.[notes on p-adic Hodge, Conrad]intro.

Remark (0.1.2.1). Delete this subsection. Cf.[Conrad].

There exists a $E = E_{K_\infty/K} \in p-\texttt{FField}$ with a canonical isomorphism

$$\mathrm{Gal}_{K_\infty} \cong \mathrm{Gal}_E .$$

Thus

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_{K_\infty}) \cong \mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_E) \cong \varphi\text{-}\mathcal{M}\mathrm{od}_{A_E} .$$

Then because a representation of Gal_K can be viewed as a representation of Gal_{K_∞} with certain descent datum, and this induces an equivalence of categories

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_K) \cong (\varphi, \Gamma)\text{-}\mathcal{M}\mathrm{od}_{A_E} .$$

3 Admissible Representations

Def. (0.1.3.1) [Admissible Representations]. Let $G \in \mathcal{G}\mathrm{rp}^{\mathrm{top}}$ and E a topological field that G acts trivially and B a topological E-algebra s.t. $B \in \mathrm{Rep}_E(G)$. Then $V \in \mathrm{Rep}_E^{\mathrm{fd}}(G)$ is called a B**-admissible** representation if $B \otimes_E V \in \mathrm{Rep}_B(G)$ is trivial.

The category $\mathrm{Rep}_E^{B\text{-adm}}(G)$ is the full subcategory of $\mathrm{Rep}_E(G)$ consisting of f.d. B-admissible E-representations of G.

Prop. (0.1.3.2) [Inclusions and Admissibility]. Let $G \in \mathcal{G}\mathrm{rp}^{\mathrm{top}}$ and E a topological field that G acts trivially and B_1, B_2, B a topological E-algebra s.t. $B_1, B_2, B \in \mathrm{Rep}_E(G)$, and $B_1 \subset B, B_2 \subset B$, $B_1 \cap B_2 = B_0$, and $B^G \subset B_0$, then

$$\mathrm{Rep}_E^{B_1\text{-adm}}(G) \cap \mathrm{Rep}_E^{B_2\text{-adm}}(G) = \mathrm{Rep}_E^{B_0\text{-adm}}(G).$$

Proof: The RHS is contained in LHS trivially. For the converse inclusion, if $V \in \mathrm{Rep}_E^{B_1\text{-adm}}(G) \cap \mathrm{Rep}_E^{B_2\text{-adm}}(G)$, there exists elements $\{u_i\} \subset (B_1 \otimes_E V)^G$ and $\{v_i\} \subset (B_2 \otimes_E V)^G$ s.t.

$$B \otimes_E V = Bu_1 \oplus \ldots Bu_n = Bv_1 \oplus \ldots Bv_n.$$

Then the transformation matrix from $\{u_i\}$ to $\{v_i\}$ is an element in $GL(n; B)$ that is invariant under G, so contained in $GL(n; B^G)$. Thus it is clear that

$$\{v_i\} \subset (B_2 \otimes_E V)^G \cap (B_1 \otimes_E V)^G = (B_0 \otimes_E V)^G,$$

and then $B_0 \otimes_E V = B_0 v_1 \oplus \ldots B_0 v_n$, and V is B_0-admissible. □

Gal_K-Regularity

Def. (0.1.3.3) [G-Regularity]. Situation as in(0.1.3.1), we want to establish a numerical criterion for recognizing B-admissible representations. B is called G**-regular** if it satisfies the following three conditions:

H1 : B is a domain.

H2 :$(\mathrm{Frac}(B))^G = B^G$, in particular, B^G is a field.

H3 : if $b \neq 0 \in B$ and Eb is stable under G-action, then $b \in B^*$.
Notice a field is clearly G-regular.

Cor. (0.1.3.4). Notice that (H3) implies $B^G \in \texttt{Field}$, because for $b \in B^{\mathrm{Gal}_K}$, Eb is clearly stable under G-action, thus b is invertible.

Also the morphism

$$\alpha_B(W) : B \otimes_{B^G} W^G \to W$$

is injective for all finite free $W \in \mathrm{Rep}_B(G)$. In particular, this is true for $W = B \otimes_E V, V \in \mathrm{Rep}_E^{\mathrm{fd}}(G)$, and we get a functor

$$D_B : \mathcal{V}\mathrm{ect}_E \to \mathcal{V}\mathrm{ect}_{B^G}$$

such that

$$\dim_{B^G} D_B(V) \leq \dim_E V.$$

Proof: To show α_W is injective, it suffices to show a linear basis $\{e_i\}$ of W^G over B^G is linearly independent over B: Suppose $\sum a_i e_i = 0$, where $a_i \in B$, with the number of nonzero coefficients minimal, and $a_1 \neq 0$, then dividing $a_1 \in \mathrm{Frac}(B)$, we assume $a_1 = 1$, and then acting by $g - \mathrm{id}$, we get

$$\sum (g(a_i) - a_i) e_i = 0$$

and this has smaller non-zero elements, unless a_i is fixed by g for any $g \in G$, so $a_i \in \mathrm{Frac}(B)^G = B^G$ by (H2), contradiction. □

Prop. (0.1.3.5). For any topological G a topological ring B with a G-action, $d \in \mathbb{N}$, there is a bijection between the set of equivalence classes of free B-representations V of G of rank d and the category of $H^1(G, \mathrm{GL}(d, B))$. Moreover, V is trivial iff it is mapped to the distinguished point of $H^1(G, \mathrm{GL}(d, B))$.

Proof: This follows by taking the matrix of $g \in G$ w.r.t. a B-basis of V. □

Cor. (0.1.3.6). Let L/K be a Galois extension of fields, then any f.d. L-representation of $\mathrm{Gal}(L/K)$ is trivial.

Proof: This follows from Hilbert's theorem90**??**. □

Prop. (0.1.3.7) [B-Admissible Representations]. If B is G-regular(0.1.3.3), $V \in \mathrm{Rep}_E^{\mathrm{fd}}(G)$ and $W = B \otimes_E V$, then the following are equivalent:

- W is trivial, i.e. V is B-admissible.
- $\alpha_B(W)$(0.1.3.4) is an isomorphism.
- $\dim_{B^G} D_B(V) = \dim_E V$.

Proof: $1, 2$ are equivalent by(0.1.3.4), as B^{Gal_K} is a field. Also $2 \to 3$ is clear.

$3 \to 2$: $\alpha_W : B \otimes_{B^G} W^G \to B \otimes_E V$ is a B-linear morphism of two finite free B-modules, then it suffices to show the determinant map is an isomorphism. Let $v_1, \ldots, v_d$ be a E-basis of V and $w_1, \ldots, w_d$ a B^G-basis of W^G. Let b be the unique element of B that

$$\alpha_W(v_1) \wedge \ldots \wedge \alpha_W(v_d) = b w_1 \wedge \ldots \wedge w_d$$

then $gb = \eta b$ for $g \in G$ where η is determined by the identity $\alpha_W(gv_1) \wedge \ldots \wedge \alpha_W(gv_d) = \eta \alpha_W(v_1) \wedge \ldots \wedge \alpha_W(v_d)$. Now the E-space of $v_1, \ldots, v_d$ is V, which is stable under G action, thus $\eta \in E$, and then by (H3) $b \in B^*$, so we are done. □

Cor. (0.1.3.8) $[\mathrm{Rep}_E^{B-\mathrm{adm}}(G)]$**.** If B is Gal_K-regular, then

- $\mathrm{Rep}_E^{B-\mathrm{adm}}(G) \subset \mathrm{Rep}_E(G)$ is stable under subobjects and quotients.
- $D_B : \mathrm{Rep}_E^B(G) \to \mathcal{V}\mathrm{ect}_{B^G}$ is exact and faithful.
- $\mathrm{Rep}_E^{B-\mathrm{adm}}(G) \subset \mathrm{Rep}_E(G)$ is stable under taking dual and tensor products. And if $V, V_1, V_2 \in \mathrm{Rep}_E^{B-\mathrm{adm}}(G)$, then there is a natural isomorphism

$$D_B(V_1) \otimes D_B(V_2) \cong D_B(V_1 \otimes V_2)$$

and

$$D_B(V) \otimes D_B(V^\vee) \cong D_B(V \otimes V^\vee) \to D_B(E) = B^G$$

is a perfect pairing between $D_B(V)$ and $D_B(V^\vee)$.

Proof: 1: Given an exact sequence $0 \to V_1 \to V \to V_2 \to 0 \in \mathrm{Rep}_E(G)$, tensoring B and taking G-fixed points, we get an exact sequence

$$0 \to D_B(V_1) \to D_B(V) \to D_B(V_2)$$

from which we derive the inequality $\dim_{B^G} D_B(V) \leq \dim_{B^G} D_B(V_1) + \dim_{B^G} D_B(V_2)$. Now we have $\dim_{B^G} D_B(V_i) \leq \dim_E V_i$ by(0.1.3.4), so

$$\dim_{B^G} D_B(V) \leq \dim_{B^G} D_B(V_1) + \dim_{B^G} D_B(V_2) \leq \dim_E V_1 + \dim_E V_2 = \dim_E V.$$

But this is an equality because V is B-admissible, thus V_1, V_2 are all B-admissible, and the exact sequence is in fact an isomorphism by dimension reason.

2: D_B is faithful because $B \otimes_{B^G} D_B(V) \cong B \otimes_E V$.

3: There is a natural map

$$D_B(V_1) \otimes_{B^G} D_B(V_2) = (B \otimes_E V_1)^G \otimes (B \otimes_E V_2)^G \to (B \otimes_E (V_1 \otimes_E V_2))^G = D_B(V_1 \otimes_E V_2),$$

and $\dim_{B^G} D_B(V_1 \otimes_E V_2) \leq \dim_E(V_1) \cdot \dim_E(V_2)$, so it suffices to show that this map is injective. For this, notice that $D_B(V_1 \otimes_E V_2) \subset B \otimes_E (V_1 \otimes_E V_2)$, and after tensoring B,

$$D_B(V_1) \otimes_{B^G} D_B(V_2) \subset B \otimes_{B^G} (D_B(V_1) \otimes_{B^G} D_B(V_2)) \cong (B \otimes_E V_1) \otimes_B (B \otimes_E V_2) \to B \otimes_E (V_1 \otimes_E V_2)$$

is an isomorphism.

To show for the dual preserves B-admissibility, notice that $\mathrm{Rep}_E^{B-\mathrm{adm}}(G)$ is also stable under exterior products, as exterior products are quotient of tensor products. Notice there is an isomorphism

$$\wedge(V^\vee) \otimes \wedge^{\dim V - 1} V \cong V^\vee,$$

so it suffices to show for $\dim V = 1$. Let v_0 be an E-basis of V, $g(v_0) = \eta(g) v_0$, then $D_B(V) = B^G(b \otimes v_0)$ for some $b \neq 0 \in B$. Thus $b/g(b) = \eta(g)$. And it is easy to show that $D_B(V^\vee) = B^G(b^{-1} \otimes v_0)$, and the natural pairing is perfect. In general, the pairing is also perfect because perfectness of a pairing can be checked after passing to the determinant space.

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4 $\mathbb{C}_K$-Admissibility

p-adic Fields

Def. (0.1.4.1) [p-adic Fields]. For $p \in \mathbf{P}$, a **p-adic field** is a field K of characteristic 0 that is complete w.r.t. a discrete valuation s.t. the residue field is perfect of characteristic p.

Def. (0.1.4.2) [Topological Completion]. If $p \in \mathbf{P}$ and K is a p-adic field, then we can define $\mathbb{C}_K = \widehat{\overline{K}}$, which is an alg.closed complete valued field. Also denote $\mathbb{C}_p = \mathbb{C}_{\mathbb{Q}_p}$.

Prop. (0.1.4.3) [Ax-Sen-Tate]. If F is a p-adic field and if $K \subset \overline{F}$, then $\widehat{\overline{F}}^{\mathrm{Gal}_K} = \widehat{K}$. Thus $\widehat{L}^{\mathrm{Gal}(L/K)} = \widehat{K}$ for any alg.ext L/K.

$\mathbb{C}_K$-Admissibility

Prop. (0.1.4.4) [Variant of Hilbert's Theorem90]. Any $V \in \mathrm{Rep}^{\mathrm{fd}}_{\widehat{K^{\mathrm{ur}}}}(\mathrm{Gal}(K^{\mathrm{ur}}/K))$ is trivial. In particular, any unramified f.d. representation of Gal_K is $\widehat{K^{\mathrm{ur}}}$-admissible thus $\mathbb{C}_p$-admissible, which is a special case of(0.1.4.5).

Proof: Denote by $\mathcal{O}$ the ring of integers of $\widehat{K^{\mathrm{ur}}}$ and $\mathfrak{m}$ the maximal ideal, Let W be a f.d. $\widehat{K^{\mathrm{ur}}}$-semi-linear representation, $(v_{1,0},\ldots,v_{d,0})$ a basis of W over $\widehat{K^{\mathrm{ur}}}$ and $\mathcal{O}_W$ the $\mathcal{O}$-span of $(v_{1,0},\ldots,v_{d,0})$, then we are going to construct a sequence of tuples $(v_{1,n},\ldots,v_{d,n})$ that $v_{i,n+1} \equiv v_{i,n} \mod \mathfrak{m}^n$ and $\mathrm{Frob}_q(v_{i,n}) \equiv v_{i,n} \mod \mathfrak{m}^n$ for all i and n.

Use induction on n: the case $n=1$ follows from the fact $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$ is trivial as a $\overline{k}$-semi-linear representation of Gal_k. To prove this, notice there is a finite extension l of k and an l-semi-linear representation W_L of $G_{l/k}$ that $\overline{k} \otimes_l W_l \cong \mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$, then the assertion follows from Hilbert's theorem90**??**.

For general n, we are looking for vectors $w_1,\ldots,w_d \in \mathcal{O}_W$ that $\mathrm{Frob}_q(v_{i,n}+\pi^n w_i) \equiv v_{i,n} + \pi^n w_i \mod \mathfrak{m}^{n+1}$, which is equivalent to $\mathrm{Frob}_q\overline{w_i} - \overline{w_i} = \frac{\mathrm{Frob}_q v_{i,n} - v_{i,n}}{\pi^n}$ in $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$. To prove this, notice $\mathrm{Frob}_q - \mathrm{id}$ is surjective on $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$, which follows from the fact $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$ is trivial as proved above and $\mathrm{Frob}_q - \mathrm{id}$ is surjective on $\overline{k}$.

Now $v_{i,n}$ are Cauchy sequences and they converges to a tuple v_i that $G_{K^{ur}/K}$ acts trivially and it is an $\mathcal{O}$-basis of $\mathcal{O}_W$, as its reduction modulo $\mathfrak{m}$ is a basis of $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$, so it is a $\widehat{K^{\mathrm{ur}}}$-basis of W. □

Prop. (0.1.4.5) [$\mathbb{C}_p$-Admissibility]. For $(\rho, V) \in \mathrm{Rep}^{\mathrm{fd}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$, the following are equivalent:

- V is $\mathbb{C}_p$-admissible.
- $\#\rho(I_K) < \infty$.
- V is $L\widehat{K^{\mathrm{ur}}}$-admissible for some finite extension L/K.

Proof: Cf.[p-adic Period Rings Intro, P18].**?**

$2 \to 3$: This follows from(0.1.4.4). □

Cor. (0.1.4.6)[$H^0_{\mathrm{conts}}(\mathrm{Gal}_K, \mathbb{C}_p(\psi))$, Sen-Tate]. $H^0_{\mathrm{conts}}(\mathrm{Gal}_K, \mathbb{C}_p(\psi)) = \begin{cases} K & ,\#\psi(I_K) < \infty \\ 0 & ,\#\psi(I_K) = \infty \end{cases}$. In particular,

$$H^0_{\mathrm{conts}}(\mathrm{Gal}_K, \mathbb{C}_p(m)) = \begin{cases} K & ,m = 0 \\ 0 & ,m \neq 0 \end{cases}.$$

Proof: This follows from(0.1.3.7) and(0.1.4.5). For the last assertion, the cyclotomic extension of K thus also the cyclotomic character of G_K is infinitely unramified, thus χ_{cycl}^s factors through a finite quotient iff $s = 0$. And $H^0(\text{Gal}_K, \mathbb{C}_p) = K$ by Ax-Sen-Tate(0.1.4.3). □

Cor. (0.1.4.7). For any $n, m \in \mathbb{Z}$,

$$\text{Hom}_{\text{Rep}_{\mathbb{C}_K}(\text{Gal}_K)}(\mathbb{C}_K(n), \mathbb{C}_K(m))$$

is of one-dimensional over K if $n = m$, and vanishes otherwise.

Proof: Let $W = \text{Hom}_{\mathbb{C}_p}(\mathbb{C}_K(n), \mathbb{C}_K(m)) = \mathbb{C}_K(m-n)$, then the desired space is W^{Gal_K}, and the assertion follows from(0.1.4.6). □

Prop. (0.1.4.8) [$H^1_{\text{conts}}(\text{Gal}_K, \mathbb{C}_p(\psi))$, Sen-Tate]. There is an inf-res exact sequence

$$0 \to H^1_{\text{conts}}(\Gamma_K, \widehat{K}_\infty(\psi)) \to H^1_{\text{conts}}(\text{Gal}_K, \mathbb{C}_p(\psi)) \to H^1_{\text{conts}}(H_K, \mathbb{C}_p(\psi)),$$

and

$$H^1_{\text{conts}}(H_K, \mathbb{C}_p(\psi)) = 0, \quad H^1_{\text{conts}}(\text{Gal}_K, \mathbb{C}_p(\psi)) = \begin{cases} 0 & , \#\psi(I_K) = \infty \\ \text{a } K\text{-vector space of dimension 1} & , \#\psi(I_K) < \infty \end{cases}.$$

5 Hodge-Tate Representations

Hodge-Tate Representations

Def. (0.1.5.1) [$B_{\text{H-T}}$]. Let $B_{\text{H-T}} = \mathbb{C}_K[t, t^{-1}], B'_{\text{H-T}} = \mathbb{C}_K((t))$, and let G_K acts on it by $g(at^i) = g(a)\chi_{\text{cycl}}(g)^i t^i$. In addition, there is a filtration on $B'_{\text{H-T}}$ given by $\text{Fil}^m B'_{\text{H-T}} = t^m\mathbb{C}_p[[t]]$, then the graded ring of $B'_{\text{H-T}}$ is isomorphic to $B_{\text{H-T}}$. (0.1.4.6) shows that $B_{\text{H-T}}^{\text{Gal}_K} = (B'_{\text{H-T}})^{\text{Gal}_K} = K$.

$B_{\text{H-T}}$ and $B'_{\text{H-T}}$ are Gal_K-regular(0.1.3.3).

Proof: $B'_{\text{H-T}}$ is Gal_K-regular because it is a field. For $B_{\text{H-T}}$, $B_{\text{H-T}} \subset \text{Frac}(B_{\text{H-T}}) \subset B'_{\text{H-T}}$, taking Gal_K-fixed points shows $(H2)$. For $(H3)$, if $\mathbb{Q}_p x$ is stable under Gal_K and x is not of the form at^i, then we can get a non-trivial Gal_K-fixed point of $\mathbb{C}_K(j-i)$, which is impossible by(0.1.4.6). □

Cor. (0.1.5.2) [Hodge-Tate Representations]. Let $W \in \text{Rep}_{\mathbb{C}_K}(\text{Gal}_K)$. For $k \in \mathbb{Z}$, let

$$W\{k\} = \{x \in W | g(x) = \chi_{\text{cycl}}^k(g)x\} \subset W(k)$$

then

$$\bigoplus_{k\in\mathbb{Z}} (\mathbb{C}_K(k) \otimes_K W\{k\}) \to W$$

is injective. W is called a **Hodge-Tate representation** if this is an isomorphism.

Proof: Notice $B_{\text{H-T}} \cong \bigoplus_{m\in\mathbb{Z}} \mathbb{C}_K(m) \in \text{Rep}_{\mathbb{C}_K}(\text{Gal}_K)$, so

$$B_{\text{H-T}} \otimes_K (B_{\text{H-T}} \otimes_{\mathbb{C}_K} W)^{\text{Gal}_K} \cong B_{\text{H-T}} \otimes_K \bigoplus_{m\in\mathbb{Z}} (\mathbb{C}_K(-m) \otimes_K W\{m\}) \hookrightarrow B_{\text{H-T}} \otimes_K \bigoplus_{m\in\mathbb{Z}} (\mathbb{C}_K(m) \otimes_K W\{m\}).$$

is injective by(0.1.3.4). □

Cor. (0.1.5.3) [Hodge-Tate Representations]. Let K be a p-adic field, then $V \in \mathrm{Rep}^{\mathrm{fd}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$ is called a **Hodge-Tate representation** if it is $B_{\text{H-T}}$-admissible. The category of Hodge-Tate representations are denoted by $\mathrm{Rep}^{\text{H-T}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$.

Then $V \in \mathrm{Rep}^{\mathrm{fd}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$ is Hodge-Tate iff it is $B'_{\text{H-T}}$-admissible iff $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ is Hodge-Tate thus decomposes as

$$\mathbb{C}_K \otimes_{\mathbb{Q}_p} V \cong \mathbb{C}_K(n_1) \oplus \ldots \oplus \mathbb{C}_K(n_d) \in \mathrm{Rep}_{\mathbb{C}_K}(\mathrm{Gal}_K).$$

Proof: If $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ is Hodge-Tate, then clearly $\dim_K(B_{\text{H-T}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}_K} = \dim_{\mathbb{Q}_p} V$, thus V is $B_{\text{H-T}}$-admissible(0.1.3.7). Conversely, $\dim_K(B_{\text{H-T}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}_K} = \dim_{\mathbb{Q}_p} V = d$ implies V is Hodge-Tate by(0.1.5.2). The equivalence with $B'_{\text{H-T}}$-admissibility is similar. □

Prop. (0.1.5.4). If $K' \in \mathtt{Field}, K' \subset \overline{K}$, then for $W \in \mathrm{Rep}^{\mathrm{fd}}_{\mathbb{C}_K}(\mathrm{Gal}_K)$, the natural maps

$$K' \otimes_K D_K(W) \to D_{K'}(W), \quad \widehat{K^{\mathrm{ur}}} \otimes_K D_K(W) \to D_{\widehat{K^{\mathrm{ur}}}}(W)$$

are isomorphisms. In particular,

$$\mathrm{Rep}^{\mathrm{fd}}_{\mathbb{Q}_p}(\mathrm{Gal}_K) \cap \mathrm{Rep}^{\text{H-T}}_{\mathbb{Q}_p}(\mathrm{Gal}_{K'}) = \mathrm{Rep}^{\mathrm{fd}}_{\mathbb{Q}_p}(\mathrm{Gal}_K) \cap \mathrm{Rep}^{\text{H-T}}_{\mathbb{Q}_p}(I_K) = \mathrm{Rep}^{\text{H-T}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$$

Proof: For $K' \subset \overline{K}$, $D_K(W) = D_{K'}(W)^{\mathrm{Gal}(K'/K)}$, thus the isomorphism follows from Galois descent**??**. For $\widehat{K^{\mathrm{ur}}}$, Cf.[Conrad, P20] □

6 Period Rings

Prop. (0.1.6.1). If $R \in \mathcal{CR}\mathrm{ing}$ is a p-adically complete, $\pi \in R^\times, p \in (\pi)$, then the map $R \to R/p$ induces an homeomorphism of monoids:

$$\varprojlim_{x \to x^p} R \cong \varprojlim_{\varphi} R/p = R^\flat$$

Proof: Injectivity: if $(a_n), (b_n) \in \lim_{x\to x^p} R$ satisfies $a_n \equiv b_n \mod \pi$ for all n, then applying power lifting**??**, $a_n \cong b_n \mod \pi^{n+k}$ for all k, so $a_n = b_n$.

Surjectivity: for $(\overline{a_n}) \in R^\flat$, choose arbitrary lifting a_n, then $a_{n+k+1}^p \equiv a_{n+k} \mod \pi$ for all $n+k$, so $k \mapsto a_{n+k}^{p^k}$ is a Cauchy sequence by power lifting**??** again, thus converging to some point b_n. then it's easily checked that $b_{n+1}^p = (\lim a_{n+1+k}^{p^k})^p = \lim a_{n+1+k}^{p^{k+1}} = b_n$. so (b_n) maps to $(\overline{a_n})$.

For the topology: it is clearly continuous, and for the reverse, if $(a_i), (b_i)$ satisfies that $a_i \equiv b_i \mod \pi$ for $i < k$, then the image in $\varprojlim_{x\to x^p} R$ satisfies $x_i \equiv y_i \mod p^{k-i}$ for $i < k$, thus it is open. □

Cor. (0.1.6.2) [Sharp Map]. From this proposition, we get a multiplicative **sharp map**:

$$\sharp : R^\flat \to R : (\overline{a}_n) \mapsto \lim_{k\to\infty} a_k^{p^k},$$

and its image is just the elements that has a compatible system of p^k-th roots $x^{\frac{1}{p^k}}$. These elements are also called **perfect**.

Cor. (0.1.6.3) [Addition in $R^\flat$]. From the isomorphism(0.1.6.1) above, we can read what the addition looks like in the presentation $\varprojlim_{\varphi} R$: if $(f_n), (g_n)$ are two elements, then their addition is given by (h_n), where $h_n = \lim_k (f_{n+k} + g_{n+k})^{p^k}$.

Cor. (0.1.6.4)[Fontaine's Functor]. By**??**, the natural map $R^\flat \to R/p$ induces a map $\theta_R : W(R^\flat) \to R$ of rings, called the **Fontaine's functor**, which writes as $\sum [a_i]p^i \mapsto \sum a_i^\sharp p^i$. And we denote $A_{\text{inf}}(R) = W(R^\flat)$ the **infinitesimal Fontaine's ring** of R.

Prop. (0.1.6.5). If R is p-complete, the Fontaine's functor θ_R is surjective iff R/p is semiperfect.

Proof: As R is p-complete, θ is surjective iff it is surjective modulo p. Because its reduction modulo p is $R^\flat \to R/p$ is surjective as $\varphi : R/p \to R/p$ does. □

Prop. (0.1.6.6). By**??**, if $K \in \text{Perfd}, \text{char}\, K = 0$, with tilt $C^\flat$, then there is a diagram

$$\begin{array}{ccc} A_{\text{inf}} & \xrightarrow{\theta} & \mathcal{O}_K \\ \downarrow & & \downarrow \\ \mathcal{O}_{C^\flat} & \xrightarrow{\overline{\sharp}} & \mathcal{O}_K/(p) \end{array}.$$

Then θ is surjective, and $\ker\theta$ is generated by some distinguished element $\xi = [t] - pu$ where $u \in A_{\text{inf}}$ is invertible and $[t]$ is the Teichmuller lift.

Proof: θ is surjective by(0.1.6.5). By**??**, there exists $t \in A_{\text{inf}}$ s.t. $t^\sharp = pu'$ for some u' invertible in $\mathcal{O}_K$. Thus $u' = \theta(u)$ for some invertible $u \in A_{\text{inf}}$, then $\theta([t] - pu) = 0$. And ξ generates the kernel because it generates after modulo ϖ, and and use the fact $\mathcal{O}_K$ is p-complete. □

Def. (0.1.6.7) [B_{dR}^+]. p is not a zero-divisor in $A_{\text{inf}}/(\xi^n)$, as in the proof of**??**, so we can define

$$B_{\text{dR}}^+ = \varprojlim_n A_{\text{inf}}/(\xi^n)[\frac{1}{p}]$$

Prop. (0.1.6.8)[Fontaine's Ring B_{dR}]. B_{dR}^+ is a complete discrete valuation ring with ξ a uniformizer and the residue field K. Hence we can define $B_{\text{dR}} = \text{Frac}(B_{\text{dR}}^+)$.

Proof: Firstly ξ is not a zero divisor in B_{dR}^+, because if $\xi x = 0, x = (x_n)$, then for any $n > 0$, and some k that $p^k x_n \in A_{\text{inf}}/(\xi^n)$, so $p^k x_n$ is annihilated by ξ in $A_{\text{inf}}/(\xi^n)$, thus $p^k x_n = \xi^{n-1} y_n$ for some y_n, because ξ is a non-zero-divisor in A_{inf}**??**. So $p^{n-1} x_{n-1} = 0 \in A_{\text{inf}}/(\xi^n)$, thus $x_{n-1} = 0$, because p is non-zero-divisor in $A_{\text{inf}}/(\xi^n)$**??**.

Next there is a map $B_{\text{dR}}^+/(\xi^m) \to A_{\text{inf}}/(\xi^m)[p^{-1}]$. This is an isomorphism: it is clearly a surjection, and if $x = (x_n)$ is mapped to 0, then for each $n \geq m$, we choose $p^{k(n)} x_n = 0 \in A_{\text{inf}}/(\xi^n)$, then $p^{k(n)} x_n = \xi^m y_n$ for a unique $y_n \in A_{\text{inf}}/(\xi^{n-m})$. So $x = \xi^m \cdot (\frac{y_n}{p^{k(n)}}) \in \xi^m B_{\text{dR}}$.(Notice the uniqueness of y_n shows $(\frac{y_n}{p^{k(n)}})$ is an element in B_{dR}^+).

Then it follows $B_{\text{dR}}^+ \cong \varprojlim_m B_{\text{dR}}^+/(\xi^m)$, which shows that B_{dR}^+ is ξ-adically complete, and $m = 1$ shows the residue field is equal to K. □

Prop. (0.1.6.9) [Topology on B_{dR}]. The Gauss norms give A_{inf} a topology, giving B_{dR} a topology. Then B_{dR} is complete in this topology, and $B_{\text{dR}} \to K$ is continuous.

With this topology, B_{dR} is abstractly isomorphic to $\mathbb{C}_p((T))$, but not topological isomorphic to it.

Proof: Cf.[Conrad, P65] or [p-adic Period Rings]P42.**?**

B_{dR} is abstractly isomorphic to $\mathbb{C}_p((T))$ by Cohn structure theorem, but [Colmez, Une construction de BdR] proved that $\overline{K}$ is dense in B_{dR}, so it cannot by topological isomorphic to $\mathbb{C}_p((T))$. □

Lemma (0.1.6.10). Let $\varepsilon = (\dots, \varepsilon_1, \varepsilon_0) \in \mathcal{O}^\flat_{\mathbb{C}_K}$ s.t. $\varepsilon_0 = 1$ and $\varepsilon_1 \neq 1$, then $|\varepsilon - 1| = \frac{p}{p-1}$.

Proof:

$$|\varepsilon - 1| = |(\varepsilon-1)^\sharp|_{\mathbb{C}_K} = |\lim_{n\to\infty}(\varepsilon_n - 1)^{p^n}|_{\mathbb{C}_K} = \lim_{n\to\infty} p^n|\varepsilon_n - 1| = \lim_{n\to\infty}\frac{p^n}{p^{n-1}(p-1)} = \frac{p}{p-1}$$

□

Prop. (0.1.6.11) [$\mathbb{Q}_p$-Line in B_{dR}]. $\theta([\varepsilon]-1) = \varepsilon_0 - 1 = 0$, so $[\varepsilon] - 1 \in \ker\theta$, and we can define

$$t_\varepsilon = \log([\varepsilon]) = \sum_{n\geq 1}(-1)^{n-1}\frac{([\varepsilon]-1)^n}{n} \in B^+_{\mathrm{dR}}.$$

Then this is a uniformizer in the CDVR B^+_{dR}. Moreover, any other choice of ε is of the form $\varepsilon' = \varepsilon^a$ for $a \in \mathbb{Z}_p$, then $t_{\varepsilon'} = at_\varepsilon$, and $\gamma(t_\varepsilon) = \chi_{\mathrm{cycl}}(\gamma)t$ for any $\gamma \in \mathrm{Gal}_K$.

Proof: t_ε is a uniformizer because $[\varepsilon] - 1$ is: $[\varepsilon^{1/p}] - 1$ is a unit in B_{dR}, and

$$\eta = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} = 1 + [\varepsilon^{1/p}] + \dots + [\varepsilon^{(p-1)/p}]$$

is distinguished, because if $\eta = \sum [c_n]p^n$, consider reducing to the residue field: $W(\mathcal{O}_{\mathbb{C}^\flat_K}) \to W(\mathcal{O}_{\mathbb{C}^\flat_K}/t)$, then $\overline{\varepsilon} = 1$ by(0.1.6.10), and $\overline{\eta} = p$, thus $|c_0| < 1, |c_1 - 1| < 1$, so it is distinguished**??**, thus a uniformizer by(0.1.6.6).

For the last assertion, by the formal property of log, it suffices to show that if $a_i \to a \in \mathbb{Z}_p$, then $[\varepsilon^{a_i}] \to [\varepsilon^a] \in B_{\mathrm{dR}}$. Then it suffices to show that for $a \in \mathbb{Z}_p, |a|$ small,

$$|[\varepsilon^a] - 1| \to 0.$$

And this can be done with the topology given in(0.1.6.9)**?** □

Cor. (0.1.6.12). $\mathrm{gr}(B_{\mathrm{dR}}) \cong B_{\text{H-T}}$.

Cor. (0.1.6.13). B_{dR} is Gal_K-regular, but B^+_{dR} is not Gal_K-regular.

Proof: B_{dR} is Gal_K-regular because it is a field. B^+_{dR} is not Gal_K-regular because $\mathbb{Q}_p t_\varepsilon$ is stable under Gal_K-action but t_ε is not invertible in B^+_{dR}. □

Prop. (0.1.6.14) [Galois Actions]. Gal_K acts on $\mathcal{O}_{\mathbb{C}_K}/(p)$ thus acts on $\mathcal{O}_{\mathbb{C}^\flat_K}$ and on A_{inf}. Then Fontaine's functor $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_K}$ is Gal_K-equivariant, thus $\ker\theta$ is Gal_K-stable, so Gal_K-acts on B^+_{dR} and B_{dR}, and $B_{\mathrm{dR}} \to \mathbb{C}_K$ is Gal_K-equivariant.

Prop. (0.1.6.15). There is a canonical lifting of $\overline{K} \to \mathbb{C}_K$ along $B^+_{\mathrm{dR}} \to \mathbb{C}_K$, and it is Gal_K-equivariant.

However, this embedding is not continuous, thus there is no embedding $\mathbb{C}_K \subset B^+_{\mathrm{dR}}$.

Proof: $K_0 = W(k)[\frac{1}{p}] \subset W(\mathcal{O}^\flat_{\mathbb{C}_K})[\frac{1}{p}] = B_{\mathrm{inf}} \subset B_{\mathrm{dR}}$, and it follows from Hensel's lemma that any element in $\overline{K}$ lifts uniquely to an element of B^+_{dR}, so $\overline{K} \subset B^+_{\mathrm{dR}}$, and is Gal_K-invariant, by uniqueness and the fact $B_{\mathrm{dR}} \to \mathbb{C}_K$ is Gal_K-equivariant(0.1.6.14).

For the last assertion, if the embedding is continuous, the $B^+_{\mathrm{dR}} \to \mathbb{C}_K$ has a section, and the filtration splits so $B_{\mathrm{dR}} \cong B_{\text{H-T}}$. □

Prop. (0.1.6.16). $K = B_{\mathrm{dR}}^{\mathrm{Gal}_K} = (B_{\mathrm{dR}}^+)^{\mathrm{Gal}_K}$

Proof: Firstly $K \subset B_{\mathrm{dR}}$ and is invariant under Gal_K by(0.1.6.15). On the other hand, the exact sequence

$$0 \to \mathrm{Fil}^{m+1} B_{\mathrm{dR}} \to \mathrm{Fil}^m B_{\mathrm{dR}} \to \mathbb{C}_p(m) \to 0\textbf{??}.$$

induces an injection

$$B_{\mathrm{dR}}^{\mathrm{Gal}_K} \cap \mathrm{Fil}^m B_{\mathrm{dR}} / B_{\mathrm{dR}}^{\mathrm{Gal}_K} \cap \mathrm{Fil}^{m+1} B_{\mathrm{dR}} \hookrightarrow \mathbb{C}_p(m)^{\mathrm{Gal}_K}.$$

Thus $B_{\mathrm{dR}}^{\mathrm{Gal}_K} = B_{\mathrm{dR}}^{\mathrm{Gal}_K} = K$. □

Def. (0.1.6.17) [deRham Representations]. Situation as in(0.1.6.16), for $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{fd}}(\mathrm{Gal}_K)$, V is called a **deRham representation** iff V is B_{dR}-admissible, or equivalently

$$\dim_K (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}_K} = \dim_{\mathbb{Q}_p} V.$$

The category of deRham representations of Gal_K are denoted by $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\mathrm{Gal}_K)$.

7 deRham Representations

Prop. (0.1.7.1)[$\mathbb{C}_K$-admissible Representations are deRham]. $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathbb{C}_K\text{-adm}}(\mathrm{Gal}_K) \subset \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\mathrm{Gal}_K)$.

Proof: For $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathbb{C}_K}$, by(0.1.4.5), there exists a finite extension L/K s.t. V is $L\widehat{K^{\mathrm{ur}}}$-admissible. Thus V is deRham as $L\widehat{K^{\mathrm{ur}}} \subset B_{\mathrm{dR}}$**??**. □

Prop. (0.1.7.2) [Potentially deRham are deRham]. Let $K' \subset \mathbb{C}_K$ be another p-adic field, then

$$\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_K) \cap \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\mathrm{Gal}_{K'}) = \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\mathrm{Gal}_K).$$

In particular, being deRham is not sensible to ramifications, which is a bad feature compared to being crystalline or semistable.

Proof: Because $\widehat{K^{\mathrm{ur}}} \subset \widehat{(K')^{\mathrm{ur}}}$ is of finite degree, it suffices to prove for two cases: K'/K is finite or $K' = \widehat{K^{\mathrm{ur}}}$. But the finite case follows from Galois descent the same as(0.1.5.4). The second case follows from [Conrad, P80]**?** □

Prop. (0.1.7.3) [Filtered D_{dR}]. For $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{fd}}(\mathrm{Gal}_K)$, there is a finite filtration Fil on $D_{\mathrm{dR}}(V)$ s.t.

$$\mathrm{Fil}^m D_{\mathrm{dR}}(V) = (t^m B_{\mathrm{dR}} \otimes_E V)^{\mathrm{Gal}_K} \subset D_{\mathrm{dR}}(V).$$

Prop. (0.1.7.4) [deRham Representations are Hodge-Tate]. For $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{fd}}(\mathrm{Gal}_K)$,

- there is an injection of graded vector spaces

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow D_{\text{H-T}}(V),$$

- If $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\mathrm{Gal}_K)$, the map in item1 is an isomorphism, and V is Hodge-Tate.
- If $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\mathrm{Gal}_K)$,

$$B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \cong B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$$

identifies filtrations.

Proof: 1: Consider the exact sequences

$$0 \to \mathrm{Fil}^{m+1} B_{\mathrm{dR}} \to \mathrm{Fil}^m B_{\mathrm{dR}} \to \mathbb{C}_p(m) \to 0.$$

Tensoring V and taking Gal_K-invariants give injections

$$h : \mathrm{gr}^m(D_{\mathrm{dR}}(V)) \hookrightarrow V(m)^{\mathrm{Gal}_K},$$

giving the injection $\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow D_{\text{H-T}}(V)$.

2: If $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$, this is an isomorphism by dimension reason, and V is Hodge-Tate by dimension reason.

3: Firstly notice $\mathrm{Fil}^m(B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V)) \subset \mathrm{Fil}^m(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ is trivial, thus it suffices to show that the induced map

$$f : \mathrm{gr}(B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V)) \to \mathrm{gr}(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) = B_{\text{H-T}} \otimes_{\mathbb{Q}_p} V$$

is an isomorphism. But notice

$$B_{\text{H-T}} \otimes \mathrm{gr}(D_{\mathrm{dR}}(V)) \xrightarrow{g} \mathrm{gr}(B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V)) \xrightarrow{f} B_{\text{H-T}} \otimes_{\mathbb{Q}_p} V$$

equals $B_{\text{H-T}} \otimes h$, so g is an isomorphism because it is surjective, and thus f is also an isomorphism. □

Cor. (0.1.7.5). 1-dimensional Hodge-Tate representations are deRham.

Proof: This is because if $V \cong \mathbb{Q}_p(\psi)$ where ψ is a character of Gal_K, and $\mathbb{C}_p \otimes_E V \cong \mathbb{C}_p(m)$, then by Sen-Tate(0.1.4.6), $\psi(-m)$ is potentially unramified, thus $\mathbb{C}_p$-admissible by(0.1.4.5), and thus deRham(0.1.7.1). □

Remark (0.1.7.6) [D_{dR} Insensitive to Ramifications]. D_{dR} is far from fully faithful. In fact, any unramified representation V is deRham by(0.1.7.2), and $D_{\mathrm{dR}}(V)$ is a simple filtration with graded ring $K^d[0]$, but V can be different from trivial representation.

Prop. (0.1.7.7). The functor $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\mathrm{Gal}_K) \to \mathrm{Fil}\,\mathcal{V}\mathrm{ect}_K$ is exact, and commutes with taking tensor products and duals.

Proof: For an exact sequence $0 \to V_1 \to V \to V_2 \to 0 \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$, there are exact sequences

$$0 \to \mathrm{Fil}^m D_{\mathrm{dR}}(V_1) \to \mathrm{Fil}^m D_{\mathrm{dR}}(V) \to \mathrm{Fil}^m D_{\mathrm{dR}}(V_2),$$

Then we can use the fact V_i are Hodge-Tate(0.1.7.4) to show that these sequences are also exact on the right.

For tensor products, notice it suffices to prove for the graded ring, but then it reduces to show that the graded structure on $D_{\text{H-T}}(V)$ is compatible with tensor products. And this is clear.

Taking dual follows from the fact the perfect pairing $D_{\mathrm{dR}}(V) \otimes D_{\mathrm{dR}}(V) \to K$ preserves filtrations. □

Prop. (0.1.7.8) [Extensions of deRham Representations]. If $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence in $\mathrm{Rep}^{\mathrm{fd}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$ s.t. V_1, V_2 are deRham, and the Hodge-Tate weights of V_1 are strictly larger than that of V_2, then V is deRham.

In particular, any upper-triangular representation with diagonal $(\mathbb{Q}_p(a_1), \mathbb{Q}_p(a_2), \ldots, \mathbb{Q}_p(a_n))$ with $a_1 > a_2 \ldots > a_n$ is deRham.

Proof: By twisting, we may assume that all Hodge-Tate weights of V_1 are positive and Hodge-Tate weights of V_2 are non-positive. There is an exact sequence

$$0 \to D_{\mathrm{dR}}(V_1) \to D_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(V_2) = \mathrm{Fil}^0 D_{\mathrm{dR}}(V_2)$$

so it suffices to show that $\mathrm{Fil}^0 D_{\mathrm{dR}}(V) \to \mathrm{Fil}^0 D_{\mathrm{dR}}(V_2)$ is surjective. But it follows from**??** that there is an exact sequence

$$\mathrm{Fil}^0 D_{\mathrm{dR}}(V) \to \mathrm{Fil}^0 D_{\mathrm{dR}}(V_2) \to H^1_{\mathrm{conts}}(\mathrm{Gal}_K, B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_1).$$

So it suffices to show that $H^1(\mathrm{Gal}_K, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_1) = 0$. The exact sequence

$$0 \to t_\varepsilon^{m+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_1 \to t_\varepsilon^m B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_1 \to \mathbb{C}_p(m) \otimes_{\mathbb{Q}_p} V_1 \to 0$$

induces a surjection $H^1_{\mathrm{conts}}(\mathrm{Gal}_K, t_\varepsilon^{m+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_1) \twoheadrightarrow H^1_{\mathrm{conts}}(\mathrm{Gal}_K, t_\varepsilon^m B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_1)$ by hypothesis. Notice B^+_{dR} is t_ε-complete, so we can use approximation technique similar to**??** to show that $H^1_{\mathrm{conts}}(\mathrm{Gal}_K, t_\varepsilon^m B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_1) = 0$. □

Remark (0.1.7.9). For an example of $V \in \mathrm{Rep}^{\text{H-T}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$ that is not deRham, Cf.[Conrad, P78].

Def. (0.1.7.10)[Hodge-Tate Weights]. For $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$, V is said to have Hodge-Tate weights $-i$ with multiplicity d_i if $\dim \mathrm{gr}^{-i}(D_{\mathrm{dR}}(V)) = d_i$.

Then $\mathbb{Q}_p(n)$ has a single Hodge-Tate weight n.

Geometric Interpretations

Prop. (0.1.7.11) [deRham Comparison for Étale Cohomologies]. If $X \in \mathcal{S}\mathrm{ch}^{\mathrm{sm},\mathit{proper}}/K$, then for any $r \in \mathbb{N}$, $\mathrm{H}^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$, and

$$H^r_{\mathrm{dR}}(X) \cong D_{B_{\mathrm{dR}}}(\mathrm{H}^r_{\text{ét}}(X_{\overline{K}}; \mathbb{Q}_p)), \quad H^{n-p}(X; \Omega^p_X) \cong \mathrm{gr}^p H^r_{\mathrm{dR}}(X).$$

This shows we can recover the de Rham cohomology of X from the étale cohomology, and the Hodge-Tate weights of $\mathrm{H}^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ lies in in $[-r, 0]$.

Conj. (0.1.7.12) [Fontaine-Mazur]. Let $F \in \texttt{NField}$ and $(\rho, V) \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_F)$ that satisfies:

- For a.e. $v \in \Sigma^{\mathrm{fin}}_F$, ρ_v is unramified.
- For any $v \in S(p)$, the representation ρ_v is deRham.

Then V appears as a subquotient of some $\mathrm{H}^r_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(m)$ where $X \in \mathcal{V}\mathrm{ar}^{\mathrm{sm},\mathit{proper}}/F$ and $m \in \mathbb{Z}$.

Proof: Emerton and Kisin proved some of the two-dimensional cases, Cf.[The Fontaine-Mazur conjecture for GL(2), Kisin], [Emerton, Local-Global Compatibility in the p-adic Langlands Programme for $\mathrm{GL}(2)_{\mathbb{Q}}$]**?**. □

Remark (0.1.7.13). It follows from proper base change**??** and(0.1.1.1) that any such cohomology group satisfies the requirement.

This conjecture is very strong, for example, the étale cohomology of smooth proper varieties are known to satisfy many good properties, like Weil conjecture, and Fontaine-Mazur conjecture implies that those properties can be derived via linear algebra data.

The local version of this conjecture is known to be false.

8 (φ, Γ)-Modules

Main References are [Fontaine90: Représentations p-adiques des corps locaux],[Fontaine94a: Le corps des périodes p-adiques] and [Fonatine94b: Repésentations p-adiques semi-stables]. [Foundations of Theory of (φ, Γ)-modules over the Robba Ring], [Berger, Galois representations and (φ, Γ)-modules], [Fontain-OuYang, p-adic Galois Representations].

Def. (0.1.8.1) [**φ-module**]. Let M be a A-module and $\sigma : A \to A$ is a ring map. Then an additive map $\varphi : M \to M$ is called **σ-semi-linear** iff $\varphi(am) = \sigma(a)\varphi(m)$ for $a \in A$. A **φ-module** over (A, σ) is an A-module M with a σ-semi-linear φ. The category of φ-modules over A is denoted by $\varphi\text{-}\mathfrak{Mod}_A$.

Giving a A-module M and a $\varphi : M \to M$, there is a map $\Phi : A \otimes_{\sigma,A} M = \sigma_* M \to M : \lambda \otimes m \to \lambda\varphi(m)$, which is a A-module map iff φ is σ-semi-linear.

If we define a ring $A_\sigma[\varphi]$ as the free group $A[X]$ modulo the relation $Xa = \sigma(a)X$ and ring relations in A, then it is a ring. Then a φ-module over (A, σ) is equivalent to a left $A_\sigma[\varphi]$-module.

Thus $\varphi\text{-}\mathfrak{Mod}_A$ is a Grothendieck Abelian category with tensor products, and moreover, the kernel as $A_\sigma[\varphi]$-module is the same as the kernel as a A-module.

Def. (0.1.8.2). If there is a map $\alpha : (A_1, \sigma_1) \to (A_2, \sigma_2)$ that commutes with σ_i, then we have a **pullback** from $\Phi\mathcal{M}_1$ to $\Phi\mathcal{M}_2$: $\alpha^*(M) = (A_2)_{\sigma_2}[\varphi] \otimes_{(A_1)_{\sigma_1}[\varphi]} M$(0.1.8.1).

Def. (0.1.8.3) [**Étale φ-Modules**]. If A is Noetherian, then a φ-module M is called **étale** iff it is f.g and the corresponding $\Phi : \sigma_* M \to M$ in(0.1.8.1) is a bijection. The subcategory of étale φ-modules is denoted by $\varphi\text{-}\mathfrak{Mod}^{\text{ét}}$.

In case when σ is a bijection, Φ is a bijection iff φ is a bijection.

Proof: Note that in this case $\sigma_* M \to M$ is a bijection by $\lambda \otimes m \to \sigma^{-1}(\lambda)m$, so the rest is easy. □

Def. (0.1.8.4) [**Dual étale φ-Modules**]. Cf.[Fontaine-Ouyang, P26].**?**

Prop. (0.1.8.5). If A is Noetherian and A_σ is flat, then $\varphi\text{-}\mathfrak{Mod}^{\text{ét}}$ is Abelian category with tensor products.

Proof: 0 is the zero object, the canonical sum&product are clearly étale . And we need to check the kernel and cokernel are étale . But we have an exact sequence $0 \to \ker \to M \to N \to \text{Coker} \to 0$ so we tensor with A_σ to get a morphism of sequences that $\sigma_* M \to M, \sigma_* N \to N$ are both bijective, so by 5-lemma, it is bijection at kernel and cokernel, so they are étale . □

Prop. (0.1.8.6) [**$\mathbb{F}_p$-Representations are étale φ-Modules**]. Let $E \in \texttt{Field}$, $\operatorname{char} E = p$, then for any $V \in \text{Rep}^{\text{fd}}_{\mathbb{F}_p}(\text{Gal}_E)$, V is E^{sep}-admissible, and

$$D_{E^{\text{sep}}}(V) = (E^{\text{sep}} \otimes V)^{\text{Gal}_E}$$

has a φ-action, and it is an étale φ-module.

Proof: V is E^{sep}-admissible by**??**. To show it is étale , it suffices to show that $\varphi : D_{E^{\text{sep}}}(V) \to D_{E^{\text{sep}}}(V)$ is bijective. Let $e_1, \ldots, e_n$ be a basis of $D_{E^{\text{sep}}}(V)$, and $v_1, \ldots, v_n$ be a basis of V, then $\mathtt{e} = \mathtt{v}B$ for some matrix $B \in \text{GL}(n; E^{\text{sep}})$. Then if $[\varphi]\mathtt{e} = A\mathtt{e}$ for $A \in \text{Mat}(n; E)$, then $A = B^{-1}\varphi(B)$, and $\det(A) = \det(B)^{p-1} \neq 0$, so φ is bijective. □

Galois Representations and Étale φ-Modules

Notations (0.1.8.7). Let $E \in \texttt{Field}^p$, denoted $\mathcal{O}_\mathcal{E} = \mathrm{Coh}(E)$**??**, and $\mathcal{E} = \mathrm{Frac}(\mathcal{O}_\mathcal{E}) = \mathcal{O}_\mathcal{E}[\frac{1}{p}]$. $\mathcal{E}$ has a natural Frobenius.

Prop. (0.1.8.8). By the functoriality of Cohen rings**?**, if $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} = \mathfrak{Coh}(\overline{E})\mathcal{E}^{\mathrm{ur}} = \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}[\frac{1}{p}]$, then there is a bijection $\mathrm{Gal}(\mathcal{E}^{\mathrm{ur}}/\mathcal{E}) \cong \mathrm{Gal}_E$. Thus there are Gal_K-action and φ-action on $\mathcal{O}_{E^{\mathrm{ur}}}, \mathcal{E}_{\mathrm{ur}}$ and by continuity extends to actions on $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}, \widehat{\mathcal{E}_{\mathrm{ur}}}$, and

$$(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{\mathrm{Gal}_E} = \mathcal{E}, \quad (\mathcal{E}^{\mathrm{ur}})^{\mathrm{Gal}_E} = \mathcal{O}_E, (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{\varphi=\mathrm{id}} = \mathbb{Q}_p, \quad (\mathcal{E}^{\mathrm{ur}})^{\varphi=\mathrm{id}} = \mathbb{Z}_p.$$

Proof: **?** □

Prop. (0.1.8.9). For $M \in \varphi\,\mathfrak{Mod}^{\mathrm{ft}}(\mathcal{O}_\mathcal{E})$, M is étale over $\mathcal{O}_E$ iff $M/(p)$ is étale over E.

Proof: □

Def. (0.1.8.10) [Effective φ-Modules]. An **effective φ-module** over $\mathcal{E}$ is a φ-module $(D, \varphi) \in \varphi\text{-}\mathfrak{Mod}_\mathcal{E}$ s.t. there is a complete $\mathcal{O}_E$-lattice M of D that $\varphi(M) \subset M$.

Def. (0.1.8.11) [Stably-Étale φ-Modules]. A **stably-étale φ-module** over $\mathcal{E}$ is a φ-module over $\mathcal{E}$ s.t. there exists a φ-stable $\mathcal{O}_\mathcal{E}$-lattice in $\mathcal{E}$ that is an étale φ-module over $\mathcal{O}_E$. Then the category of stably-étale φ-modules is a Tannakian category, denoted by $\varphi\text{-}\mathfrak{Mod}^{\mathrm{st\,.\,\acute{e}t}}(\mathcal{O}_\mathcal{E})$.

Proof: **?** □

Prop. (0.1.8.12). Any $V \in \mathrm{Rep}_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}(\mathrm{Gal}_E)$ is trivial.

Proof: Cf.[Fontaine-Ouyang]P34.**?** □

Thm. (0.1.8.13) [Classification of $\mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_E)$]. For $V \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_E)$,

$$\mathbb{M}(V) = (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} V)^{\mathrm{Gal}_E}$$

is an étale φ-module over $\mathcal{O}_\mathcal{E}$, and for any $M \in \varphi\text{-}\mathfrak{Mod}^{\mathrm{\acute{e}t}}(\mathcal{O}_\mathcal{E})$,

$$\mathbb{V}(M) = (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} M)^{\varphi=\mathrm{id}}$$

is a $\mathbb{Z}_p$-representation of Gal_E. And these two functors define an equivalence of categories:

$$\mathbb{M} : \mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_E) \cong \varphi\text{-}\mathfrak{Mod}^{\mathrm{\acute{e}t}}(\mathcal{O}_\mathcal{E}) : \mathbb{V}.$$

Proof: To show $\mathbb{M}(V)$ is étale , Cf.[Fontaine]P35.

By(0.1.8.14), we have an isomorphism

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \cong \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} M,$$

and by(0.1.8.12),

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathbb{M}(V) \cong \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} V.$$

Thus

$$\mathbb{V}(\mathbb{M}(V)) = (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathbb{M}(V))^{\varphi=\mathrm{id}} \cong (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} V)^{\varphi=\mathrm{id}} = V(0.1.8.8),$$

$$\mathbb{M}(\mathbb{V}(M)) = (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M))^{\mathrm{Gal}_E} \cong (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_\mathcal{E}} M)^{\mathrm{Gal}_E} = M(0.1.8.8).$$

□

Lemma (0.1.8.14). Situation as in(0.1.8.13),

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} V(M) \cong \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M.$$

Proof: Cf.[Fontaine-Ouyang]P36.**?** □

Thm. (0.1.8.15) [Classification of $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_E)$].

- For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_E)$,
$$\mathbb{D}(V) = (\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}_E}$$
is an stably-étale φ-module over $\mathcal{E}$, and there is a natural isomorphism
$$\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathcal{E}} \mathbb{D}(V) \cong \widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} V.$$
- for any $D \in \varphi\text{-}\mathcal{M}\mathrm{od}^{sst.\,\text{ét}}(\mathcal{E})$,
$$\mathbb{V}(D) = (\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathcal{E}} D)^{\varphi=\mathrm{id}}$$
is a $\mathbb{Q}_p$-representation of Gal_E, and there is a natural isomorphism
$$\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} \mathbb{V}(D) \cong \mathcal{E}^{\mathrm{ur}} \otimes_{\mathcal{E}} D,$$
- These two functors define an equivalence of Tannakian categories:
$$\mathbb{M} : \mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_E) \cong \varphi\text{-}\mathcal{M}\mathrm{od}^{sst.\,\text{ét}}(\mathcal{E}) : \mathbb{D}.$$

Proof: Cf.[Fontaine-Ouyang]P37.**?** □

Cor. (0.1.8.16). Isomorphism classes of d-dimensional $\mathbb{Q}_p$-representations of Gal_E are in bijection with the isomorphism classes of matrixes $\mathrm{GL}(d; \mathcal{O}_{\mathcal{E}})$ where

$$A \sim B \iff \exists P \in \mathrm{GL}(d; \mathcal{E}), B = P^{-1} A \varphi(P).$$

<u>(φ, Γ)**-Modules**</u>

Def. (0.1.8.17) [(φ, Γ)-modules]. If A is a topological ring with a Frobenius φ, and A has an action of a topological group Γ that commutes with σ, then a (φ, Γ)**-module** M is a φ-module M over A with a semi-linear action of Γ that commutes with φ.

If A is complete and φ is flat, then an **étale** (φ, Γ)**-module** M is a (φ, Γ)-module that the φ-module structure is étale (0.1.8.3).

Similar to φ-modules, (φ, Γ)-modules forms a Grothendieck Abelian category with tensor products.**?**

Thm. (0.1.8.18) [Classification of $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$].