# Substitution

· Review:

· Rings and fields: $(R, +, \cdot)$ and in a field $F$ every nonzero $x \in F$ has a multiplicative inverse.

· Some equations don't have solutions in a given ring but do in a larger ring.

E.g $x^2+1$ has no solutions in $\mathbb{R}$ or $\mathbb{Z}/3\mathbb{Z}$ but does in $\mathbb{C}$ and $\mathbb{Z}[i]/3\mathbb{Z}[i]$.

· Definition:

A ring $R$ is algebraically closed if for any polynomial $p(x) = \sum_{i=0}^{d} a_i x^i$ with $a_i \in R$ and $d > 0$, the equation $p(x) = 0$ has a solution in $R$

· Remarks

1) We saw above that $\mathbb{R}$ and $\mathbb{Z}/3\mathbb{Z}$ are not alg. closed.

2) We will see below that $\mathbb{C}$ is algebraically closed ("the fundamental theorem of algebra")

3) $R$ algebraically $\Rightarrow R$ field Pf: $ax-1=0$ has a solution $\forall a \in R \Rightarrow a$ has a mult inverse

The converse isn't true, as mentioned in 1) ($\mathbb{R}$ and $\mathbb{Z}/3\mathbb{Z}$).

In fact $R = \mathbb{Z}[i]/3\mathbb{Z}[i]$ is not algebraically closed either because $x^3-x-1=0$ has no solutions in $R$

· Assume for a contradiction that $x = a+bi$ is a solution to $x^3-x-1=0$, with $a, b \in \mathbb{Z}$

· $x^3 = (a+bi)^3 = a^3 + 3a^2bi + 3a(bi)^2 + (bi)^3 \equiv a^3 - b^3 i \mod 3$

Therefore $0 = x^3 - x - 1 \equiv (a^3-a-1) - (b^3+b)i \mod 3$ and so $a^3-a-1 \equiv 0 \mod 3$ for $a \in \mathbb{Z}$ $\lightning$

In the remainder of the lecture we outline the proof of the fundamental thm of algebra

· Theorem

The field $\mathbb{C}$ is algebraically closed.

We outline the ingredients of the proof.

· Definition:

A function $f: \Omega \to \mathbb{C}$ ($\Omega$ open $\subset \mathbb{C}$) is meromorphic around $\alpha \in \Omega$ with order $\mathrm{ord}_\alpha(f) = N \in \mathbb{Z}$

$$f(z) = \sum_{k \geq N} c_k z^k \qquad \text{with } c_N \neq 0 \qquad \text{on some } B_r(\alpha) \setminus \{\alpha\}.$$

If $\mathrm{ord}_\alpha(f) \geq 0$, we say $f$ is holomorphic around $\alpha$. $f$ is holomorphic (meromorphic) if it is holomorphic (meromorphic) around every $\alpha \in \Omega$.

A pole of a meromorphic $f$ is a zero of $\frac{1}{f}$

· Example

1) $\frac{1}{z-3}$ is meromorphic around 3 with order -1. Also 3 is a pole of $\frac{1}{z-3}$

2) A polynomial $H(z) = \sum_{h=0}^{d} a_h z^h$ of degree $d$ is holomorphic around 0 of order $\mathrm{ord}_0(a_d z^d) = d$ (H).

3) In fact polynomials are holomorphic on every $\Omega \subset \mathbb{C}$: $\sum_{h=0}^{d} a_h z^d = \sum_{h=0}^{d} a_k (z-\alpha+\alpha)^k$ = expand ....

We now require three facts from $\mathbb{C}$-analysis.

· Fact 1: It is possible to define $\mathbb{C}$-integration of $f: \Omega \to \mathbb{C}$ continuous along a smooth path $\gamma: [0,1] \to \mathbb{C}$. $\mathrm{Int}(\gamma) \subset \Omega$

$$\int_\gamma f(z)\,dz := \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$$

"follows from Cauchy's thm" (residue thm) Fact 2: If $h: \Omega \to \mathbb{C}$ is holomorphic and $\frac{h'}{h}$ has neither zero nor poles along $\gamma$, $\mathrm{Int}(\gamma) \subset \Omega$, encircles the zeroes of $H$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{h'}{h}\,dz = \sum_{\alpha \in \{\text{zeroes of } h \text{ in } \Omega\}} \mathrm{ord}_\alpha(h).$$

$\mathrm{Int}(\gamma) \subset \Omega$

"Rouché's principle" Fact 3: If $H: \Omega \to \mathbb{C}$ can be written as $H(z) = h(z) + \varepsilon(z)$ with $h, \varepsilon$ holom on $\Omega$ and $|h(z)| > |\varepsilon(z)|$ along $\gamma$ (also $\frac{h'}{h}, \frac{H'}{H}$ non zero/pole on $\gamma$).

then $\frac{1}{2\pi i} \int_\gamma \frac{H'}{H}\,dz = \frac{1}{2\pi i} \int_\gamma \frac{h'}{h}\,dz.$

We now "prove" the fundamental thm of algebra.

proof:

Given a degree $d > 0$ polynomial $H(z) = \sum_{i=0}^{d} a_i z^i$, we will show that $\sum_{\text{zeroes } \alpha \text{ of } H \text{ in } \mathbb{C}} \mathrm{ord}_\alpha(H) = d$

Since $d > 0$, this implies that $H$ must have a zero (otherwise the LHS sum couldn't be $> 0$).

· Now we prove it:

· Write $H(z) = a_d z^d + \sum_{i=0}^{d-1} a_i z^i =: h(z) + \varepsilon(z)$

· Pick $\gamma$ a circle around zero large enough so that $|h(z)| > |\varepsilon(z)|$ along $\gamma$ + containing all zeroes of $H$

(possible because $x^d$ grows quicker than degrees $< d$).

· We then have $\sum_{\alpha \in \{\text{zeroes of } H\}} \mathrm{ord}_\alpha(H) = \frac{1}{2\pi i} \int_\gamma \frac{H'}{H}\,dz$ Fact 2:

$= \frac{1}{2\pi i} \int_\gamma \frac{h'}{h}\,dz$ Fact 3

$= \mathrm{ord}_0(h)$ Fact 2: 0 only zero of $h(z) = a_d z^d$

$= d$

□

If time permits: Cauchy's residue thm says

"If $f: \Omega \to \mathbb{C}$ is meromorphic on $\Omega$ and has neither zeros nor poles along $\gamma$, encircling all poles, then $\frac{1}{2\pi i} \int_\gamma f\,dz = \sum_{\alpha \in \{\text{poles of } f\}} \overbrace{\mathrm{res}_\alpha(f)}^{\in \mathbb{C}}$"

Here $\overset{\in \mathbb{C}}{\boxed{\mathrm{res}_\alpha(f)}}$ is defined as $c_{-1}$ in $f(z) = \sum_{h \geq N} c_h (z-h)^h$ so e.g. $\mathrm{res}_\alpha(f) = 0$ if holo.

Fact 2 follows by applying this to $f = \frac{h'}{h}$: $\{\text{poles of } f\} = \{\text{zeroes of } h\} \ni \alpha$ satisfies $\mathrm{res}_\alpha(f) = \mathrm{ord}_\alpha(h)$.