18.708 SPRING 2022

LECTURES ON OPERATOR ALGEBRAS

MINH-TÂM TRINH

Instructions. You may consult books, papers, and websites as long as you cite them and write up your solutions in your own words. Do not request answers on forums online. The problem set is due **Thursday, April 28**. **Update (April 26):** Corrections have been marked in red.

1. Hilbert Spaces

Let $\mathbf{H}$ be a Hilbert space. We always write $\mathscr{B}(\mathbf{H})$ to denote the $\mathbf{C}$-algebra of bounded operators on $\mathbf{H}$.

Problem 1. Let $a \in \mathscr{B}(\mathbf{H})$. Show that:

(1) If a is a projection, then a is positive.

(2) If a is positive, then a is self-adjoint. *Hint:* It suffices to check $\langle av, w\rangle = \langle v, aw\rangle$ for all v, w. Write

$$4\langle av, w\rangle = \lambda_{v+w}(a) - i\lambda_{v+iw}(a) - \lambda_{v-w}(a) + i\lambda_{v-iw}(a),$$

where $\lambda_v(a) = \langle av, v\rangle$.

(3) a is positive if and only if $a = b^*b$ for some $b \in \mathscr{B}(\mathbf{H})$. *Hint:* If a is positive, then $\mathrm{Spec}(a) \subseteq \mathbf{R}_{\geq 0}$. Now (2) shows that a has a positive square root.

Problem 2. Suppose that $\mathbf{H} = L^2([0,1])$ under the inner product

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx.$$

Let $\mathbf{H}_0 \subseteq \mathbf{H}$ be the subspace of piecewise-continuous functions f such that $f' \in \mathbf{H}$. There are linear maps $X, P : \mathbf{H}_0 \to \mathbf{H}$ defined by

$$[Xf](x) = xf(x) \quad \text{and} \quad [Pf](x) = f'(x).$$

The operators X and P may be interpreted as the position and momentum of a particle on the interval.

(1) Show that the map $\mathscr{F} : \mathbf{H} \to \ell^2(\mathbf{Z})$ given by

$$[\mathscr{F}f](n) = \int_0^1 f(x)e^{-2\pi i n x}\,dx$$

is well-defined, and by calculating $\mathscr{F}^{-1}$ explicitly on a Hilbert basis, show that $\mathscr{F}$ is an isometry.

(2) Show that $\mathscr{F}X\mathscr{F}^{-1}$ and $\mathscr{F}P\mathscr{F}^{-1}$ extend to linear operators on $\ell^2(\mathbf{Z})$ by calculating them explicitly.

(3) Using (1) and (2), deduce that X and P extend to linear operators on $\mathbf{H}$ such that both are self-adjoint, but P is not bounded.

2. Von Neumann Algebras

Problem 3. Show that the strong operator topology on $\mathscr{B}(\mathbf{H})$ is coarser than the norm topology.

Suppose $\mathbf{H}$ is separable and infinite-dimensional. Let $M \subseteq \mathscr{B}(\mathbf{H})$ be a type II_1 factor. This means that $1 \in M$ is a finite projection, but M does not contain any minimal nonzero projections.

The *division algorithm* for M states that for any projections $e, f \in M$, there exist finitely many pairwise-orthogonal projections $f_1, \dots, f_k \in M$ all equivalent to f and a projection $r \prec f$ such that

$$e = f_1 + \cdots + f_k + r.$$

(The proof uses Zorn's lemma and the fact that $\prec$ is a total order.)

Problem 4. The goal of this problem is to construct the function $\bar{\tau}$ on projections in M that gives rise to the trace $\tau : M \to \mathbf{C}$.

(1) Show that for any projection $e \in M$, there is a projection $f \in M$ such that

$$f \leq e \quad \text{and} \quad f \sim e - f.$$

Hint: To do the $e = 1$ case, consider families $\{f_i, g_i\}_i$ of pairwise-orthogonal projections such that $f_i \sim g_i$ for all i. By Zorn's lemma, there is a maximal such family. Set $f = \sum_i f_i$ and $g = \sum_i g_i$, so that $f \perp g$ and $f \sim g$. We claim that $g = 1 - f$.

Indeed, if $h = 1 - f - g$ is nonzero, then by Corollary 4.1.2 from Jones's 2015 notes, we can find nonzero projections $f_0, g_0 \in hMh$ such that $f_0 \perp g_0$ and $f_0 \sim g_0$, contradicting maximality of the original family.

(2) Deduce from (1) that there is a sequence of projections

$$e_1 \geq e_2 \geq \cdots \geq e_i \geq \cdots$$

in M such that $e_i \sim e_{i-1} - e_i$.

(3) By combining (2) with the division algorithm above, show that there is a unique norm-continuous map

$$\bar{\tau} : (\{\text{projections in } M\}/{\sim}, \preceq) \to ([0,1], \leq)$$

such that $\bar{\tau}(e + f) = \bar{\tau}(e) + \bar{\tau}(f)$ for all $e \perp f$.

3. Subfactors and Index

Let $N \subseteq M$ be a (type II_1) subfactor such that $[M : N] < \infty$. We write τ for the usual trace on M, as well as its restriction to N. Let $e_1 \in \mathscr{B}(L^2(M))$ be the orthogonal projection onto $L^2(N)$, and let

$$M_1 = (M + Me_1M)'' \subseteq \mathscr{B}(L^2(M)).$$

Umegaki's theorem states that left multiplication by e_1 defines a positive $\mathbf{C}$-linear map $M \to N$ such that $\tau(e_1 a) = \tau(a)$ for all $a \in M$.

Problem 5. We will verify that the trace τ_{M_1} satisfies what knot theorists call the "second Markov move". In what follows, let

$$\lambda = \tau_{M_1}(e_1).$$

(1) Using the uniqueness of traces on type II_1 factors, show that the $\mathbf{C}$-linear map on N defined by $b \mapsto \tau_{M_1}(be_1)$ is a rescaling of $\tau|_N$, namely, $\lambda\tau|_N$.
(2) Combine (1) with Umegaki's theorem above to deduce that

$$\tau_{M_1}(ae_1) = \lambda\tau(a)$$

for all $a \in M$. *Hint:* $e_1 = e_1^2$.

Remark. In lecture, we (should have) checked that $\lambda = \frac{1}{[M:N]}$.

Problem 6. Let the polynomials $P_n(q) \in \mathbf{Z}[q]$ be defined by

$$P_{i+1}(q) = P_i(q) - qP_{i-1}(q)$$

and $P_0(q) = P_1(q) = 1$. Show that

$$\frac{1}{4\cos^2(\frac{\pi}{n+2})} < \lambda < \frac{1}{4\cos^2(\frac{\pi}{n+1})} \implies \begin{cases} P_i(\lambda) > 0 \text{ for } i \leq n, \\ P_{n+1}(\lambda) < 0 \end{cases}$$

for all integers $n \geq 1$.

4. The Temperley–Lieb and Iwahori–Hecke Algebras

Recall that in our conventions, the Iwahori–Hecke algebra H_n is the $\mathbf{Z}[z]$-algebra generated by $\sigma_1, \ldots, \sigma_{n-1}$ subject to

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i & \text{for } |j-i| \geq 2, \\ \sigma_i^2 &= 1 + z\sigma_i. \end{aligned}$$

We view H_n as a quotient of the group algebra $\mathbf{Z}[z][Br_n]$. Ocneanu's traces $\tau_n : H_n \to \mathbf{Z}[z^{\pm 1}, a^{\pm 1}]$ are uniquely defined by the trace axioms and the identities:

$$\beta \in Br_{n-1} \implies \begin{cases} \tau_n(\beta) = \tau_{n-1}(\beta), \\ \tau_n(\sigma_{n-1}^{\pm 1}\beta) = -\dfrac{a^{\mp 1}z}{a - a^{-1}}\,\tau_{n-1}(\beta). \end{cases}$$

The (reduced) HOMFLY series of the link $\hat{\beta}$ is $\mathbf{P}(\hat{\beta}) = (-a)^{|\beta|}\left(\frac{a-a^{-1}}{z}\right)^{n-1}\tau_n^H(\beta)$ for all n and $\beta \in Br_n$, where $|\beta|$ is the length of β as a word in the σ_i.

Problem 7. Let $\alpha = \sigma_1^3 \in Br_2$ and $\beta = (\sigma_1\sigma_2)^2 \in Br_3$.

(1) Using pictures, show that α and β have isotopic link closures.
(2) Show that $\mathbf{P}(\hat{\alpha}) = \mathbf{P}(\hat{\beta})$ using Ocneanu's traces.
(3) Show that α and α^{-1} cannot have isotopic link closures.

Jones defines the Temperley–Lieb algebra TL_n to be the $\mathbf{Z}[\lambda]$-algebra generated by $e_1, \ldots, e_{n-1}$ subject to

$$\begin{aligned} e_i e_{i\pm 1} e_i &= \lambda e_i, \\ e_i e_j &= e_j e_i && \text{for } |j-i| \geq 2, \\ e_i^2 &= e_i. \end{aligned}$$

Henceforth we set $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ and $\lambda = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$.

Problem 8. Set $C_i' = \sigma_i + q^{-\frac{1}{2}}$. Show that:

(1) $(C_i')^2 = \lambda C_i'$.

(2) The map $H_n \to TL_n$ that sends $C_i' \mapsto \lambda e_i$ extends to a ring homomorphism. *Hint:* First, rewrite the defining relations of H_n in terms of the C_i'.