Meeting readers' needs

to help them

follow complicated logic

18.704 Spring 2022
Susan Ruff
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Newman, 1980

The American Mathematical Monthly

SIMPLE ANALYTIC PROOF OF THE PRIME NUMBER THEOREM

D. J. NEWMAN

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The magnificent prime number theorem has received much attention and many proofs throughout the past century. If we ignore the (beautiful) elementary proofs of Erdős **[1]** and Selberg **[6]** and focus on the analytical ones, we find that they all have some drawback. The original proofs **[7]** of Hadamard and de la Vallée Poussin were based, to be sure, on the nonvanishing of $\zeta(z)$ in Re $z \geq 1$, but they also required annoying estimates of $\zeta(z)$ at ∞, the reason being that formulas for coefficients of Dirichlet series involve integrals over *infinite* contours (unlike the situation for power series) and so effective evaluation requires estimates at ∞.

The more modern proofs, due to Wiener **[2]** and Ikehara **[8]** (see also Heins's book **[3]**) do get around the necessity of estimating at ∞ and are indeed based only on the appropriate nonvanishing of $\zeta(z)$, but they are tied to certain results on Fourier transforms.

We propose to return to contour integral methods so as to avoid Fourier analysis, and also to use finite contours so as to avoid estimates at ∞. Of course certain errors are introduced thereby, but the point is that these can be effectively estimated away by elementary arguments.

So let us begin with the well-known fact **[7]** about the ζ-function:

$$(z-1)\zeta(z) \text{ is analytic and zero free throughout Re } z \geq 1. \qquad (1)$$

This will be assumed throughout and will allow us to give our proof of the prime number theorem.

In fact we give two proofs. The first one is the shorter and simpler of the two, but we pay a price in that we obtain one of Landau's equivalent forms of the theorem rather than the standard form, $\pi(N) \sim N/\log N$. Our second proof is a more direct assault on $\pi(N)$ but is somewhat more intricate than the first. Here we find some of Tchebychev's elementary ideas very useful.

Basically our novelty consists in using a modified contour integral,

$$\int_\Gamma f(z) N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz,$$

rather than the classical one, $\int_C f(z) N^z z^{-1} dz$. The method is rather flexible, and we could use it to directly obtain $\pi(N)$ by choosing $f(z) = \log \zeta(z)$. We prefer, however, to derive both proofs from the following convergence theorem. Actually, this theorem dates back to Ingham **[9]**, but his proof is à la Fourier analysis and is much more complicated than the contour integral method we now give.

THEOREM. *Suppose $|a_n| \leq 1$ and form the series $\sum a_n n^{-z}$ which clearly converges to an analytic function $F(z)$ for* Re $z > 1$. *If, in fact, $F(z)$ is analytic throughout* Re $z \geq 1$, *then $\sum a_n n^{-z}$ converges throughout* Re $z \geq 1$.

Proof of the convergence theorem. Fix a w in Re $w \geq 1$. Thus $F(z+w)$ is analytic in Re $z \geq 0$. We choose an $R \geq 1$ and determine $\delta = \delta(R) > 0$, $\delta \leq \frac{1}{2}$ and an $M = M(R)$ so that

$$F(z+w) \text{ is analytic and bounded by } M \text{ in } -\delta \leq \text{Re } z, |z| \leq R. \qquad (2)$$

Now form the counterclockwise contour Γ, bounded by the arc $|z| = R$, Re $z > -\delta$, and the

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Korevaar, 1982

The Mathematical Intelligencer

On Newman's Quick Way to the Prime Number Theorem

J. Korevaar

1. Introduction and Overview

There are several interesting functions in number theory whose tables look quite irregular, but which exhibit surprising asymptotic regularity as $x \to \infty$. A notable example is the function $\pi(x)$ which counts the number of primes p not exceeding x.

1.1. The Famous Prime Number Theorem

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty, \qquad (1.1)$$

was surmised already by Legendre and Gauss. However, it took a hundred years before the first proofs appeared, one by Hadamard and one by de la Vallée Poussin (1896). Their and all but one of the subsequent proofs make heavy use of the Riemann zeta function. (The one exception is the long so-called elementary proof by Selberg [11] and Erdös [4].)

For Re $s > 1$ the zeta function is given by the Dirichlet series

D. J. Newman

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s}. \qquad (1.2a)$$

By the unique representation of positive integers n as products of prime powers, the series may be converted to the Euler product (cf. [5])

$$\zeta(s) = \left(1 + \frac{1}{p_1^s} + \frac{1}{p_1^{2s}} + \ldots\right)\left(1 + \frac{1}{p_2^s} + \frac{1}{p_2^{2s}} + \ldots\right)\ldots$$
$$= \prod_p \frac{1}{1 - p^{-s}}. \qquad (1.2b)$$

The above function element is analytic for Re $s > 1$ and can be continued across the line Re $s = 1$ (Fig. 1). More precisely, the difference

$$\zeta(s) - \frac{1}{s-1}$$

can be continued analytically to the half-plane Re $s > 0$ (cf. § B.1 in the box on p. 111) and in fact to all of $\mathbb{C}$. The essential property of $\zeta(s)$ in the proofs of the prime number theorem is its non-vanishing on the line Re $s = 1$

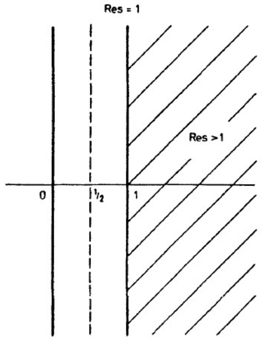

Figure 1

Zagier, 1997

The American Mathematical Monthly

Newman's Short Proof of the Prime Number Theorem

D. Zagier

Dedicated to the Prime Number Theorem on the occasion of its 100th birthday

The prime number theorem, that the number of primes $\leq x$ is asymptotic to $x/\log x$, was proved (independently) by Hadamard and de la Vallée Poussin in 1896. Their proof had two elements: showing that Riemann's zeta function $\zeta(s)$ has no zeros with $\Re(s) = 1$, and deducing the prime number theorem from this. An ingenious short proof of the first assertion was found soon afterwards by the same authors and by Mertens and is reproduced here, but the deduction of the prime number theorem continued to involve difficult analysis. A proof that was elementary in a technical sense—it avoided the use of complex analysis—was found in 1949 by Selberg and Erdös, but this proof is very intricate and much less clearly motivated than the analytic one. A few years ago, however, D. J. Newman found a very simple version of the Tauberian argument needed for an analytic proof of the prime number theorem. We describe the resulting proof, which has a beautifully simple structure and uses hardly anything beyond Cauchy's theorem.

Recall that the notation $f(x) \sim g(x)$ ("f and g are asymptotically equal") means that $\lim_{x\to\infty} f(x)/g(x) = 1$, and that $O(f)$ denotes a quantity bounded in absolute value by a fixed multiple of f. We denote by $\pi(x)$ the number of primes $\leq x$.

Prime Number Theorem. $\pi(x) \sim \dfrac{x}{\log x}$ as $x \to \infty$.

We present the argument in a series of steps. Specifically, we prove a sequence of properties of the three functions

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \quad \Phi(s) = \sum_p \frac{\log p}{p^s}, \quad \vartheta(x) = \sum_{p \leq x} \log p \quad (s \in \mathbb{C},\ x \in \mathbb{R});$$

we always use p to denote a prime. The series defining $\zeta(s)$ (the Riemann zeta-function) and $\Phi(s)$ are easily seen to be absolutely and locally uniformly convergent for $\Re(s) > 1$, so they define holomorphic functions in that domain.

(I). $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ *for* $\Re(s) > 1$.

Proof: From unique factorization and the absolute convergence of $\zeta(s)$ we have

$$\zeta(s) = \sum_{r_2, r_3, \cdots \geq 0} (2^{r_2} 3^{r_3} \cdots)^{-s} = \prod_p \left(\sum_{r \geq 0} p^{-rs}\right) = \prod_p \frac{1}{1 - p^{-s}} \quad (\Re(s) > 1).$$

(II). $\zeta(s) - \dfrac{1}{s-1}$ *extends holomorphically to* $\Re(s) > 0$.

For sections specified by email: What is the goal? How does it fit into the larger proof? How do you know?

What is the goal?
How does it fit into the larger proof?

Newman, 1980
The American Mathematical Monthly

Second Proof of the Prime Number Theorem. In this section we begin with Tchebychev's observation [5] that

$$\sum_{p<n} \frac{\log p}{p} - \log n \quad \text{is bounded,} \tag{12}$$

which he derives in a direct elementary way from the prime factorization of $n!$.

The point is that the prime number theorem is easily derived from

$$\sum_{p<n} \frac{\log p}{p} - \log n \quad \text{converges to a limit,} \tag{13}$$

by a simple summation by parts, which we leave to the reader. Nevertheless the transition from (12) to (13) is not a simple one and we turn to this now.

So form, for Re $z>1$, the function

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}\left(\sum_{p<n} \frac{\log p}{p}\right) = \sum_{p} \frac{\log p}{p}\left(\sum_{n>p} \frac{1}{n^z}\right).$$

Now

$$\sum_{n>p} \frac{1}{n^z} = \frac{1}{(z-1)p^{z-1}} + z\int_p^{\infty} \frac{1-\{t\}}{t^{z+1}}\,dt = \frac{p}{(z-1)}\left(\frac{1}{p^z-1} + A_p(z)\right)$$

where $A_p(z)$ is analytic for Re$z>0$ and is bounded by

$$\frac{1}{p^x(p^x-1)} + \frac{|z(z-1)|}{xp^{x+1}}.$$

Hence

$$f(z) = \frac{1}{z-1}\left(\sum_{p} \frac{\log p}{p^z-1} + A(z)\right),$$

where $A(z)$ is analytic for Re $z>\frac{1}{2}$ by the Weierstrass M-test.

By Euler's factorization formula, however, we recognize that

$$\sum_{p} \frac{\log p}{p^z-1} = \frac{-d}{dz}\log \zeta(z); \tag{14}$$

and so we deduce, by (1), that $f(z)$ is analytic in Re $z \leqslant 1$ except for a double pole with principal part $1/(z-1)^2 + c/(z-1)$, at $z=1$. Thus if we set

What is the goal?
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Korevaar, 1982
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First idea. We try to estimate $G(0) - G_\lambda(0)$ with the aid of Cauchy's formula. Thus we look for a suitable path of integration W around 0. The simplest choice would be a circle, but we can not go too far into the left half-plane because we know nothing about $G(z)$ there. So for given $R > 0$, the positively oriented path W will consist of an arc of the circle $|z| = R$ and a segment of the vertical line $\operatorname{Re} z = -\delta$ (Fig. 2). Here the number $\delta = \delta(R) > 0$ is chosen so small that $G(z)$ is analytic on and inside W. We denote the part of W in $\operatorname{Re} z > 0$ by W_+, the part in $\operatorname{Re} z < 0$ by W_-. By Cauchy's formula,

$$G(0) - G_\lambda(0) = \frac{1}{2\pi i} \int_W \{G(z) - G_\lambda(z)\} \frac{1}{z}\, dz. \qquad (2.2)$$

We have the following simple estimates:

for $x = \operatorname{Re} z > 0$,

$$|G(z) - G_\lambda(z)| = \left|\int_\lambda^\infty F(t) e^{-zt} dt\right| \leqslant \int_\lambda^\infty e^{-xt} dt = \frac{1}{x} e^{-\lambda x};$$

$$(2.3)$$

for $x = \operatorname{Re} z < 0$,

$$|G_\lambda(z)| = \left|\int_0^\lambda F(t) e^{-zt} dt\right| \leqslant \int_0^\lambda e^{-xt} dt < \frac{1}{|x|} e^{-\lambda x}. \quad (2.4)$$

$$\vartheta(x) \geq \sum_{x^{1-\epsilon} \leq p \leq x} \log p \geq \sum_{x^{1-\epsilon} \leq p \leq x} (1-\epsilon) \log x$$
$$= (1-\epsilon) \log x \left[\pi(x) + O(x^{1-\epsilon})\right].$$

What is the goal?
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Zagier, 1997
The American Mathematical Monthly

Proof of the Analytic Theorem. For $T > 0$ set $g_T(z) = \int_0^T f(t) e^{-zt}\, dt$. This is clearly holomorphic for all z. We must show that $\lim_{T\to\infty} g_T(0) = g(0)$.

Let R be large and let C be the boundary of the region $\{z \in \mathbb{C} \mid |z| \leq R, \Re(z) \geq -\delta\}$, where $\delta > 0$ is small enough (depending on R) so that $g(z)$ is holomorphic in and on C. Then

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C (g(z) - g_T(z)) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

by Cauchy's theorem. On the semicircle $C_+ = C \cap \{\Re(z) > 0\}$ the integrand is bounded by $2B/R^2$, where $B = \max_{t \geq 0} |f(t)|$, because

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t) e^{-zt}\, dt\right| \leq B \int_T^\infty |e^{-zt}|\, dt = \frac{Be^{-\Re(z)T}}{\Re(z)} \quad (\Re(z) > 0)$$

and

$$\left| e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z} \right| = e^{\Re(z)T} \cdot \frac{2\Re(z)}{R^2}.$$

Hence the contribution to $g(0) - g_T(0)$ from the integral over C_+ is bounded in absolute value by B/R. For the integral over $C_- = C \cap \{\Re(z) < 0\}$ we look at $g(z)$ and $g_T(z)$ separately. Since g_T is entire, the path of integration for the integral involving g_T can be replaced by the semicircle $C'_- = \{z \in \mathbb{C} \mid |z| = R, \Re(z) < 0\}$, and the integral over C'_- is then bounded in absolute value by $2\pi B/R$

Meet readers' needs with

GUIDING TEXT

- Ensure readers know
  - WHAT you're doing
  - WHY you're doing it
  - HOW you're doing it

- ~~Always tell readers~~
  - ~~WHAT you're doing~~
  - ~~WHY you're doing it~~
  - ~~HOW you're doing it~~

contours (unlike the situation for power series) and so effective evaluation requires estimates at ∞.

The more modern proofs, due to Wiener [2] and Ikehara [8] (see also Heins's book [3]) do get around the necessity of estimating at ∞ and are indeed based only on the appropriate nonvanishing of $\zeta(z)$, but they are tied to certain results on Fourier transforms.

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SENTENCE STRUCTURE

Which sentence would you expect to come next? 1 or 2?

"Some astonishing questions about the nature of the universe have been raised by scientists exploring the nature of black holes in space."

1. The collapse of a dead star into a point perhaps no larger than a marble creates a black hole.

2. A black hole is created by the collapse of a dead star into a point perhaps no larger than a marble.

"***Put in the topic position the old information that links backward***…"

Gopen & Swan, "The Science of Scientific Writing"

Meet readers’ needs with

SENTENCE STRUCTURE

Which sentence would you expect to come next? 1 or 2?

“I ate cookies yesterday.”

1. Yesterday it started to rain at 2pm.

2. The cookies were green.

“Put in the topic position the old information that links backward; ***put in the stress position the new information you want the reader to emphasize.****”*

Gopen & Swan, “The Science of Scientific Writing”

“Put in the topic position the old information that links backward; put in the stress position the new information you want the reader to emphasize.”

Known-to-new structure creates “flow.”

Here, **Blue = known**; **Red = new**

interact with thin neighborhoods of algebraic varieties.

To describe a broom, we recall the wave packet decomposition of Ef introduced by Bourgain (cite!). The wave packet decomposition says that inside a large ball of radius R, we can decompose Ef into a sum over wave packets $Ef_{\theta,v}$. Each wave packet $Ef_{\theta,v}$ is essentially supported in a tube $T_{\theta,v}$ of length R, radius $R^{1/2}$. The axis of $T_{\theta,v}$ points in a direction depending only on θ and the location of $T_{\theta,v}$ is described by v. The absolute value of a wave packet $|Ef_{\theta,v}|$ is approximately a constant function on $T_{\theta,v}$.

Revise to improve known-to-new flow.

The Catalan numbers enumerate many things. For example, the n^{th} Catalan number counts the number of Dyck paths of length $2n$. Full binary trees with n internal vertices are in bijection with Dyck paths of length $2n$. Therefore full binary trees are also enumerated by the Catalan numbers.

Sample revision:

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Look for and fill gaps in flow that may slow (or prohibit!) comprehension.

Plugging our initial values into (2) we see the expression simplifies to **2+2**. **Four** is less than…

Plugging our initial values into (2) we see the expression simplifies to **2+2**. **Perfect squares** have the property…

Plugging our initial values into (2) we see the expression simplifies to **2+2, which is a perfect square. Perfect squares** have the property…

Which gaps slow reading depends on the audience.

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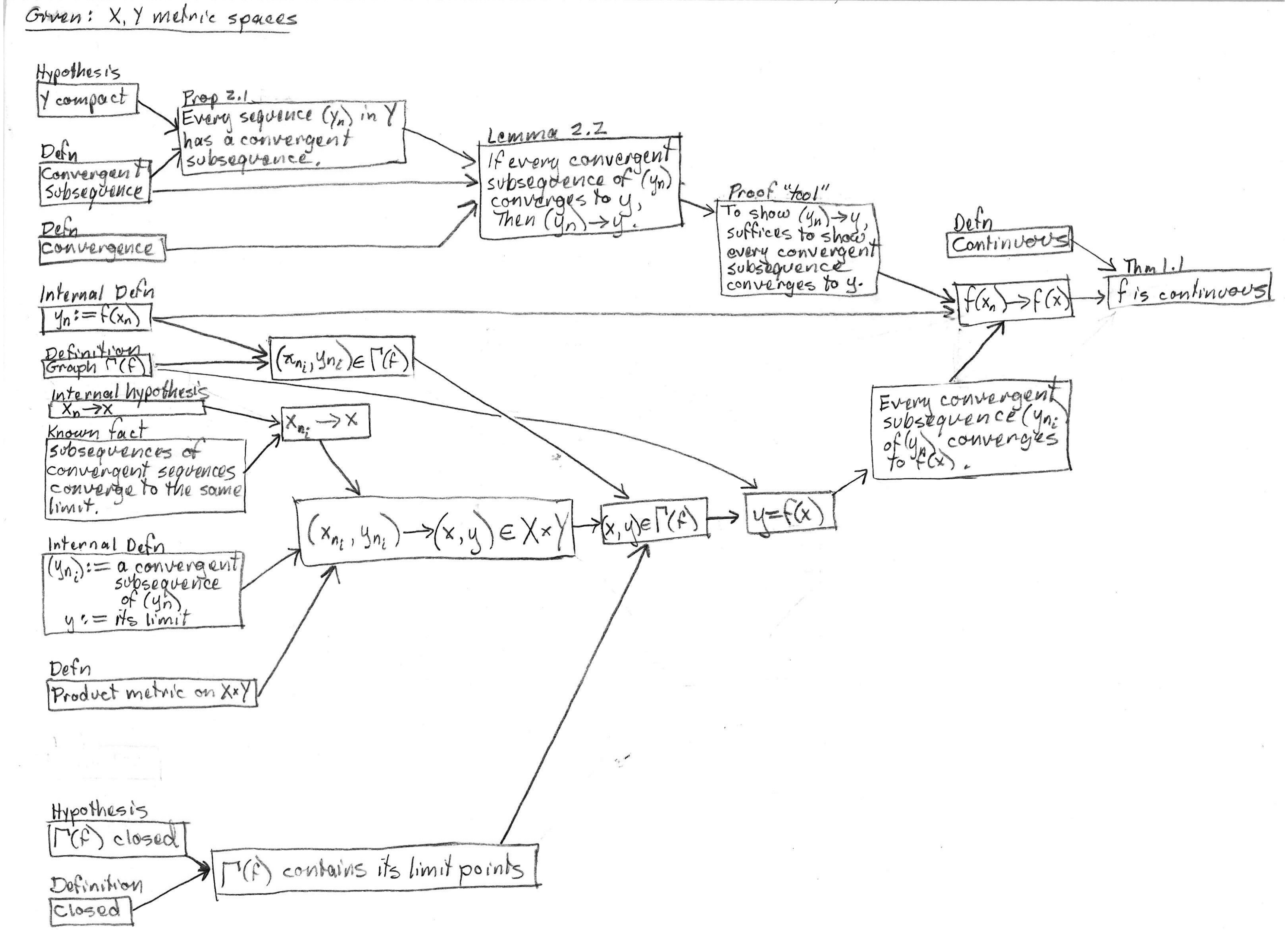
Text is inherently one dimensional: one long thread of text. But logic is not!

A proof's logic



One thread could use known-to-new flow.

How can we pull the various threads together?

Connect threads via Guiding text

- Ensure readers know
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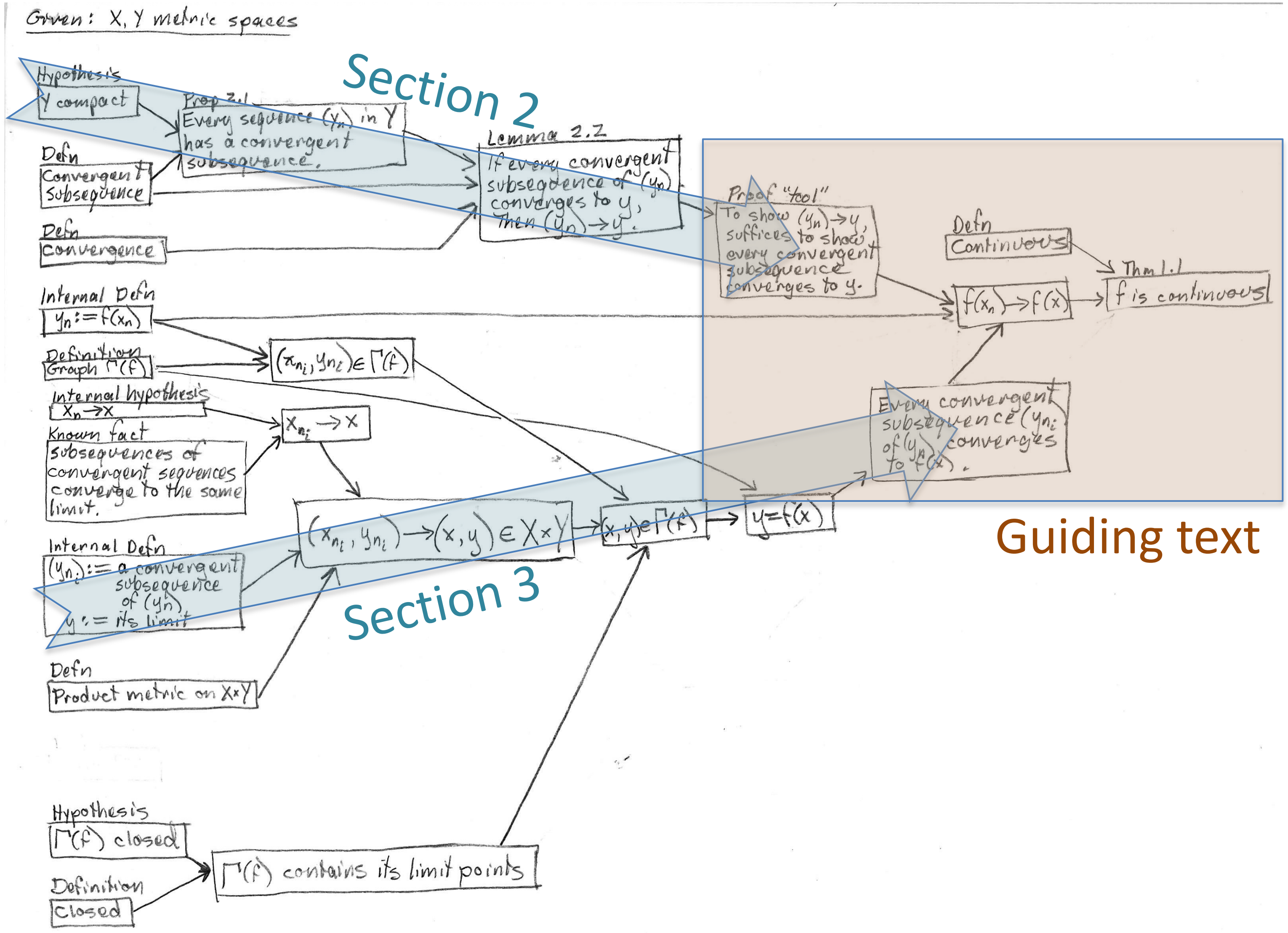
Connect known-to-new threads via Guiding text

Guiding text:
To show f is continuous,
we prove in Section 2 that
it suffices to show…;
then in Section 3...



Summary: As you revise

- Who is the audience? What do they know and care about?
- For each chunk of text, will your audience know
  - What you're doing?
  - Why you're doing it?

  If not, add guiding text.
- Does each sentence use known-to-new structure?

  If not, turn sentences around &/or fill gaps in flow.

The goal: to craft text that reveals to your audience the flow and structure of the underlying logic.

Beware “superficial flow”

The market-determined price of a bond with fixed, known cash flows determines the bond’s internal rate of return, or yield. Different yields are typically approximately equal. Approximations can be provided by Taylor series. The Taylor series is due to James Gregory of Scotland. Scotland has 790 islands, including the Northern Isles and the Hebrides, according to Wikipedia. Wikipedia occasionally asks for donations.

“FLOW” SHOULD HELP READERS FOLLOW THE FLOW OF THE LOGIC