# CONJUGACY CLASSES AND IRREDUCIBLE REPRESENTATIONS OF HEIS($q$)

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## 1. Introduction

The *Heisenberg group*, named after the physicist Werner Heisenberg, has its profound implications in applied mathematics and quantum mechanics. We first start with the discretized version of the Heisenberg group:

**Definition 1.1.** *The finite Heisenberg group is the group of* $3 \times 3$ *upper triangular matrices of the following form*:

$$\mathrm{Heis}(q) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{F}_q \right\} \tag{1.1}$$

Note that we adopt the convention $q = p^r$, where $p$ is prime. The continuous version of the Heisenberg group, also known as the *real Heisenberg group*, can be obtained when entries are taken to be real numbers. The real Heisenberg group is closely related to the formulation of the Heisenberg's uncertainty principle and fundamental aspects of quantum mechanics, which we summarize in Sec. 5. In the rest of this paper, we will be mainly focused on the discussion of the finite Heisenberg group and its irreducible representations and characters. We will first revisit the idea of the induced representation in Sec. 2. Then, we derive the conjugacy classes of Heis($q$) in Sec. 3. The characters and irreducible representations will be discussed in Sec. 4. Lastly, we link it to its importance in modern quantum mechanics in Sec. 5.

## 2. Induced representation

Induced representations are a way to construct a representation of the group from the representation of its subgroup by "extending" the subgroup representation [T, Chapter 16].

**Definition 2.1.** *Suppose that $H$ is a subgroup of the finite group $G$ and $\sigma : H \to GL(W)$ is a representation of $H$. The induced representation from $H$ up to $G$, denoted as $\pi = Ind_H^G \sigma$ is a group homomorphism $\pi : G \to GL(V)$, where*

$$V = \{f : G \to W \,|\, f(hg) = \sigma(h)f(g), \text{ for all } h \in H, g \in G\} \tag{2.1}$$

*The representation $\pi(g)$ of $G$ is defined by*

$$[\pi(g)f](x) = f(xg), \text{ for all } x, g \in G \tag{2.2}$$

The *Frobenius formula* gives the characters for the induced representation of $G$.

**Theorem 2.2** (Frobenius formula)**.** *Using the above notation, we have*

$$\chi_\pi(g) = \frac{1}{|H|}\sum_{x\in G}\tilde{\chi_\sigma}(xgx^{-1}) = \sum_{a\in H\backslash G}\tilde{\chi_\sigma}(aga^{-1})$$

where

$$\tilde{\chi_\sigma}(x) = \begin{cases}\chi_\sigma(x), \text{ if } x \in H \\ 0, \text{ if } x \notin H\end{cases}$$

## 3. Conjugacy classes of the Heisenberg group

We now turn to the main discussion of the finite Heisenberg group, following the construction in [T, Chapter 18]. To begin with, we compute the conjugacy classes of Heis$(q)$. Notice that

$$\begin{aligned}&\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}^{-1}\\ &= \begin{pmatrix}1 & a & c+(bx-ay)\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\end{aligned} \tag{3.1}$$

Thus, the $a$ and $b$ entries remain unchanged after the conjugation. For $a = b = 0$, the conjugacy class has only one element since $bx - ay = 0$ and the upper right entry remains unchanged as well. Since $c$ can be taken to be any values in $\mathbb{F}_q$, there are $q$ conjugacy classes of this type. If $(a, b) \neq (0, 0)$, then the matrix $\left(\begin{smallmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{smallmatrix}\right)$ is conjugate to all matrices with the same $a, b$ entries. In other words, there are $q$ elements in this conjugacy class, and there are $q^2 - 1$ conjugacy classes of this type for different choices of $a$ and $b$. The class equation of Heis$(q)$ then can be written as

$$|\mathrm{Heis}(q)| = q^3 = q \times 1 + (q^2 - 1) \times q \tag{3.2}$$

## 4. Irreducible Representations and their characters of Heis$(q)$

From the discussion on conjugacy classes in Heis$(q)$ and the well-known fact that the number of conjugacy classes corresponds to the number of irreducible representations, we can infer that there are in total $q + q^2 - 1$ irreducible representations of Heis$(q)$.

### 4.1. **One-dimensional representations.**

We first try to find the one-dimensional characters of Heis$(q)$. Let $\rho$ be a one-dimensional representation and $\chi$ be the character of $\rho$. Since representations are homomorphic by definition, the following equation must be satisfied:

$$\chi(gh) = \rho_{gh} = \rho_g\rho_h = \chi(g)\chi(h) \tag{4.1}$$

The exact formula of the character $\chi$ can be motivated by observing the multiplication of two matrices:

$$\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & m & n\\ 0 & 1 & r\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & x+m & n+xr+y\\ 0 & 1 & y+r\\ 0 & 0 & 1\end{pmatrix} \tag{4.2}$$

Focusing our attention to the $e_{1,2}$ and $e_{2,3}$ entries, we can define the character $\chi_{a,b}$ so that Eqn. 4.1 is satisfied.

**Definition 4.1.** *The one-dimensional representations (and characters) of Heis(q) are*

$$\chi_{a,b}\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = \exp\,(\frac{2\pi i \mathrm{Tr}\,(\mathrm{ax}+\mathrm{by})}{p}) \tag{4.3}$$

*where* $a, b \in \mathbb{F}_q$

One can check that $\chi_{a,b}$ indeed satisfy Eqn. 4.1:

$$\begin{aligned}\chi_{a,b}(g)\,\chi_{a,b}(h) &= e^{\frac{2\pi i}{p}(ax+by)}\,e^{\frac{2\pi i}{p}(am+br)}\\ &= e^{\frac{2\pi i}{p}[a(x+m)+b(y+r)]}\\ &= \chi_{a,b}(gh)\end{aligned} \tag{4.4}$$

Therefore, we have found $q^2$ one-dimensional representations of Heis($q$) for different choices of $a$ and $b$, and their corresponding characters can be directly calculated.

4.2. **Induced representations of Heis($q$).** The next irreducible representation of Heis($q$) cannot be seen readily, instead, it has to be constructed using the induced representation from the following subgroup $H$ of Heis($q$):

$$H = \left\{ \begin{pmatrix}1 & 0 & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} \middle|\, y, z \in \mathbb{F}_q \right\} \tag{4.5}$$

To obtain the induced representation of Heis($q$), we define the representations on this subgroup:

**Definition 4.2.** *The one-dimensional representations of the subgroup $H$ of Heis(q) are*

$$\sigma_s\begin{pmatrix}1 & 0 & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = \exp\,(\frac{2\pi i \mathrm{Tr}(sz)}{p}), \text{ for all } s \neq 0 \tag{4.6}$$

Using the idea of the induced representation, we obtain $q-1$ representations $\pi_s = \mathrm{Ind}_H^G \sigma_s$ of Heis($q$) for each $\sigma_s$ as discussed in Sec. 2. The following proposition gives the characters of this representation:

**Proposition 4.3.** *Let* $g = \left(\begin{smallmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{smallmatrix}\right) \in$ *Heis(q). The characters of the representation* $\pi_s$ *are:*

$$\chi_{\pi_s}(g) = \begin{cases} q\exp(\frac{2\pi \mathrm{iTr(sz)}}{\mathrm{p}}), \text{ if } \; \textit{(x,y)=(0,0)}\\ 0, \text{ otherwise}\end{cases}$$

| Character table for Heis($q$) | | |
|---|---|---|
| | $\left\{\begin{pmatrix}1&0&z\\0&1&0\\0&0&1\end{pmatrix}\right\}$ | $\left\{\begin{pmatrix}1&x&*\\0&1&y\\0&0&1\end{pmatrix}\right\}$ |
| # Classes | $q$ | $q^2-1$ |
| # Elements in class | 1 | q |
| $\chi_{a,b}$ | 1 | $\exp\left(\frac{2\pi i \mathrm{Tr(ax+by)}}{p}\right)$ |
| $\chi_{\pi_s}, s \in \mathbb{F}_q, s \neq 0$ | $q\exp\left(\frac{2\pi i \mathrm{Tr}(sz)}{p}\right)$ | 0 |

TABLE 1. The character table of Heis($q$).

*Proof.* To get the character values of elements in $G$, we first determine the representatives of the quotient group $H\backslash G$ to be

$$H\backslash G = \left\{ \begin{pmatrix}1&c&0\\0&1&0\\0&0&1\end{pmatrix} \middle| c \in \mathbb{F}_q \right\} \tag{4.7}$$

Then, apply Thm 2.2 for $g = \left(\begin{smallmatrix}1&0&z\\0&1&0\\0&0&1\end{smallmatrix}\right)$

$$\begin{aligned} \chi_{\pi_s}(g) &= \sum_{a\in H\backslash G} \tilde{\chi}_{\sigma_s}(aga^{-1}) \\ &= \sum_{a\in H\backslash G} \tilde{\chi}_{\sigma_s}(g) \\ &= \sum_{a\in H\backslash G} \sigma_s(g) \\ &= q\exp(\frac{2\pi i \mathrm{Tr}\,(sz)}{p}) \end{aligned} \tag{4.8}$$

We have used the fact $g \in H$ in the third equality and Def. 4.2 in the last equality. Repeating the same calculation for $g = \left(\begin{smallmatrix}1&x&z\\0&1&y\\0&0&1\end{smallmatrix}\right)$, we obtain

$$\begin{aligned} \chi_{\pi}(g) &= \sum_{a\in H\backslash G} \tilde{\chi}_{\sigma}(aga^{-1}) \\ &= 0 \end{aligned} \tag{4.9}$$

since $aga^{-1} \notin H$, for all $a \in H\backslash G$. $\square$

Applying Prop. 4.2 on the identity matrix, we immediately know that the dimension of the induced representation is $q$. Furthermore, the representations above are all irreducible as can be verified by taking the Hermitian product. Finally, since the following dimension formula is satisfied, we have found all the irreducible representations of Heis($q$):

$$|\mathrm{Heis}(q)| = q^3 = q^2 \times 1^2 + q^2 \times (q-1) \tag{4.10}$$

Our work can be summarized in the character table 1.

## 5. Relation to quantum mechanics

In this last section, we discuss the key application of Heis($q$) in quantum mechanics. The main reference for this section is [W, Chapter 13].

5.1. **Heisenberg Lie algebra.** Consider the Lie algebra $\eta_3$ with basis (X,Y,Z) and the Lie bracket operation defined by its values on the basis $(X, Y, Z)$:

$$[X, Y] = Z, [Y, Z] = 0, [X, Z] = 0. \tag{5.1}$$

This is called the *Heisenberg Lie Algebra* $\eta_3$. The Lie bracket operation resembles the canonical commutator relation in quantum mechanics:

$$[\hat{x}, \hat{p}] = i\hbar I, [\hat{x}, i\hbar I] = 0, [\hat{p}, i\hbar I] = 0. \tag{5.2}$$

where $I$ is the identity and $\hat{x}, \hat{p}$ are the operators of position and momentum, respectively. The Heisenberg Lie algebra is isomorphic to the Lie algebra of $3 \times 3$ strictly upper triangular real matrices when we take the X,Y,Z to be the following $3 \times 3$ matrices and the Lie brackets to be the usual matrix commutator:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.3}$$

This is related to the real Heisenberg group by the isomorphism generated by the following exponential map:

$$\exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{5.4}$$

5.2. **Schrödinger representation.**

**Definition 5.1.** *The Schrödinger representation* is another representation $(\Gamma_S, L^2(\mathbb{R}))$ of the Heisenberg Lie algebra $\eta_3$ satisfying:

$$\Gamma_S(X)\psi(q) = -iq\psi(q), \Gamma_S(Y)\psi(q) = -\frac{d}{dq}\psi(q), \Gamma_S(Z)\psi(q) = -i\psi(q)$$

where $\psi(q) \in L^2(\mathbb{R})$. This representation encodes the essence of the Schrodinger equation, the differential equation that governs the evolution of quantum states. In 1932, Marshall Stone and John von Neumann proved the following theorem about the uniqueness of the representation of the Heisenberg group.

**Theorem 5.2** (Stone-von Neumann)**.** *There is a unique irreducible unitary representation (up to isomorphism) of the Heisenberg group on finitely many generators.*

This implies that the Schrödinger representation is the unique (up to isomorphism) irreducible representation of the Heisenberg group in finite dimensions. In particular, Schrödinger's formulation of quantum mechanics using the Schrodinger equation is the only equivalent formulation of Heisenberg's matrix mechanics, where the position and momentum operators are taken to be matrices that satisfy certain commutation relations.

The importance of the real Heisenberg group lies in the fact that the formulation of the Stone-von Neumann theorem is based on the Heisenberg group representations instead of the Heisenberg Lie algebra representations. In addition, the Stone-von Neumann theorem is only valid in finite dimensional cases. In quantum field theory, where infinite degrees of freedom are present, it is no longer true that only one irreducible representation can be found.

## References

[T] A. Terras. *Fourier Analysis on Finite Groups and Applications*. Cambridge University Press (1999).

[W] P. Woit. *Quantum Theory, Groups and Representations: An Introduction*. Springer (2017).

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