ON THE FINITE UPPER HALF PLANE

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ABSTRACT. Many continuous spaces have well-studied finite analogues, the most obvious example being the finite fields $\mathbb{F}_q^n$ for the hyperspace $\mathbb{R}^n$. These finite analogues often pave the way for new discoveries, both in the continuous realm and in other areas of study. This paper will define a finite version of the Poincaré half plane model using finite fields, and examine a few noteworthy constructions involving the finite upper half plane.

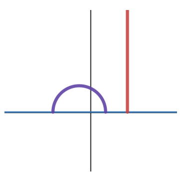
1. THE POINCARÉ HALF PLANE

Before we discuss the finite upper half plane, we first define the Poincaré half plane model, also simply the upper half plane, to explain what we are discretizing.

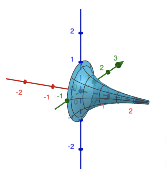
Definition 1.1. The *Poincaré half plane* is the set $H = \{x + iy \in \mathbb{C} \mid y > 0\}$ of complex numbers with positive imaginary part, equipped with the distance metric $(ds)^2 = \frac{(dx)^2+(dy)^2}{y^2}$.

This metric defines a hyperbolic, non-Euclidean space. The *geodesics*, or curves minimizing arc length, are either vertical rays or open half-circles centered on the real axis. This paper will not prove why these curves minimize distance. However, we offer some intuition from the metric itself for why this is the case. As we move closer to the real axis, distances stretch longer; two vertical rays become infinitely far apart as the imaginary part goes to 0. Thus, the shortest path between two points is not a straight line, but rather an arc.

For those unfamiliar with non-Euclidean metrics, it may be difficult to visualize exactly how this space behaves. Hyperbolic spaces are defined by their constant negative curvature; if we were to imagine this space with a Euclidean metric, this would result in every point being a saddle point, with two principal curves with opposite curvatures. The right diagram shows the *pseudosphere*, the surface of revolution of the curve parametrized by $(t - \tanh t, \operatorname{sech} t)$, which has constant negative curvature everywhere except the cusp.



(A) The geodesics of the half plane.



(B) An example of hyperbolic space in Euclidean $\mathbb{R}^3$.

2. The Finite Upper Half Plane

We now define the finite upper half plane as follows. Let $\mathbb{F}_q$ be a field of odd characteristic p with $q = p^r$, and let $\delta \in \mathbb{F}_q$ be a nonsquare. Note that we need q to be odd in order for such a nonsquare element to exist.

Definition 2.1. The *finite upper half plane* is the set

$$H_q = \{x + y\sqrt{\delta} \mid x, y \in \mathbb{F}_q, y \neq 0\}. \tag{2.1}$$

Since we are working with $\mathbb{F}_q$, the condition $y > 0$ translates to the condition $y \neq 0$; thus, this might be considered the union of two finite half-planes. However, this is not a meaningful distinction in the finite case, as we can simply translate the points as needed.

We define a few terms from complex analysis in the way one would expect. For a point $z \in H_q$ with $z = x + y\sqrt{\delta}$, we define the *real part* $\mathrm{Re}(z) = x$ and the imaginary part $\mathrm{Im}(z) = y$. The *conjugate* $\overline{z}$ is defined as $x - y\sqrt{\delta}$, and we define the *norm* by $N(z) = z\overline{z}$ and the trace by $Tr(z) = z + \overline{z}$.

Now, given a matrix $g \in GL(2, \mathbb{F}_q)$ with nonzero determinant with entries $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we can define a corresponding action on H_q. Specifically, for any $z \in H_q$, we write

$$gz = \frac{az + b}{cz + d}.$$

We can explicitly see that this forms a bijective map from H_q onto itself. Letting $z = x + y\sqrt{\delta}$, we have

$$\begin{aligned}
\mathrm{Im}(gz) &= \mathrm{Im}\left(\frac{az+b}{cz+d} \cdot \frac{\overline{cz+d}}{\overline{cz+d}}\right) \\
&= \frac{1}{N(cz+d)} \mathrm{Im}\left((az+b)(\overline{cz+d})\right) \\
&= \frac{1}{N(cz+d)} \left[\mathrm{Re}(az+b)\mathrm{Im}(\overline{cz+d}) + \mathrm{Im}(az+b)\mathrm{Re}(\overline{cz+d})\right] \\
&= \frac{1}{N(cz+d)} \left[(ax+b)(-cy) + (ay)(cx+d)\right] \\
&= \frac{(ad-bc)y}{N(cz+d)},
\end{aligned}$$

which gives us

$$\mathrm{Im}(gz) = \frac{\mathrm{Im}(z)\det g}{N(cz+d)}. \tag{2.2}$$

This must be nonzero, as neither $\det g$ nor the imaginary part of z are 0. Note also that, since the imaginary part of $cz + d$ is not zero, it has nonzero norm as well.

Remark 2.2. For those familiar with complex analysis, this construction is analogous to Möbius transformations on the complex plane with real coefficients, which are the maps $z \to \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$. These maps preserve the metric of the Poincaré half plane, and therefore preserve the geodesics as well.

3. Graphs on the finite half-plane

One interesting construction on the finite upper half plane is the finite upper half plane graph, which is a type of Cayley graph involving a specific subset of H_q. To do so, we define a kind of "distance" between two points in H_q as follows.

Definition 3.1. The *distance* between two points $z, w \in H_q$ is defined by

$$d(z,w) = \frac{N(z-w)}{\mathrm{Im}(z)\mathrm{Im}(w)}. \tag{3.1}$$

This definition of distance is similar to the Poincaré metric, where we divide by the square of the imaginary part. Just as the Möbius transformations preserved the distance metric in the Poincaré half plane, we can show that the action of $GL(2,\mathbb{F}_q)$ on H_q preserves this notion of distance.

Theorem 3.2. *For all $g \in GL(2,\mathbb{F}_q)$, and for all $z, w \in H_q$, $d(gz, gw) = d(z,w)$.*

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have that

$$\begin{aligned} N(gz-gw) &= N\left(\frac{az+b}{cz+d} - \frac{aw+b}{cw+d}\right) \\ &= \frac{N\left(\det(g)(z-w)\right)}{N(cz+d)N(cw+d)} \\ &= \frac{\det(g)^2 N\left(z-w\right)}{N(cz+d)N(cw+d)} \end{aligned}$$

Combining this with (2.2), the conclusion follows. □

Now that we have defined a distance function, we can define a graph on the finite half-plane.

Definition 3.3. For $a \in \mathbb{F}_q$, the finite upper half plane graph $X_q(\delta, a)$ is the graph with vertices in H_q and two vertices $z, w \in H_q$ are adjacent if and only if $d(z,w) = a$.

The graph $X_3(-1,1)$ is an octahedron, a diagram of which can be found in Terras [T, page 313].

We conclude this paper by explore some other interesting constructions involving the finite upper half plane. These analyses closely follow Shaheen [S, page 14].

4. Lines and Circles On the Finite Upper Half Plane

We do not have any meaningful sense of geodesics in the finite half plane; in particular, $\mathbb{F}_q^*$ has no clear ordering, and so it doesn't make sense to discuss paths of minimal distance. Nevertheless, it is still interesting to wonder how the Euclidean definitions of lines and circles behave in this finite half plane. More precisely, let $x = \mathrm{Re}(z)$ and let $y = \mathrm{Im}(z)$. We can consider the solutions to the equation $ax + by + c = 0$, with $a, b, c \in \mathbb{F}_q$, to be the analogs of lines, and solutions to $(x-h)^2 - \delta(y-k)^2 = r$, with $h, k, r \in \mathbb{F}_q$, to be the analogs of circles. Note that in the complex case with $\delta = -1$, the second equation matches our usual definition.

Using the above notation, we know that $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{\sqrt{\delta}}{2\delta}(z - \overline{z})$. We can therefore simplify the equation for a line as follows:

$$\begin{aligned}0 &= ax + by + c \\ &= (\frac{a}{2} + \frac{b}{2\delta}\sqrt{\delta})z + (\frac{a}{2} - \frac{b}{2\delta}\sqrt{\delta})\overline{z} + c \\ &= \beta z + \overline{\beta}\overline{z} + c,\end{aligned}$$

where $\beta = \frac{a}{2} + \frac{b}{2\delta}\sqrt{\delta}$ is another element of $\mathbb{F}_q(\sqrt{\delta})$. We can also rewrite the equation for a circle. Let $z_0 = h + k\sqrt{\delta}$. Then, the equation for a circle becomes

$$\begin{aligned}r &= (x-h)^2 - \delta(y-k)^2 \\ &= (x^2 - 2xh + h^2) + \delta(-y^2 + 2yk - k^2) \\ &= z\overline{z} + z_0\overline{z_0} - z_0\overline{z} - z\overline{z_0}.\end{aligned}$$

This further simplifies to $r = N(z - z_0)$, showing how this agrees with our usual idea of a circle. Finally, looking at both of these formulas, we see that both can be written in the follow way:

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0,$$

with $\alpha, \gamma \in \mathbb{F}_q$ and $\beta \in \mathbb{F}_q(\sqrt{\delta})$. When $\alpha = 0$, we recover the equation for a line, and when $\alpha \neq 0$, we recover the equation for a circle. Thus, this expression encompasses both lines and circles in the finite upper half plane. This agrees with the equation for lines and circles in the complex plane; in fact, most of the same analysis holds when working in $\mathbb{C}$.

5. Extending the Finite Half Plane

As H_q can be thought of a subset of $\mathbb{F}_q(\sqrt{\delta})$, with δ being a nonsquare element of $\mathbb{F}_q$, we can also ask whether or not we can continue extending the field and create a half-plane using $\mathbb{F}_q(\sqrt{\delta})$ as the base field. In this section, we will show that there exists a sequence of finite fields

$$\mathbb{F}_q \subset \mathbb{F}_q(\sqrt{\delta_0}) \subset \mathbb{F}_q(\sqrt{\delta_0})(\sqrt{\delta_1}) \subset \mathbb{F}_q(\sqrt{\delta_0})(\sqrt{\delta_1})(\sqrt{\delta_2}) \subset \dots$$

such that each δ_i is a multiplicative generator of $\mathbb{F}_q(\sqrt{\delta_0})\dots(\sqrt{\delta_{i-1}})$ and the norm map N from $\mathbb{F}_q(\sqrt{\delta_0})\dots(\sqrt{\delta_i})$ to $\mathbb{F}_q(\sqrt{\delta_0})\dots(\sqrt{\delta_{i-1}})$ maps δ_i to δ_{i-1}. We will prove this in two parts.

Lemma 5.1. *Given $u \in \mathbb{F}_q(\sqrt{\delta})^*$ is a multiplicative generator of $\mathbb{F}_q(\sqrt{\delta})^*$, then $N(u)$ is a multiplicative generator of $\mathbb{F}_q^*$.*

Proof. The norm map for $\mathbb{F}_q(\sqrt{\delta})^*$ is $N(x) = xx^q$ for all $x \in \mathbb{F}_q(\sqrt{\delta})^*$. Now, since u is a generator, $u^k = 1$ if and only if k is a multiple of $q^2 - 1$. As a result, $N(u)^l = u^{(q+1)l} = 1$ if and only if l is a multiple of $q - 1$. Thus, $N(u)$ is a generator for $\mathbb{F}_q^*$. □

We use this lemma to show that for a generator α of $\mathbb{F}_q^*$, there always exists some generator of $\mathbb{F}_q(\sqrt{\delta})^*$ whose norm is α.

Lemma 5.2. *Given $\alpha \in \mathbb{F}_q^*$ is a multiplicative generator of $\mathbb{F}_q^*$, there exists some multiplicative generator β of $\mathbb{F}_q(\sqrt{\delta})^*$ with $N(\beta) = \alpha$.*

Proof. Let γ be a multiplicative generator of $\mathbb{F}_q(\sqrt{\delta})^*$. By the previous lemma, $u = N(\gamma)$ is a generator of $\mathbb{F}_q^*$. Any other generator of $\mathbb{F}_q(\sqrt{\delta})^*$ takes the form of γ^c, where $\gcd(c, q^2 - 1) = 1$; similarly, all generators of $\mathbb{F}_q^*$ are of the form u^b, where $\gcd(b, q - 1) = 1$. Thus, it is enough to show that, for every b with $\gcd(b, q - 1) = 1$, there exists some c with $\gcd(c, q^2 - 1) = 1$ such that $N(\gamma^c) = u^c = u^b$. This happens if and only if

$$c \equiv b \mod (q-1) \quad \text{and} \quad \gcd(c, q^2-1) = 1. \tag{5.1}$$

Since $\gcd(c, q - 1) = \gcd(b, q - 1) = 1$, we can simplify the LHS of (4.1) further, to

$$c \equiv b \mod (q-1) \quad \text{and} \quad \gcd(c, q+1) = 1. \tag{5.2}$$

Now, since q is odd, we further know that $\gcd(q - 1, q + 1) = 2$; thus, by the Chinese Remainder Theorem we can take the c that solves the system

$$\begin{aligned} c &\equiv b \mod (q-1) \\ c &\equiv 1 \mod (\frac{q+1}{2}). \end{aligned}$$

Thus, we have shown that each generator in $\mathbb{F}_q^*$ is the norm of at least one generator of $\mathbb{F}_q(\sqrt{\delta})^*$. □

Using the above, we see that we can iteratively construct the desired sequence of fields by selecting the appropriate generator at each stage. Thus, we can keep extending the finite to higher dimensions in this fashion.

References

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