EXCEPTIONAL ISOMORPHISMS OF SL, PSL IN RANK 2

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The study of complex representations of finite, (projective) linear groups finds applications in discrete Fourier analysis, graph theory, and chemistry. [T]. The problem is greatly simplified on several special cases in rank 2. In this paper, we exhibit isomorphisms from $\mathrm{SL}(2,2), \mathrm{PSL}(2,3), \mathrm{PSL}(2,5), \mathrm{PGL}(2,5)$ to symmetric and alternating groups, which do not generalize to fields of higher order. We then compute character tables for these symmetric groups, which is made straightforward by their conjugacy structure; the isomorphisms allow us to transfer this information to the linear groups.

1. Linear groups and orders

1.1. **Linear groups.** We begin by recalling definitions for our groups of interest. Let F be a field and $n \geq 1$ an integer.

Definition 1.1. The **general linear group** $\mathrm{GL}_n(F)$ is the group of invertible $n \times n$ matrices with entries in F, under matrix multiplication. The **special linear group** $\mathrm{SL}_n(F)$ is the subgroup of $\mathrm{GL}_n(F)$ consisting of matrices with determinant 1. When F is a finite field of order q, we denote these groups as $\mathrm{GL}(n,q)$ and $\mathrm{SL}(n,q)$, respectively.

Proposition 1.2. *Let $Z(G)$ denote the center of a group G. Then*

$$Z(GL_n(F)) = \{\lambda I : \lambda \neq 0\}, \quad Z(SL_n(F)) = \{\lambda I : \lambda^n = 1\}.$$

Proof. Clearly, any scalar multiple of the identity commutes with all matrices. On the other hand, for $i \neq j$, let $E_{i,j} \in GL_n(F)$ be equal to the identity matrix except for a 1 in the (i,j)-th entry. Then $\det E_{i,j} = 1$. Suppose A commutes with every matrix $E_{i,j}$. Say $i < j$; then $AE_{i,j} = E_{i,j}A$ implies $a_{ji} = 0$ and $a_{ii} = a_{jj}$ by comparing entries. Similarly, $AE_{j,i} = E_{j,i}A$ implies $a_{ij} = 0$, and these hold for all $i \neq j$. Hence A is diagonal with equal diagonal entries, a scalar multiple of I. Since every matrix $E_{i,j}$ is in $SL_n(F)$, this shows that both centers are as claimed. □

1.2. **Projective groups and orders.** The main groups we will work with are quotients of linear groups. Recall that the center Z of a group G is normal, so the quotient group G/Z is well-defined.

Definition 1.3. The **projective general linear group** $\mathrm{PGL}_n(F)$ is the quotient group $\mathrm{GL}_n(F)/Z(\mathrm{GL}_n(F))$. Its elements are equivalence classes of $n \times n$ matrices, of the form $\overline{A} = \{\lambda A : \lambda \neq 0\}$ for $A \in \mathrm{GL}_n(F)$. The **projective special linear group** $\mathrm{PSL}_n(F)$ is the quotient group $\mathrm{SL}_n(F)/Z(\mathrm{SL}_n(F))$. Its elements are equivalence classes of the form $\overline{S} = \{\lambda S : \lambda^n = 1\}$ for $S \in \mathrm{SL}_n(F)$. When F is finite of order q, we denote these groups as $\mathrm{PGL}(n,q)$ and $\mathrm{PSL}(n,q)$, respectively.

There is a natural embedding $\mathrm{PSL}_n(F) \to \mathrm{PGL}_n(F)$ that sends each class $\overline{S} \in \mathrm{PSL}_n(F)$ to the class $\overline{S} \in \mathrm{PGL}_n(F)$. This is simply the map that makes the diagram

$$\begin{tikzcd} SL_n(F) \arrow[r] \arrow[d] & PSL_n(F) \arrow[d, dotted] \\ GL_n(F) \arrow[r] & PGL_n(F) \end{tikzcd}$$

commute; one easily checks it is well-defined and an homomorphism. Its kernel consists of classes $\overline{S}$ such that $\overline{S} \in \mathrm{PGL}_n(F)$ is the identity, i.e. such that $S = \lambda I$ for some nonzero λ. Then $\det S = 1$ forces $\lambda^n = 1$, so that $\overline{S} \in \mathrm{PSL}_n(F)$ is also the identity. Hence this embedding is an isomorphism, and we identify $\mathrm{PSL}_n(F)$ as a subgroup of $\mathrm{PGL}_n(F)$.

For the rest of this paper, we will work with matrices of rank $n = 2$, and we will impose that $F = \mathbb{F}_q$ is finite of order q and prime characteristic $p > 0$. Of particular interest is the case $q = p$.

Proposition 1.4. $|\mathrm{GL}(2, q)| = (q^2 - 1)(q^2 - q)$. *Therefore* $|\mathrm{SL}(2, q)| = (q^2 - 1)q$, $|\mathrm{PGL}(2, q)| = (q^2 - 1)q$, *and*

$$|\mathrm{PSL}(2, q)| = \begin{cases} (q^2 - 1)q & \text{if } q \text{ is even,} \\ \frac{1}{2}(q^2 - 1)q & \text{otherwise.} \end{cases}$$

Proof. A 2×2 matrix A with entries in $\mathbb{F}_q$ is invertible if and only if its columns are nonzero and neither is a scalar multiple of the other. There are $q^2 - 1$ ways to choose a nonzero first column v. Then v has $q - 1$ scalar multiples, so there are $(q^2-1)-(q-1) = (q^2-q)$ ways to choose the second column. Thus, $\mathrm{GL}(2, q)$ has order $(q^2 - 1)(q^2 - q)$. Moreover, $\mathrm{SL}(2, q)$ is the kernel of the surjective homomorphism $\det : \mathrm{GL}(2, q) \to (\mathbb{F}_q)^\times$, so its order is $|\mathrm{GL}(2, q)|/|q - 1| = (q^2 - 1)q$.

For the projective groups, we use Proposition 1.2. The center $Z(\mathrm{GL}(2, q))$ has order $|(\mathbb{F}_q)^\times| = q-1$. On the other hand, the polynomial x^2-1 always splits over $\mathbb{F}_q$ as $(x-1)(x+1)$, so its two roots are $\{\pm 1\}$, which are only distinct when $\mathbb{F}_q$ has odd characteristic. So the center $Z(\mathrm{PSL}(2, q)) = \{\pm I\}$ has order 1 if q is even and order 2 otherwise. It follows that the orders of the quotient groups $\mathrm{PGL}(2, q), \mathrm{PSL}(2, q)$ are as claimed. ☐

2. Exceptional isomorphisms

2.1. **Group actions and the projective line.** To establish isomorphisms from projective groups to permutation groups, our strategy is to look at the action of these groups on either $\mathbb{F}_p^2$ or the projective line $\mathbf{P}^1(\mathbb{F}_p)$. We first recall some concepts.

Definition 2.1. Let G be a group and X a set; let $\mathrm{Sym}(X)$ be the group of bijections $X \to X$ under composition. An **action** of G on X is an homomorphism $\alpha : G \to \mathrm{Sym}(X)$. We say G *acts on* X, and we usually write $gx = y$ when $\alpha(g)$ maps $x \mapsto y$. An action is *faithful* if the map α is injective, and we say G acts *faithfully.*

Of interest is when X is finite of size $n \geq 1$. If we index the elements of X as $x_1, \dots, x_n$, we specify an isomorphism $\mathrm{Sym}(X) \to S_n$, and an action on X can be identified with the composition $G \to \mathrm{Sym}(X) \to S_n$: the image of each $g \in G$ is the permutation $\pi \in S_n$ such that $gx_i = x_{\pi(i)}$ for all i. When the action is faithful, this becomes an isomorphism from G to a subgroup of S_n. We can further compose this with the (injective) homomorphism $S_n \to \mathrm{GL}_n(\mathbb{C})$ that sends each permutation π to its permutation matrix A_π, whose entries are

$$(A_\pi)_{ij} = \begin{cases} 1 & \text{if } \pi(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

This yields a representation $G \to \mathrm{GL}_n(\mathbb{C})$ of dimension n. We call it the *permutation representation* corresponding to said action.

Definition 2.2. The **projective line over** $\mathbb{F}_p$, denoted $\mathbf{P}^1(\mathbb{F}_p)$, is the set of 1-dimensional subspaces of $\mathbb{F}_p^2$. We denote the subspace spanned by $v \in \mathbb{F}_p^2$ as $\langle v \rangle \in \mathbf{P}^1(\mathbb{F}_p)$. Notice that $|\mathbf{P}^1(\mathbb{F}_p)| = p+1$: the elements are $\left\langle \binom{1}{x} \right\rangle$ for each $x \in \mathbb{F}_p$, together with $\left\langle \binom{0}{1} \right\rangle$.

Proposition 2.3. *The groups* $\mathrm{PGL}(2,p)$ *and* $\mathrm{PSL}(2,p)$ *act faithfully on* $\mathbf{P}^1(\mathbb{F}_p)$*; each class* $\overline{A}$ *acts by sending* $\langle v \rangle \mapsto \langle Av \rangle$*.*

Proof. Let $A \in \mathrm{GL}(2,p)$. For any $\lambda \neq 0$ and $v \in \mathbb{F}_p^2$, the vectors $Av, \lambda(Av)$ are multiples of each other, so $\langle Av \rangle = \langle (\lambda A)v \rangle$. Thus the map sending $\langle v \rangle \mapsto \langle Av \rangle$ is independent of the choice of representative $A \in \overline{A}$, and we see that the action of $\mathrm{PGL}(2,p)$ is well-defined.

Suppose the class $\overline{A}$ fixes all 1-dimensional subspaces, that is, every $v \in \mathbb{F}_p^2$ is an eigenvector of A. Let λ_1, λ_2 be the eigenvalues of $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$. Then A is diagonal with diagonal entries λ_1, λ_2. So A sends $\binom{1}{1} \mapsto \binom{\lambda_1}{\lambda_2}$, and we must have $\lambda_1 = \lambda_2$. It follows that $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) = \lambda I$ for some λ, and $\overline{A} = 1 \in \mathrm{PGL}(2,p)$. The action is faithful.

An analogous argument applies for the action of $\mathrm{PSL}(2,p)$. $\square$

2.2. **The isomorphisms for** $\mathrm{SL}(2,2)$ **and** $\mathrm{PSL}(2,3)$**.** The first two exceptional isomorphisms are straightforward once we establish the appropriate group actions.

Lemma 2.4. A_n *is the only subgroup of* S_n *of index 2.*

Proof. Let $G < S_n$ have index 2. Then G has prime index, hence is normal. If it contains any 2-cycle, it will contain the conjugacy class of all 2-cycles, which generate S_n; this contradicts G being proper. So G contains no 2-cycles, and the quotient map $\pi : S_n \to S_n/G = \{\pm 1\}$ sends all 2-cycles to -1. Therefore π is the sign homomorphism, and $G = \ker \pi = A_n$. $\square$

Theorem 2.5. $\mathrm{SL}(2,2) \cong S_3$ *and* $\mathrm{PSL}(2,3) \cong A_4$*.*

Proof. For the first isomorphism, notice that each $M \in \mathrm{SL}(2,p)$ is invertible, so it fixes the origin and permutes the other points of $\mathbb{F}_p^2$. In other words, $\mathrm{SL}(2,2)$ acts on the three nonzero points of $\mathbb{F}_2^2$ (by left matrix multiplication). Only the identity matrix fixes all nonzero points (and thus all of $\mathbb{F}_2^2$), so this action is faithful. If we index these 3 nonzero points, we get an injective homomorphism $\mathrm{SL}(2,2) \to S_3$, i.e. an isomorphism from $\mathrm{SL}(2,2)$ to a subgroup $H \leq S_3$ of order $|\mathrm{SL}(2,2)| = 6$. But $|S_3| = 6$, so $H = S_3$.

For the second homomorphism, we consider the faithful action of $\mathrm{PSL}(2,3)$ on $\mathbf{P}^1(\mathbb{F}_3)$ from Proposition 2.3. If we label the 4 elements of $\mathbf{P}^1(\mathbb{F}_3)$, then it yields an injective homomorphism $\mathrm{PSL}(2,3) \to S_4$, i.e. an isomorphism from $\mathrm{PSL}(2,3)$ to a subgroup $H \leq S_4$ of order $|\mathrm{PSL}(2,3)| = 60$. But $|S_4| = 120$, so H has index 2, and thus $H = A_5$ by Lemma 2.4. $\square$

2.3. **The isomorphism for** $\mathrm{PSL}(2,5)$**.** For the remaining isomorphism, our strategy is slightly different. There is no obvious set of 5 elements on which $\mathrm{PSL}(2,5)$ acts faithfully; we will instead find two actions and compose them to get an homomorphism of $\mathrm{PGL}(2,5)$ to S_5.

Theorem 2.6. $\mathrm{PGL}(2,5) \cong S_5$ *and* $\mathrm{PSL}(2,5) \cong A_5$*.*

Proof. The second isomorphism will follow from the first: the subgroup $\mathrm{PSL}(2,5)$ has index 2, so it will be isomorphic to an index 2 subgroup of S_5, hence A_5.

We first consider the faithful action of $\mathrm{PGL}(2,5)$ on $\mathbf{P}^1(\mathbb{F}_5)$ from Proposition 2.3. Let us index the elements of $\mathbf{P}^1(\mathbb{F}_5)$ as

$$\begin{aligned} p_1 &= \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \quad p_2 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle, \quad p_3 = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle, \\ p_4 &= \left\langle \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\rangle, \quad p_5 = \left\langle \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\rangle, \quad p_6 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle. \end{aligned} \tag{2.1}$$

Then we obtain an isomorphism $\rho : \mathrm{PGL}(2,5) \to H$ to some subgroup $H \leq S_6$. Denote the image of $\overline{M} \in \mathrm{PGL}(2,5)$ as $\rho_{\overline{M}} \in S_6$.

Consider the $(2,2,2)$-cycles in S_6; there are 15 of them. Call them *flips*. They are the permutations of order 2 that fix no indices.

Lemma 2.7. *H contains 10 flips. They are the images under ρ of the elements of* $\mathrm{PGL}(2,5)$ *represented by the 10 matrices*

$$\begin{aligned} U = &\left\{ \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & b \\ b^{-1} & -1 \end{pmatrix} : b \neq 0 \right\} \\ &\cup \left\{ \begin{pmatrix} 1 & b \\ 2b^{-1} & -1 \end{pmatrix} : b \neq 0 \right\}. \end{aligned} \tag{2.2}$$

Proof (of the lemma). Let $\overline{M} \in \mathrm{PGL}(2,5)$ be such that $\rho_{\overline{M}}$ is a flip. Via the isomorphism ρ, this is to say that $\overline{M}$ has order 2 and fixes no element of $\mathbf{P}^1(\mathbb{F}_5)$. In turn, this is equivalent to the following conditions on any representative $M \in \overline{M}$:

(i) $\det M \neq 0$.
(ii) $M^2 = \lambda I$ for some nonzero $\lambda \in \mathbb{F}_5$.
(iii) M has no eigenvalues (over $\mathbb{F}_5$).

These respectively mean that $M \in \mathrm{GL}(2,5)$, that $\overline{M}$ has order 2, and that $\overline{M}$ does not fix any 1-dimensional subspaces. We consider two cases. Suppose that $\overline{M}$ sends $\left\langle \binom{1}{0} \right\rangle \mapsto \left\langle \binom{0}{1} \right\rangle$; choose a representative M such that $M\binom{1}{0} = \binom{0}{1}$. Then M has the form

$$M = \begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix} \tag{2.3}$$

for some $b, d \in \mathbb{F}_5$. We compute $M^2 = \left(\begin{smallmatrix} b & bd \\ d & b+d^2 \end{smallmatrix}\right)$. Here (ii) implies $d = 0$ and $b \neq 0$. Now, recall that the eigenvalues of M are precisely the roots (over $\mathbb{F}_5$) of the characteristic polynomial of M. We have $\operatorname{tr} M = 0$ and $\det M = -b$, so this polynomial is $p(x) = x^2 - b$. It has roots in $\mathbb{F}_5$ if and only if $b \in \{-1, 0, 1\}$. Hence (iii) implies $b \in \{\pm 2\}$. These choices of b yield the first two matrices in (2.2), and we promptly check that they satisfy conditions (i)-(iii).

Suppose instead that $\overline{M}$ sends $\langle \binom{1}{0} \rangle \mapsto \langle \binom{1}{c} \rangle$ for some $c \in \mathbb{F}_5$; choose M such that $M\binom{1}{0} = \binom{1}{c}$. Then

$$M = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \tag{2.4}$$

for some $b, d \in \mathbb{F}_5$. We compute $M^2 = \left(\begin{smallmatrix} 1+bc & b(1+d) \\ c(1+d) & d^2+bc \end{smallmatrix}\right)$. Then (ii) implies that $1 + bc = d^2 + bc$, i.e. $d \in \{\pm 1\}$, as well as $b(1 + d) = 0 = c(1 + d)$. If $d = 1$, we must have $b = c = 0$ and $M = I$, contradicting (iii). Hence $d = -1$ and $M = \left(\begin{smallmatrix} 1 & b \\ c & -1 \end{smallmatrix}\right)$.

Now we have $\operatorname{tr} M = 0$ and $\det M = -1 - bc$, so the characteristic polynomial of M is $p(x) = x^2 + (1 + bc)$, which has roots in $\mathbb{F}_5$ if and only if $bc \in \{-2, -1, 0\}$. Like earlier, (iii) implies $bc \in \{1, 2\}$. We are free to choose $b \neq 0$, and each choice fixes c as either b^{-1} or $2b^{-1}$. These choices of b and c yield the remaining 8 matrices in (2.2), and we can verify that they satisfy conditions (i)-(iii), as desired. It is also clear that all 10 matrices correspond to different classes in $\mathrm{PGL}(2, 5)$. □

Let $T \subset S_6$ be the set of the $5 = 15 - 10$ remaining flips which are not in H. Using the indexing (2.1), we can explicitly compute (in cycle notation) which flips lie in H, as well as which lie in T, by inspecting the action of each matrix from Lemma 2.7. We find that

$$\begin{aligned} T = \{&(1\,6)(2\,5)(3\,4), (2\,6)(4\,5)(1\,3), (3\,6)(2\,4)(1\,5), \\ &(4\,6)(3\,5)(1\,2), (5\,6)(2\,3)(1\,4)\}. \end{aligned} \tag{2.5}$$

We claim that H acts on T by conjugation. Indeed, for any $h \in H$ and $t \in T$, the conjugate hth^{-1} is again a $(2, 2, 2)$-cycle, and it is not in H, as otherwise $t = h^{-1}(hth^{-1})h \in H$, absurd. This way, for any $h \in H$, the conjugation map by h restricts to a bijection $T \to T$. The action is well-defined. It is also faithful: suppose $h \in H$ fixes all elements of T. We must have

$$\begin{aligned} (h(1)\,h(6))(h(2)\,h(5))(h(3)\,h(4)) &= (1\,6)(2\,5)(3\,4), \\ (h(2)\,h(6))(h(4)\,h(5))(h(1)\,h(3)) &= (2\,6)(4\,5)(1\,3), \\ (h(3)\,h(6))(h(2)\,h(4))(h(1)\,h(4)) &= (3\,6)(2\,4)(1\,5), \\ (h(4)\,h(6))(h(3)\,h(5))(h(1)\,h(2)) &= (4\,6)(3\,5)(1\,2), \\ (h(5)\,h(6))(h(2)\,h(3))(h(1)\,h(4)) &= (5\,6)(2\,3)(1\,4). \end{aligned}$$

We compare cycle notations. Suppose h does not fix 1. Say $h(1) = 2$; then the fourth equation forces $h(2) = 1$, and now the second forces $h(3) = 6$ while the fifth forces $h(3) = 4$, absurd. A symmetrical argument applies for $h(1) = 3, 4, 5, 6$, and it follows that h must fix 1. Then the fourth equation forces $h(2) = 2$, the fifth forces $h(3) = 3$, the first forces $h(4) = 4$, and hence h must be the identity.

If we label the elements of T, this action of H finally gives us an isomorphism from H to a subgroup of S_5 of order $|H| = |\operatorname{PGL}(2,5)| = 120 = |S_5|$, i.e. to S_5 itself. Composing isomorphisms $\operatorname{PGL}(2,5) \to H \to S_5$ yields $\operatorname{PGL}(2,5) \cong S_5$, which proves the theorem. □

Remark 2.8. When we identify $\operatorname{PGL}(2,5)$ with S_5, the second isomorphism above describes an *exotic* embedding $S_5 \to S_6$. Unlike the usual embeddings that fix one index, this copy of S_5 is transitive on the six indices, since the action of $\operatorname{PGL}(2,5)$ is transitive. One can show that S_6 acts faithfully on its 6 cosets by (left) multiplication, yielding an automorphism $S_6 \to S_6$. In fact, this turns out to be the only *outer automorphism* $S_n \to S_n$ for any $n \geq 1$.

3. Character tables

The isomorphisms above tell us that we may understand more about the representations of $\operatorname{SL}(2,2)$, $\operatorname{PSL}(2,3)$ and $\operatorname{PGL}(2,5)$ by studying those of the isomorphic groups S_3, A_4 and A_5. This motivates us to compute character tables by focusing exclusively on the latter groups; the advantage is that their conjugacy classes are significantly easier to work with.

3.1. **Characters and conjugacy classes.** Let G be a finite group. Recall that there is a natural inner product $\langle \cdot, \cdot \rangle$ for (complex) characters of G: if χ, χ' are characters, we define

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi'(g)} = \frac{1}{|G|} \sum_{g \in \mathcal{C}} |C(g)|\chi(g)\overline{\chi'(g)}, \tag{3.1}$$

where $C(g)$ is the conjugacy class of g in G, and the last sum is taken over a complete set $\mathcal{C}$ of representatives of conjugacy classes of G.

Proposition 3.1. *Let G be a finite group; for $g \in G$, let $C(g)$ be the conjugacy class of g in G. Let $\chi_1, \ldots, \chi_r$ be the irreducible (complex) characters of G, and $d_1, \ldots, d_r$ their dimensions.*

(i) *Let $g \in G$, and let χ be a character of G of dimension d. Then $\chi(g^{-1}) = \overline{\chi(g)}$. If g has order k in G, then $\chi(g)$ is a sum of d terms, each a k-th root of unity.*

(ii) (Row orthogonality)
$$\langle \chi_i, \chi_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) (Column orthogonality)
$$\sum_{i=1}^{r} \chi_i(g)\overline{\chi_i(g')} = \begin{cases} |G|/|C(g)| & \text{if } C(g) = C(g'), \\ 0 & \text{otherwise.} \end{cases}$$

(iv) (Regular character) *Let $\chi_{reg} = d_1\chi_1 + \ldots + d_r\chi_r$. Then $\chi_{reg}(1) = |G|$ and $\chi_{reg}(g) = 0$ for $g \neq 1$. In particular, $d_1^2 + \ldots + d_r^2 = |G|$.*

(v) (Permutation representations) *Let ρ be a permutation representation with character χ. Then for each $g \in G$, $\chi(g)$ is the number of fixed points of the action of g.*

(vi) (Irreducible decomposition) *Let ρ be a representation with character χ that decomposes as $\rho = n_1\rho_1 \oplus \ldots \oplus n_r\rho_r$, where each ρ_i has character χ_i. Then $\langle \chi, \chi_i \rangle = n_i$ and $\langle \chi, \chi \rangle = n_1^2 + \ldots + n_2^2$. In particular, if $\langle \chi, \chi \rangle = 1$ then χ is irreducible, and if $\langle \chi, \chi \rangle = 2$ then χ is the sum of two irreducible characters.*

Proof. See [A] pp. 298-306, and also [J-L] pp. 117-167 for an approach using $\mathbb{C}G$-modules. We prove (i) and (v) here.

(i) Let $g \in G$ have order k, let $\rho : G \to GL(V)$ have character χ, and let $\rho(g)$ have eigenvalues $\lambda_1, \ldots, \lambda_d$. Then $\rho(g)^k = \rho(g^k) = I$, and each λ_i^k must be an eigenvalue of I, i.e. $\lambda_i^k = 1$. So each λ_i must be a k-th root of unity and $\chi(g) = \operatorname{tr} \rho(g) = \lambda_1 + \ldots + \lambda_d$ is the claimed sum.

Moreover, the eigenvalues of $\rho(g^{-1}) = \rho(g)^{-1}$ are $\lambda_1^{-1}, \ldots, \lambda_d^{-1}$. Since $|\lambda_i| = 1$, it follows that $\lambda_i^{-1} = \overline{\lambda_i}$ and $\chi(g^{-1}) = \overline{\lambda_1^{-1}} + \ldots + \overline{\lambda_d^{-1}} = \overline{\chi(g)}$.

(v) Say ρ comes from the action of G on elements $x_1, \ldots, x_n$. Let $A = \rho(g)$ have entries $\{a_{ij}\}$. Then a_{ij} is 1 if $gx_i = x_j$ and 0 otherwise. This means that the sum $\chi(g) = \operatorname{tr} A = a_{11} + \ldots + a_{nn}$ counts the number of indices i such that $gx_i = x_i$, that is, the elements fixed by g. $\square$

We know that the conjugacy classes in S_n are given by cycle type, but we will also need to know what they are in alternating groups. Let $n \geq 1$, and let $C_G(x), Z_G(x)$ denote the conjugacy class and centralizer (respectively) of x in the group G.

Proposition 3.2. *Let $x \in A_n$. If $Z_{S_n}(x)$ contains an odd permutation, then $C_{A_n}(x) = C_{S_n}(x)$. Otherwise, $C_{A_n}(x)$ and $C_{A_n}((1\,2)^{-1}x(1\,2))$ are distinct, of equal-size, and their union is $C_{S_n}(x)$.*

Proof. We follow [J-L] pp. 111-112. Suppose some $g \in Z_{S_n}(x)$ is odd. Say $y \in C_{S_n}(x)$, i.e. $y = h^{-1}xh$ for some h. If h is even, then $y \in C_{A_n}(x)$ already. Otherwise, gh is even and $(gh)^{-1}h(gh) = h^{-1}(g^{-1}xg)h = h^{-1}xh = y$, so that $y \in C_{A_n}(x)$ as well. Then $C_{S_n}(x) \subseteq C_{A_n}(x)$ and equality follows.

Now suppose $Z_{S_n}(x)$ only contains even permutations, that is, $Z_{S_n}(x) = Z_{A_n}(x)$. We have

$$C_{S_n}(x) = \{hxh^{-1} : h \in S_n \text{ odd}\} \cup \{hxh^{-1} : h \in S_n \text{ even}\}. \tag{3.2}$$

The second set in the union is $C_{A_n}(x)$. Let $t = (1\,2)$; we claim the first set is $C_{A_n}(t^{-1}xt)$. Indeed, write each odd h as $h = t\ell$ for each even ℓ. (The map $\ell \to t\ell$ is a bijection from even to odd permutations.) Then

$$\{hxh^{-1} : h \in S_n \text{ odd}\} = \{\ell^{-1}(t^{-1}xt)\ell : \ell \in S_n \text{ even}\} = C_{A_n}(t^{-1}xt).$$

Hence (3.2) becomes

$$C_{S_n}(x) = C_{A_n}(x) \cup C_{A_n}(t^{-1}xt). \tag{3.3}$$

Moreover, by the orbit-stabilizer theorem, we have $|C_{A_n}(x)| = |A_n|/|Z_{A_n}(x)| = |S_n|/2|Z_{S_n}(x)| = |C_{S_n}(x)|/2$, and similarly $|C_{A_n}(t^{-1}xt)| = |C_{S_n}(t^{-1}xt)|/2 = |C_{S_n}(x)|/2$. It follows that the union (3.3) partitions $C_{S_n}(x)$ in half. $\square$

3.2. **The character tables of S_3 and A_4.** Let us first look at S_3. There are three conjugacy classes: the trivial class, the 2-cycles and the 3-cycles, of sizes 1, 3, 2, respectively. So there are three distinct irreducible characters χ_1, χ_2, χ_3, with dimensions d_1, d_2, d_3. Proposition 3.1(iv) yields $d_1^2 + d_2^2 + d_3^2 = 6$, which forces (without loss of generality) $d_1 = d_2 = 1$ and $d_3 = 2$.

Now, the trivial and sign homomorphisms $S_3 \to \{\pm 1\} \to \mathbb{C}$ give two distinct representations of S_3 of dimension 1, hence irreducible and equal to their characters. These must be χ_1, χ_2 in some order. Say, let χ_1 be trivial; then χ_3 is determined by the regular character decomposition $\chi_{\text{reg}} = \chi_1 + \chi_2 + 2\chi_3$.

Theorem 3.3. *The character table of S_3 is given by table (3.4) below.*

(3.4)

	[1]	[3]	[2]
	1	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

□

Next, we look at A_4. Its permutations consist of the identity, the 3-cycles, and the $(2,2)$-cycles. Clearly, $(1\,2)(3\,4)$ commutes with $(1\,2)$. On the other hand, suppose $c = (1\,2\,3)$ commuted with some odd t, so that $tct^{-1} = c$. Then $(1\,2\,3) = (t(1)\,t(2)\,t(3))$. Comparing cycle notations, we must have $t(4) = 4$, and $t(1) = 1, 2, 3$ forces $t(2) = 2, 3, 1$, respectively. This means that $t \in \{1, c, c^2\}$. So $t^3 = 1$ is even, and t must be even, a contradiction.

It follows from Proposition 3.2 that A_4 has four conjugacy classes, which are represented by 1, $(1\,2)(3\,4)$, $(1\,2\,3)$ and $(2\,1\,3)$. Their sizes are 1, 3, 4, 4, respectively (there are 8 3-cycles in total). So there are 4 distinct irreducible characters, say $\chi_1, \chi_2, \chi_3, \chi_4$, with dimensions d_i; choose labels such that χ_1 is trivial and $d_1 \leq \ldots \leq d_4$. By Proposition 3.1(iv), we have $d_1^2 + \ldots + d_4^2 = 12$; by inspection, this forces $d_1 = d_2 = d_3 = 1$ and $d_4 = 3$.

	[1]	[3]	[4]	[4]
	1	(1 2)(3 4)	(1 2 3)	(2 1 3)
χ_1	1	1	1	1
χ_2	1	a	b	c
χ_3	1	a'	b'	c'
χ_4	3			

Let us find the second and third rows, which correspond to 1-dimensional characters. Let ζ be a complex cube root of unity. The $(2,2)$-cycles have order 2 and the 3-cycles have order 3, so by Proposition 3.1(i), we have $a, a' \in \{\pm 1\}$ and $b, b', c, c' \in \{1, \zeta, \zeta^2\}$. By row orthogonality, we have $0 = \langle \chi_2, \chi_1 \rangle = 1 + 3a + 4b + 4c$. In particular, $1 + 3a$ is real, so $4b + 4c$ is real, and thus $b + c$ is real. By inspection, this can only happen when b, c are ζ, ζ^2 in some order. This will also imply $b + c = \zeta + \zeta^2 = -1$ and $0 = 1 + 3a - 4$, i.e. $a = 1$.

An identical argument applies when replacing χ_2, a, b, c with χ_3, a', b', c'. Thus $a = a' = 1$ and b, b' are complex cube roots of unity, while $c = b^2, c' = (b')^2$. This leaves only two possibilities for the nontrivial 1-dimensional characters, so they must correspond to χ_2, χ_3 in some order. The remaining character χ_4 is completely determined by the regular character decomposition $\chi_{\text{reg}} = \chi_1 + \chi_2 + \chi_3 + 3\chi_4$.

Theorem 3.4. *The character table of A_4 is given by table (3.5) below.*

(3.5)

	[1] 1	[3] (1 2)(3 4)	[4] (1 2 3)	[4] (2 1 3)
χ_1	1	1	1	1
χ_2	1	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$
χ_3	1	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$
χ_4	3	-1	0	0

□

3.3. **The character table of A_5.** A_5 consists of the identity, the 3-cycles, the $(2, 2)$-cycles, and the 5-cycles. Clearly $(1\,2\,3)$ and $(1\,2)(4\,5)$ both commute with $(4\,5)$. Meanwhile, suppose $c = (1\,2\,3\,4\,5)$ commuted with some odd t, so $t^{-1}ct = c$, i.e. $(1\,2\,3\,4\,5) = (t(1)\,t(2)\,t(3)\,t(4)\,t(5))$. By trying values for $t(1)$ and requiring that the cycle notations agree, we see that $t \in \{1, c, c^2, c^3, c^4\}$, hence $t^5 = 1$ is even, a contradiction: t odd implies t^5 odd.

Thus, by Proposition 3.2, A_5 has five conjugacy classes: the trivial class, the $(2, 2)$-cycles, the 3-cycles, and two classes of 5-cycles. Their sizes are $1, 15, 20, 12, 12$, respectively (note that there are 24 5-cycles in total), and they are represented by $1, (1\,2)(3\,4), (1\,2\,3), (1\,2\,3\,4\,5), (2\,1\,3\,4\,5)$.

So there are 5 distinct irreducible characters of A_5, say $\chi_1, \chi_2, \ldots, \chi_5$; let χ_1 be the trivial character. Let d_i be their dimensions, and label so that $d_1 \leq \ldots \leq d_5$. By Proposition 3.1(iv), we have $d_1^2 + d_2^2 + \ldots + d_5^2 = |A_5| = 60$, where $d_1 = 1$. Since $8^2 > 60$ and each d_i divides 60, we have $d_i \leq 6$ for each i. By inspection, we find that the unique values satisfying the sum are $d_1 = 1, d_2 = 3, d_3 = 3, d_4 = 4, d_5 = 5$.

	[1] 1	[15] (1 2)(3 4)	[20] (1 2 3)	[12] (1 2 3 4 5)	[12] (2 1 3 4 5)
χ_1	1	1	1	1	1
χ_2	3				
χ_3	3				
χ_4	4				
χ_5	5				

To get started, we find some nontrivial representation and decompose it. Consider the usual action of A_5 on the set of 5 indices by permutation; by Proposition 3.1(v), this yields a representation $\rho_{\text{perm}} : A_5 \to GL_5(\mathbb{C})$ of dimension 5, with character

χ_{perm}. Furthermore, for all $g \in A_5$, $\chi_{\text{perm}}(g)$ is the number of indices fixed by g.

(3.6)

	[1] 1	[15] (1 2)(3 4)	[20] (1 2 3)	[12] (1 2 3 4 5)	[12] (2 1 3 4 5)
χ_1	1	1	1	1	1
χ_{perm}	5	1	2	0	0

We compute $\langle \chi_{\text{perm}}, \chi_{\text{perm}} \rangle = 2$ and $\langle \chi_{\text{perm}}, \chi_1 \rangle = 1$. By Proposition 3.1(vi), these respectively mean that χ_{perm} is the sum of two (distinct) irreducible characters, and that χ_1 is one of them. Therefore, $\chi_{\text{perm}} - \chi_1$ is an irreducible character. It has dimension 4, so it must be χ_4.

	[1] 1	[15] (1 2)(3 4)	[20] (1 2 3)	[12] (1 2 3 4 5)	[12] (2 1 3 4 5)
χ_1	1	1	1	1	1
χ_2	3	a'	a		
χ_3	3	b'	b		
χ_4	4	0	1	-1	-1
χ_5	5	c'	c		

Proposition 3.5. *Every element of A_5 is conjugate (in A_5) to its inverse.*

Proof. The inverse of a permutation has the same cycle type, so we must only check for the 5-cycles. And indeed: if $(a\,b\,c\,d\,e)$ is a 5-cycle, then conjugating by $(a\,e)(b\,d) \in A_5$ yields its inverse $(e\,d\,c\,b\,a)$. □

Propositions 3.1(i) and 3.5 imply that $\chi_i(g) = \overline{\chi_i(g)}$ for all $g \in A_5$ and all χ_i, so all the entries in the character table are real. In particular, consider the entries a, b, c in the column of the class of 3-cycles. A 3-cycle has order 3, so by Proposition 3.1(i), a, b, c are real sums of three cube roots of unity. If ζ is a complex cube root of unity, then $\zeta^2 = \overline{\zeta}$ and $\zeta + \overline{\zeta} = -1$, and it follows that for each sum to be real, it must be integer. By column orthogonality, we have $1^2 + a^2 + b^2 + 1^2 + c^2 = 60/20 = 3$, forcing two of a, b, c to be zero and the other to be ± 1. By the regular character decomposition, we also have $1 + 3a + 3b + 4 + 5c = 0$, and it follows that $a = b = 0$ and $c = -1$.

Now we look at the column of $(2,2)$-cycles, with entries a', b', c'. A $(2,2)$-cycle has order 2, so a', b', c' are (real) sums of ± 1, in fact integers. Column orthogonality yields $1^2 + (a')^2 + (b')^2 + 0^2 + (c')^2 = 60/15 = 4$, so $a', b', c' \in \{\pm 1\}$. The regular character decomposition yields $1 + 3a' + 3b' + 0 + 5c' = 0$, implying $a' = b' = -1$ and $c' = 1$.

	[1] 1	[15] (1 2)(3 4)	[20] (1 2 3)	[12] (1 2 3 4 5)	[12] (2 1 3 4 5)
χ_1	1	1	1	1	1
χ_2	3	-1	0	d	d'
χ_3	3	-1	0	e	e'
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	f	f'

Six entries remain. By row orthogonality, we have $1 = \langle \chi_5, \chi_5 \rangle = 1 + (f^2 + (f')^2)/60$. But f, f' are real, which forces $f^2 + (f')^2 = 0$ and $f = f' = 0$. On the other hand, row orthogonality again gives $0 = \langle \chi_1, \chi_2 \rangle = 12d + 12e - 12$, so $d' = 1 - d$. It also gives $1 = \langle \chi_2, \chi_2 \rangle = (24 + 12d^2 + 12(d')^2)/60$, so $d^2 + (1-d)^2 = 3$, or $d^2 - d - 1 = 0$. Let

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} \tag{3.7}$$

be the two distinct (roots) of this last polynomial. Then $d \in \{\alpha, \beta\}$ and $d' = 1 - d$. Moreover, we may replace χ_2, d, d' with χ_3, e, e' in the above argument to deduce that $e \in \{\alpha, \beta\}$ and $e' = 1 - e$. This completely determines χ_2 and χ_3. Without loss of generality, we may let $d = \alpha$ and $e = \beta$.

Theorem 3.6. *The character table of A_5 is given in Table (3.8) below.*

(3.8)

	[1] 1	[15] (1 2)(3 4)	[20] (1 2 3)	[12] (1 2 3 4 5)	[12] (2 1 3 4 5)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

□

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