CONSTRUCTING THE DISCRETE SERIES REPRESENTATION OF $\mathrm{GL}(2, \mathbb{F}_q)$

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1. Introduction

In this paper we construct the *discrete series representations*, a collection of irreducible representations of $\mathrm{GL}(2, \mathbb{F}_q)$. The representations arise from one-dimensional representations ν of a subgroup $K \leq \mathrm{GL}(2, \mathbb{F}_q)$ which can be likened to the orthogonal subgroup $\mathrm{O}(2, \mathbb{R}) \leq \mathrm{GL}(2, \mathbb{R})$. The manner in which the discrete series representation is obtained from a representation of K is not immediately obvious; for example, taking the induced representation from K to $\mathrm{GL}(2, \mathbb{F}_q)$ yields a reducible representation. However, for a certain class of "indecomposable" representations $\nu\colon K \to \mathbb{C}^\times$, one obtains an irreducible representation $\sigma_\nu\colon \mathrm{GL}(2, \mathbb{F}_q) \to \mathrm{GL}(V)$ where V is the vector space of functions $\mathbb{F}_q^\times \to \mathbb{C}$ (see Chapter 21 of [T]). To construct this representation, we define σ_ν on the generators of a particular presentation of $\mathrm{GL}(2, \mathbb{F}_q)$, and check that σ_ν respects the relations of the presentation.

2. Representations of a Subgroup of $\mathrm{GL}(2, \mathbb{F}_q)$

Suppose $q \neq 2$. One may pick a nonsquare element $\delta \in \mathbb{F}_q$[1] and write $\mathbb{F}_{q^2} = \mathbb{F}_q[\sqrt{\delta}]$, defining the subgroup

$$K = \left\{ \begin{bmatrix} a & b\delta \\ b & a \end{bmatrix} \in \mathrm{GL}(2, \mathbb{F}_q) \right\}. \tag{2.1}$$

One has an isomorphism $\phi\colon \mathbb{F}_{q^2}^\times \to K$ given by

$$a + b\delta \mapsto \begin{bmatrix} a & b\delta \\ b & a \end{bmatrix}. \tag{2.2}$$

Checking this is simple; observe that $\left[\begin{smallmatrix} a & b\delta \\ b & a \end{smallmatrix}\right]$ is the matrix representation of the linear map $\mathbb{F}_q[\sqrt{\delta}]$ given by multiplication by $a + b\delta$. Here, the basis of $\mathbb{F}_q[\sqrt{\delta}]$ as an $\mathbb{F}_q$-vector space is chosen to be $\{1, \delta\}$.

Definition 2.1. A multiplicative character $\nu\colon \mathbb{F}_q[\sqrt{\delta}]^\times \to \mathbb{C}$ is said to be **decomposable** if $\nu = \chi \circ N$, where χ is a multiplicative character of $\mathbb{F}_q^\times$ and N is the norm, defined by $N = \det \circ \phi$.

Essentially, a decomposable character is given by the "pullback" of a representation π of $\mathbb{F}_q^\times$ to a representation $\pi \circ N$ of $\mathbb{F}_{q^2}^\times$. In order to determine when a given multiplicative character is decomposable we introduce the following lemmas:

Proposition 2.2. *The norm map $N\colon \mathbb{F}_q[\sqrt{\delta}]^\times \to \mathbb{F}_q^\times$ is a surjective homomorphism.*

[1] Such δ exists for $q > 2$. To see, note that the squaring map is not injective since $(-x)^2 = x^2$, hence its image is not the entire field.

Proof. It is well known that the Galois group $\mathrm{Gal}(\mathbb{F}_q[\sqrt{\delta}]^\times/\mathbb{F}_q^\times)$ is generated by the Frobenius automorphism $\alpha \mapsto \alpha^q$. But the Galois group only contains two elements: the identity and the conjugation map swapping $\delta \mapsto -\delta$. Thus $N(\alpha) = \alpha\overline{\alpha} = \alpha^{q+1}$. By considering primitive elements, can view $N\colon \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$, as a map $\mathbb{Z}/(q^2-1)\mathbb{Z} \to \mathbb{Z}/(q-1)\mathbb{Z}$ given by multiplication by $q+1$. Hence, the kernel of N consists of $(q^2-1)/(q-1) = q+1$ elements. Consequently the image of N consists of $(q^2-1)/(q+1) = q-1$ elements, which is the order of $\mathbb{F}_q^\times$. Hence N is surjective. □

Proposition 2.3. *The map $\psi\colon \mathbb{F}_q[\sqrt{\delta}]^\times \to \ker N$ taking $\alpha \mapsto \overline{\alpha}\alpha^{-1}$ is surjective.*

Proof. The map $\psi\colon \mathbb{F}_q[\sqrt{\delta}]^\times \to \ker N$ taking $\alpha \mapsto \overline{\alpha}\alpha^{-1}$ is a homomorphism, as a product of homomorphisms into an abelian group. The kernel is given by

$$\ker \phi = \{\alpha \mid \overline{\alpha} = \alpha\} = \mathbb{F}_q^\times,$$

which has order $q-1$. Thus the image of ψ has order $(q^2-1)/(q-1) = q+1$, which is the order of $\ker N$(see Proposition 2.2). Hence ψ is surjective. □

Lemma 2.4. *A multiplicative character ν of $\mathbb{F}_q[\sqrt{\delta}]^\times$ is decomposable if and only if $\nu(\alpha) = \nu(\overline{\alpha})$ for all $\alpha \in \mathbb{F}_q[\sqrt{\delta}]$.*

Proof. Suppose that $\nu = \chi \circ N$, where N is the norm from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$. Since the norm is preserved under conjugation, one obtains that ν is preserved under conjugation as well.

Conversely, suppose that ν is preserved under conjugation. Suppose that $N(\alpha) = N(\beta)$ for $\alpha, \beta \in \mathbb{F}_q[\sqrt{\delta}]$. Then $N(\alpha\beta^{-1}) = 1$, so by Proposition 2.3, there exists $\gamma \in \mathbb{F}_q[\sqrt{\delta}]$ such $\overline{\gamma}\gamma^{-1} = \beta$. Thus $\nu(\alpha\beta^{-1}) = \nu(\overline{\gamma}\gamma^{-1}) = 1$. Hence $\nu(\alpha) = \nu(\beta)$ by multiplicativity and the fact that ν is preserved under conjugation. Hence, using the fact N is surjective, one may factor ν as $\nu = \chi \circ N$, where $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ is some function. We finish by showing that χ is a multiplicative character, implying ν is decomposable. To see this, observe that for $x, y \in \mathbb{F}_q$, there exists $\alpha, \beta \in \mathbb{F}_q[\sqrt{\delta}]$ such that $N(\alpha) = x$ and $N(\beta) = y$, yielding

$$\chi(xy) = \chi(N(\alpha\beta)) = \nu(\alpha\beta) = \nu(\alpha)\nu(\beta) = \chi(x)\chi(y). \tag{2.3}$$

□

The "obvious" way to obtain a representation from a one-dimensional representation ν defined on a subgroup of K, taking the induced representation, unfortunately does not yield an irreducible representation.

Proposition 2.5. *Two elements $g, g' \in K$ are conjugate in $\mathrm{GL}(2, \mathbb{F}_q)$ if and only if g, g' correspond to conjugates in $\mathbb{F}_q[\sqrt{\delta}]$.*

Proof. Suppose $g, g' \in K$ are conjugates in G, that is, there exists $x \in G$ such that $xgx^{-1} = g'$. Then g, g' are similar matrices, and hence share the same trace and determinant. As elements of a quadratic extension with the same trace and norm, this characterizes $\phi^{-1}(g)$ and $\phi^{-1}(g')$ up to conjugacy.

The converse follows similarly, as matrices of dimension 2 with the same trace and determinant are characterized up to similarity. □

Lemma 2.6. *Let ν be a (indecomposable) multiplicative character of $\mathbb{F}_{q^2}^\times \cong K$. Then the induced representation $\pi = \operatorname{Ind}_K^G \nu$ is reducible, where $G = \mathrm{GL}(2, \mathbb{F}_q)$.*

Proof. First, note that by Proposition 2.5, two elements g, g' in the subgroup K are conjugate in G, if and only if g, g' correspond to conjugates in $\mathbb{F}_q[\sqrt{\delta}]$. Again, we shall denote the conjugate of g by $\overline{g}$. Then $\chi_\pi(g) = \chi_\pi(\overline{g})$, which implies that $\chi_\nu(g) = \chi_\nu(\overline{g})$, so by Lemma 2.4, ν is decomposable, which contradicts the assumption of indecomposability. Relaxing this assumption still leads to a reducible representation, however.

We shall use the Frobenius formula to calculate $\langle \chi_\pi, \chi_\pi \rangle$. Given $g \in K$, one has by the Frobenius formula

$$\chi_\pi(g) = \frac{1}{|K|} \sum_{x \in G} \widetilde{\chi_\nu}(xgx^{-1}) = \begin{cases} \chi_\nu(g) + \chi_\nu(\overline{g}) & \text{if } g \text{ is elliptic} \\ (q^2 - q)\chi_\nu(g) & \text{if } g \text{ is central} \end{cases}. \tag{2.4}$$

In the first case, we used the fact that $xgx^{-1} \in K$ if and only if $xgx^{-1} \in \{g, \overline{g}\}$, and that the centralizer of g has order $|G|/(q^2 - q) = q^2 - 1$. In the second case, we used the fact that $xgx^{-1} \in K$ if and only if $xgx^{-1} = g$, and that the centralizer has size $|G|/1 = (q^2 - q)(q^2 - 1)$. One may further write $\chi_\pi(g) = 2\chi_\nu(g)$, since $g, \overline{g}$ are conjugate.

From the Frobenius formula, one sees that χ_π vanishes for elements not in an elliptic or central conjugacy class, hence

$$\begin{aligned} \langle \chi_\pi, \chi_\pi \rangle &= \sum_{g \in K_{\text{central}}} (q^2 - q)^2 \chi_\nu(g)^2 + \sum_{g \in K_{\text{elliptic}}} \frac{q(q-1)}{2} \cdot 4\chi_\nu(g)^2 \\ &> \sum_{g \in K_{\text{central}}} (q^2 - q)\chi_\nu(g)^2 + \sum_{g \in K_{\text{elliptic}}} (q^2 - q)\chi_\nu(g)^2 \\ &= (q^2 - q)\langle \chi_\nu, \chi_\nu \rangle \\ &= (q^2 - q)(q^2 - 1) \\ &= |G|. \end{aligned} \tag{2.5}$$

Since $\langle \chi_\pi, \chi_\pi \rangle \neq |G|$, the induced representation is not irreducible. ☐

3. The Discrete Series Representation

Definition 3.1. The **Borel subgroup** B is the subgroup of $\mathrm{GL}(2, \mathbb{F}_q)$ defined by

$$B = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathrm{GL}(2, \mathbb{F}_q) \right\}. \tag{3.1}$$

One can easily check that the set B indeed is a subgroup. In order to define the discrete series representation, we first express $\mathrm{GL}(2, \mathbb{F}_q)$ as a group presentation involving the Borel subgroup. To start, consider the matrices

$$t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad v_{r,s} = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}.$$

Theorem 3.2. *The general linear group $\mathrm{GL}(2, \mathbb{F}_q)$ has a group presentation*

$$\mathrm{GL}(2, \mathbb{F}_q) \cong \langle B, w \mid wv_{r,s} = v_{s,r}w,\ w^2 = -\mathrm{id},\ wtw = (twt)^{-1} \rangle.$$

Proof. Let $G' = \langle B, w \mid wv_{r,s} = v_{s,r}w,\ w^2 = -\operatorname{id},\ (wt)^3 = \operatorname{id}\rangle$. We wish to show that $G' \cong \mathrm{GL}(2,\mathbb{F}_q)$. Consider the surjective homomorphism $\theta\colon G' \to \mathrm{GL}(2,\mathbb{F}_q)$ which is the identity on B and maps w to the **Weyl element**,

$$w \mapsto w' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Checking that the Weyl element satisfies the given relations in the presentation confirms that θ is a homomorphism. By straightforward calculation, one verifies that if $c \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (bc-ad)c^{-1} & -a \\ 0 & -c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & c^{-1}d \\ 0 & 1 \end{bmatrix}.$$

Thus θ is surjective. It suffices to show that θ has trivial kernel. We make the following claim and defer its proof to the end of this proof:

Proposition 3.3. *Let $D = \{v_{r,s} \mid r, s \in \mathbb{F}_q\}$ be the diagonal subgroup. For each $b \in B - D$, there exists $b_1, b_2 \in B$ such that $w'bw' = b_1 w' b_2$.*

Now suppose, for the sake of contradiction, that there exists some $g \in \ker\theta$ such that $g \neq \operatorname{id}$. Then g may be written as $g = b_1wb_2w\cdots b_{n-1}wb_n$, a word of length $2n-1$ for $n \geq 2$. We show that if $n > 2$, then g may be rewitten as a word of length $\leq 2n-3$; induction then shows that $g = a_1wa_2$ for some $a_1, a_2 \in B$. To do this, note that if $b_2 \in D$, then $wb_2w \in B$ via the group relations. Thus $g = b'\cdots wb_{n-1}wb_n$, where $b' = b_1wb_2wb_3 \in B$, which is a word of length $2n-5$. Otherwise, if $b_2 \in B - D$, applying Proposition 3.3 allows us to write $wb_2w = c_1wc_2$ for $c_1, c_2 \in B$. Thus we may write $g = b_1'wb_2'\cdots wb_{n-1}wb_n$, where $b_1' = b_1c_1 \in B$ and $b_2' = b_2c_3 \in B$, which is a word of length $2n-3$.

Thus g is ultimately of the form a_1wa_2 for some $a_1, a_2 \in B$. One may check that such an element gets mapped by θ to matrix of the form

$$\theta(g) = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha' & \beta' \\ 0 & \gamma' \end{bmatrix} = \begin{bmatrix} -\alpha'\beta & \alpha\gamma' - \beta\beta' \\ -\alpha'\gamma & -\beta'\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.2}$$

where $\alpha, \gamma \neq 0$ and $\alpha', \gamma' \neq 0$, since B consists of invertible matrices. But this yields a contradiction since the previous equation implies $\alpha'\gamma = 0$. Thus the kernel of θ is trivial. □

Proof of Proposition 3.3. Suppose that $b \in B - D$. Then one may write $b = \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]$ where $b \neq 0$, which in turn is equal to the product $v_{1,cb^{-1}}tv_{a,c}$. Since $wtw = (twt)^{-1}$, one has

$$wbw = (wv_{1,cb^{-1}}w^{-1})(wtw)(w^{-1}v_{a,c}w) = (wv_{1,cb^{-1}}w^{-1})t^{-1}v_{1,-1}wt^{-1}(w^{-1}v_{a,c}w), \tag{3.3}$$

which puts wbw in the desired form, with $b_1 = (wv_{1,cb^{-1}}w^{-1})t^{-1}v_{1,-1}$ and $b_2 = t^{-1}(w^{-1}v_{a,c}w)$. □

We may finally construct the discrete series representation σ_ν associated to a indecomposable multiplicative character $\nu\colon K \to \mathbb{C}^\times$. To do this, we shall define σ_ν on B and the Weyl element w, and verify σ_ν is a homomorphism by checking that it respects the group presentation of $\mathrm{GL}(2,\mathbb{F}_q)$ given by Theorem 3.2. Consider the $\mathbb{C}$-vector space V of functions $\mathbb{F}_q^\times \to \mathbb{C}$, and let $\psi\colon \mathbb{F}_q^+ \to \mathbb{C}^\times$ be a nontrivial additive character[2]. Now define

$$[(\sigma_\nu[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}])\, f](x) = \nu(c)\psi(bc^{-1}x)f(ac^{-1}x), \tag{3.4}$$

and

$$[(\sigma_\nu w)f](x) = -\sum_{y\in\mathbb{F}_q^\times} \nu(y^{-1})j_\psi(xy)f(y),$$

where $j_\psi\colon \mathbb{F}_q^\times \to \mathbb{C}$ is the *generalized Kloosterman sum*

$$j_\psi(x) = \frac{1}{q}\sum_{N(t)=x} \psi(t+\bar{t})\nu(t).$$

Here, the sum is taken over all $t \in \mathbb{F}_q[\sqrt{\delta}]$ with norm x.

Before continuing, we state some facts about generalized Kloosterman sums; the proofs of these can be verified by computation and can be found in [PS].

Lemma 3.4. *Let $\nu\colon K \to \mathbb{C}^\times$ be a multiplicative character and $\psi\colon \mathbb{F}_q^+ \to \mathbb{C}^\times$ be a nontrivial additive character, and $j_\psi\colon \mathbb{F}_q^\times \to \mathbb{C}$ be the generalized Kloosterman sum associated to ψ and ν. Then the following identities hold:*

$$\sum_{v\in\mathbb{F}_q^\times} j_\psi(uv)j_\psi(v)\nu(v^{-1}) = \begin{cases} \nu(-1) & \text{if } u = 1 \\ 0 & \text{if } u \neq 1 \end{cases}, \tag{3.5}$$

$$\sum_{v\in\mathbb{F}_q^\times} j(xv)j(yv)\nu(v^{-1})\psi(v) = \nu(-1)\psi(-x-y)j(xy) \tag{3.6}$$

Theorem 3.5. *σ_ν defines a representation of* $\mathrm{GL}(2,\mathbb{F}_q)$.

Proof. Let us first check that σ_ν defines a homomorphism from B to $\mathrm{GL}(V)$. For this, we simply calculate

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} = \begin{bmatrix} a\alpha & a\beta + b\gamma \\ 0 & c\gamma \end{bmatrix}, \tag{3.7}$$

and write

$$\left[\sigma_\nu\left(\left[\begin{smallmatrix} a\alpha & a\beta+b\gamma \\ 0 & c\gamma \end{smallmatrix}\right]\right) f\right](x) = \nu(c\gamma)\psi((a\beta+b\gamma)c^{-1}\gamma^{-1}x)f(a\alpha c^{-1}\gamma^{-1}x) \tag{3.8}$$

Writing $g(x) = \left[\sigma_\nu\left(\left[\begin{smallmatrix} \alpha & \beta \\ 0 & \gamma \end{smallmatrix}\right]\right) f\right](x)$, one has

$$\begin{aligned}
[(\sigma_\nu[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}])\, g](x) &= \nu(c)\psi(bc^{-1}x)g(ac^{-1}x) && (3.9)\\
&= \nu(c)\psi(bc^{-1}x)\nu(\gamma)\psi(\beta\gamma^{-1}(ac^{-1}x))f(\alpha\gamma^{-1}(ac^{-1}x)) && (3.10)\\
&= \nu(c\gamma)\psi((a\beta+b\gamma)c^{-1}\gamma^{-1}x)f(a\alpha c^{-1}\gamma^{-1}x) && (3.11)\\
&= \left[\sigma_\nu\left(\left[\begin{smallmatrix} a\alpha & a\beta+b\gamma \\ 0 & c\gamma \end{smallmatrix}\right]\right) f\right](x), && (3.12)
\end{aligned}$$

where we used the fact that ν and ψ are homomorphisms into $\mathbb{C}^\times$. Thus σ_ν defines a homomorphism on B. To extend this to a homomorphism on $\mathrm{GL}(2,\mathbb{F}_q)$, we must

[2] such characters can be obtained by pulling back a character through the trace map $\mathrm{tr}\colon \mathbb{F}_q \to \mathbb{F}_p$, where $q = p^r$.

check that the three relations in the group presentation are respected by σ_ν. We check that $\sigma_\nu(w')\sigma_\nu(v_{r,s}) = \sigma_\nu(v_{s,r})\sigma_\nu(w')$, by first computing

$$\big[\sigma_\nu(v_{r,s})f\big](x) = \nu(s)f(rs^{-1}x). \tag{3.13}$$

Then

$$\begin{aligned}\big[\sigma_\nu(w')\sigma_\nu(v_{r,s})f\big](x) &= -\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j_\psi(xy)\nu(s)f(rs^{-1}y)\\ &= -\sum_{x\in\mathbb{F}_q^\times}\nu(sy^{-1})j_\psi(xy)f(rs^{-1}y).\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\big[\sigma_\nu(v_{s,r})\sigma_\nu(w')f\big](x) &= -\nu(r)\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j_\psi(sr^{-1}xy)f(y)\\ &= -\sum_{y\in\mathbb{F}_q^\times}\nu(sy^{-1})j_\psi(xy)f(rs^{-1}y).\end{aligned} \tag{3.15}$$

Here, the fact that $y \mapsto rs^{-1}y$ is an automorphism of $\mathbb{F}_q^\times$ was used in the last equality. Thus the first relation is respected by σ_ν. For the second relation, we calculate

$$\begin{aligned}\big[\sigma_\nu(w')\sigma_\nu(w')f\big](x) &= -\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j_\psi(xy)[\sigma_\nu(w')f\big](y)\\ &= \sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j_\psi(xy)\left[\sum_{z\in\mathbb{F}_q^\times}\nu(z^{-1})j_\psi(yz)f(z)\right]\\ &= \sum_{z\in\mathbb{F}_q^\times}\nu(xz^{-1})f(z)\left[\sum_{y\in\mathbb{F}_q^\times}\nu(x^{-1}y^{-1})j_\psi(yz)j_\psi(xy)\right],\end{aligned}$$

Where we used the multiplicativity of ν in the second equality. Now consider the automorphisms $y \mapsto x^{-1}y$ and $z \mapsto xz$, which allows us to rewrite

$$\begin{aligned}\big[\sigma_\nu(w')\sigma_\nu(w')f\big](x) &= \sum_{z\in\mathbb{F}_q^\times}\nu(z^{-1})f(xz)\left[\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j_\psi(zy)j_\psi(y)\right]\\ &= \nu(-1)f(x)\\ &= [\sigma_\nu(v_{-1,-1})f](x),\end{aligned} \tag{3.16}$$

where (3.5) was used in the second equality. Thus the second relation is respected by σ_ν. For the third relation, it suffices to check that $\sigma_\nu(w')\sigma_\nu(t)\sigma_\nu(w') =$

$\sigma_\nu(-t^{-1})\sigma_\nu(w')\sigma_\nu(t^{-1})$. The former is given by

$$\begin{aligned}
[\sigma_\nu(w')\sigma_\nu(t)\sigma_\nu(w')f]\,(x) &= -\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j_\psi(xy)[\sigma_\nu(t)\sigma_\nu(w')f](y)\\
&= -\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j_\psi(xy)\psi(y)[\sigma_\nu(w')f](y)\\
&= \sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j_\psi(xy)\psi(y)\left[\sum_{z\in\mathbb{F}_q^\times}\nu(z^{-1})j_\psi(yz)f(z)\right]. \qquad (3.17)
\end{aligned}$$

On the other hand, one may write

$$\begin{aligned}
\left[\sigma_\nu(-t^{-1})\sigma_\nu(w')\sigma_\nu(t^{-1})f\right](x) &= -\left[\sigma_\nu(t^{-1})\sigma_\nu(w')\right]\psi(-x)f(x)\\
&= \nu(-1)\psi(-x)\sum_{z\in\mathbb{F}_q^\times}\nu(z^{-1})j_\psi(xz)\psi(-z)f(z)\\
&= \sum_{z\in\mathbb{F}_q^\times}\nu(z^{-1})f(z)\left[\nu(-1)\psi(-x-z)j_\psi(xz)\right]\\
&= \sum_{z\in\mathbb{F}_q^\times}\nu(z^{-1})f(z)\left[\sum_{y\in\mathbb{F}_q^\times}j_\psi(xy)j_\psi(yz)\nu(y^{-1})\psi(y)\right], \qquad (3.18)
\end{aligned}$$

where (3.6) was used in the last equality. Comparison of (3.17) and (3.18) shows that the third relation is respected by σ_ν. Thus σ_ν is a well defined representation. □

This establishes the discrete series representation σ_ν of $\mathrm{GL}(2,\mathbb{F}_q)$ which is the desired irreducible "extension" of ν; showing that this representation is irreducible involves computing its character, which is unfortunately outside of the scope of this paper(see Chapter 21 of [T]). In particular, it turns out that the indecomposability of the character $\nu\colon K\to\mathbb{C}^\times$ is crucial to the irreducible of the discrete series representation.

References

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