$k-$BESSEL FUNCTIONS AS EIGENFUNCTIONS OF THE LAPLACIAN

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1. Introduction

In this paper, we will discuss the analogue of the Laplacian for the finite upper half plane and provide an explicit form for its eigenfunctions, which is something that is not possible for the real upper half plane case. As we will see, the Laplacian on the finite upper half plane is motivated from the combinatorial Laplacian ([Terras(1999), p. 51]) and is equal to $A_a-(q+1)I$ for A_a the adjacency operator of an appropriately defined Cayley graph with degree $q+1$ (we are working in the field $\mathbb{F}_q$). We then define the $k-$Bessel functions by $k(z|\chi,\psi) = \sum_{u\in\mathbb{F}_q} \chi\left[\mathrm{Im}\left(\frac{-1}{z+u}\right)\right]\psi(-u)$ where χ,ψ are multiplicative (for $\mathbb{F}_q^*$) and additive (for $\mathbb{F}_q$) characters respectively. We show that the $k-$Bessel functions are indeed eigenfunctions of the Laplacian and that they transform under horizontal translations of the input according to the additive character of the translation. In Section 2, we present a summary of the real and finite upper half planes as well as fractional linear transformations and some elementary results. In section 3, we build on the discussion of finite upper half planes to define the family of graphs $X_q(\delta,a)$ which we in fact show to be Cayley graphs with a suitable generating set $S_q(\delta,a)$. We then use these graphs to define Laplacians on the finite upper half plane. In Section 4, we first discuss the finite power function which is an eigenfunction of the Laplacian on the finite upper half plane and then discuss the k-Bessel functions, which are a family of eigenfunctions of the Laplacian with nice properties under horizontal translations of the input. The $k-$Bessel functions are tremendously useful in constructing the discrete series representation of $GL(2,\mathbb{F}_q)$, which we will not discuss in this paper (we refer the reader to [Piatetski-Shapiro et al.(1983), p. 34-40] and [Terras(1999), p. 370-374] for a discussion of the Discrete Series).

2. Upper Half Planes and Fractional Linear Transformations

In this section, we discuss some prelimiaries, including the real upper half plane H and the finite upper half plane H_q. We define the notion of distance on the Laplacian on H and the notion of distance on H_q. We also discuss fractional linear transformations and the fact that they are isometries. We will use these facts in later sections to further explore H_q, specifically a discrete analog of the Laplacian for H_q and its eigenfunctions.

2.1. **Real Poincaré Upper Half Plane.** The Poincaré upper half plane is

$$H = \{z = x+iy \in \mathbb{C} \mid y > 0\}$$

with noneuclidean arc element $ds^2 = y^{-2}(dx^2 + dy^2)$ and corresponding Laplacian

$$\Delta u = y^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Recall the action of elements $SL(2, \mathbb{R})$ acting on $z \in H$ is given by the fractional linear/Mobius transformation: if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $gz = \frac{az+b}{cz+d}$. Mobius transforms preserve both ds and Δ (a proof of conservation of distances is given in Theorem 2.1) .

2.2. **Finite Upper Half Plane.** We model the finite upper half plane on the real one from above. Let $\mathbb{F}_q$ be a finite field of odd characteristic p, where $q = p^r$ and let $\delta \in \mathbb{F}_q$ be nonsquare. Then the finite upper half plane is

$$H_q = \{z = x + y\sqrt{\delta} \mid x, y \in \mathbb{F}_q, y \neq 0\}$$

We can view the finite upper half plane as a subset of the quadratic field extension $\mathbb{F}_q(\sqrt{\delta})$ of $\mathbb{F}_q$. We note that H_q is not strictly a half plane (as we exclude the real axis) but is useful as a parallel to H.
Mobius transformations apply to elements of H_q as well, with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $z \in H_q$ giving $gz = \frac{az+b}{cz+d}$. Notation: If $z = x + y\sqrt{\delta} \in H_q$, we write $x = \text{Re } z$, $y = \text{Im } z$, $z^q = \bar{z} = x - y\sqrt{\delta}$. The norm of z is $Nz = z\bar{z}$ and the trace is $z + \bar{z}$. We will often refer to $\sqrt{\delta}$ as the *origin*.

2.3. **Finite Noneuclidean Geometry.** We need a notion of distance to obtain a finite analog of noneuclidean geometry (from Section 2.1) on the finite upper half plane. We define a distance (not a metric) between two points in H_q:

$$d(z, w) = \frac{N(z - w)}{\text{Im } z \text{ Im } w} = \frac{(x - u)^2 - \delta(y - v)^2}{yv}$$

where $z = x + y\sqrt{\delta}$, $w = u + v\sqrt{\delta}$, with $x, y, u, v \in \mathbb{F}_q, yv \neq 0$. We note that $d(z, w) \in \mathbb{F}_q$, so there is certainly no possibility of a triangle inequality and hence this can't be a metric. However, by taking $\delta = -1$, we can recover $ds^2 = y^{-2}(dx^2 + dy^2)$, the hyperbolic distance for H, making $d(z, w)$ a natural extension.

We have the following important result:

Theorem 2.1. *$d(z, w) = d(gz, gw)$ for all $g \in GL(2, \mathbb{F}_q)$ (or $SL(2, \mathbb{R})$), and for all $z, w \in H_q$ (or H).*

Proof. We provide the proof for $g \in GL(2, \mathbb{F}_q)$ and $z, w \in H_q$; the other proof for $SL(2, \mathbb{R})$ and H is the exact same, by taking $\delta = -1$.

Let $z = x+y\sqrt{\delta}$, $w = u+v\sqrt{\delta}$. Consider a general element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $GL(2, \mathbb{F}_q)$. It is well known that Mobius transformations can be thought of as compositions of three elementary transformations: $f(z) = z + b, f(z) = az, f(z) = 1/z$. We can easily see from the definition that $d(z, w) = d(z + b, w + b)$ and $d(z, w) = d(az, aw)$. We now show $d(z, w) = d(1/z, 1/w)$. We have

$$
\begin{aligned}
N\left(\frac{1}{z}-\frac{1}{w}\right) &= \frac{(w-z)(\bar{w}-\bar{z})}{(wz)(\bar{w}\bar{z})} = \frac{u^2-\delta v^2+x^2-\delta y^2-2ux+2vy\delta}{(u^2-\delta v^2)(x^2-\delta y^2)} \\
&= \frac{(x-u)^2-\delta(y-v)^2}{(u^2-\delta v^2)(x^2-\delta y^2)}
\end{aligned}
$$

and

$$\mathrm{Im}\frac{1}{z}\ \mathrm{Im}\frac{1}{w} = \mathrm{Im}\frac{x-y\sqrt{\delta}}{x^2-\delta y^2}\mathrm{Im}\frac{u-v\sqrt{\delta}}{u^2-\delta v^2} = \frac{yv}{(u^2-\delta v^2)(x^2-\delta y^2)}$$

and hence $d(z,w) = d(1/z, 1/w)$.

Thus, for a general Mobius transformation on the finite (or real) upper half plane, we have $d(z,w) = d(gz, gw)$ as we can compose the elementary transformations and use the fact that each of these preserves distances. $\square$

3. The Graphs $X_q(\delta, a)$ and their properties

In this section, we will define a family of graphs $X_q(\delta, a)$ on H_q and prove that these graphs can be thought of as Cayley graphs with the set of generators $S_q(\delta, a)$ which are the "circles" of radius a around the origin in H_q. We then use these graphs to motivate a "discretization" of the Laplacian on H for the H_q case using the combinatorial Laplacian for $X_q(\delta, a)$.

3.1. Defining the graphs $X_q(\delta, a)$.

Definition 1. *For $a \in H_q$, the graph $X_q(\delta, a)$ has vertices given by the elements of H_q and $z, w \in H_q$ connected by an edge when $d(z,w) = a$.*

Example 3.1. Consider the graph $X_3(-1, 1)$. We can take $\delta = -1$ because $-1 \equiv 2 \pmod 3$ is not a square in $\mathbb{F}_3$. We write $i = \sqrt{-1}$ in $\mathbb{F}_9$. We begin by identifying the neighbors of i in the graph. We want $z = x + iy$ such that

$$d(z,i) = \frac{N(z-i)}{y} = \frac{x^2+(y-1)^2}{y} = 1$$

which gives points $\pm 1 \pm i$ as the neighbors of i. To get the neighbors (distance 1 away) of $z = x + iy$, we apply the matrix $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$ to $\pm 1 \pm i$ (this follows from Theorem 2.1). In doing so, we have obtained a 4-regular graph.

3.2. Properties of $X_q(\delta, a)$.

Theorem 3.2. *$X_q(\delta, a)$ is a $(q+1)$-regular graph if $a \neq 0$ or 4δ.*

Proof. It suffices to show that there are $q+1$ points adjacent to the origin $\sqrt{\delta}$. This is because we can denote a general element $x + y\sqrt{\delta}$ of H_q as $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\sqrt{\delta}$. Along with Theorem 2.1, this will allow us to conclude the result.

We thus look for $z = x + y\sqrt{\delta}$ such that $N(z-\sqrt{\delta}) = ay$, that is, $x^2 = ay + \delta(y-1)^2$. Noting that $\bar{z} = z^q$ and that $\mathrm{Im}\, z = \frac{z-\bar{z}}{2\sqrt{\delta}} = \frac{z-z^q}{2\sqrt{\delta}}$, we can also write the condition as

$$(z-\sqrt{\delta})^{q+1} = a(2\sqrt{\delta})^{-1}(z-z^q)$$

which is a degree $q+1$ polynomial over $\mathbb{F}_q(\sqrt{\delta})$ and hence has at most $q+1$ solutions in $\mathbb{F}_q(\sqrt{\delta})$. Note that that if any solution had $y=0$, this would imply $x^2=\delta$, contradicting the nonsquare nature of δ. Thus, all solutions will lie in H_q.

Some computation shows that the desired condition is equivalent to $N(z+c)=r$, where $c=(a/2\delta-1)\sqrt{\delta}$ and $r=a(1-a/4\delta)$ and we are thus solving $w^{q+1}=r$ where $w=z+c$. We wish to show that this equation has exactly $q+1$ solutions. We note that for $r\neq 0$, that is for $a\neq 0, 4\delta$, we can solve $N(z+c)=r$ as the norm is simply a homomorphism from $\mathbb{F}_q(\sqrt{\delta})^*$ to $\mathbb{F}_q^*$. We know that the multiplicative group $\mathbb{F}_q(\sqrt{\delta})^*$ is cyclic with some generator, say γ. As $N(z)=z^{q+1}$, the kernel of the norm will have $q+1$ elements given of the form γ^k where $(q+1)k\equiv 0 \pmod{q^2-1}$. Thus, $q-1$ must divide k, and we have $k=(q-1)j$ for $j=0,1,...,q$. The image of the norm map thus has $(q^2-1)/(q+1)=q-1=|\mathbb{F}_q^*|$ elements, and thus $w^{q+1}=r$ has $q+1$ solutions w given any $r\neq 0$ in $\mathbb{F}_q$. This implies that the origin has $q+1$ neighbors and by extension, that $X_q(\delta,a)$ is $(q+1)$-regular. □

In fact, we have a stronger result. We can show that $X_q(\delta,a)$ can be thought of as a Cayley graph, with an explicit formula for the generating set.

Theorem 3.3. *$X_q(\delta,a)$ is connected for $a\neq 0,4\delta$. In fact, it is a Cayley graph for the affine group*

$$\mathrm{Aff}(q)=\left\{\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\middle|\ x,y\in\mathbb{F}_q, y\neq 0\right\}$$

using the generators

$$S_q(\delta,a)=\left\{\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\middle|\ x,y\in\mathbb{F}_q, y\neq 0, x^2=ay+\delta(y-1)^2\right\}$$

We accept the above theorem without proof and refer the reader to [Terras(1999), p. 319-321] for a proof.

We note that $x^2=ay+\delta(y-1)^2$ is equivalent to $d(x+y\sqrt{\delta},\sqrt{\delta})=a$, and thus $S_q(\delta,a)$ corresponds to "circles" around the origin. It can also be seen that $S_q(\delta,a)$ contains $(q+1)$ elements, thereby verifying Theorem 3.2.

We now define an analogue of the Laplacian discussed for H in the case of H_q. This is not immediately obvious, as H_q consists of discrete points. We turn to the combinatorial Laplacian $\Delta=A-2I$ for the circle graph $X(\mathbb{Z}/n\mathbb{Z},\{\pm1\})$ in the context of the Laplacian of the real line for motivation. As described above, the $S_q(\delta,a)$ are "circles" around the origin, the analogue of our circle graphs in the $1-$dimensional case. We thus consider the combinatorial Laplacians for the Cayley graphs $X_q(\delta,a)$ given by $\Delta_a=A_a-(q+1)I$, where A_a is the adjacency operator for $X_q(\delta,a)$, and $q+1$ is the degree of $X_q(\delta,a)$.

4. The k-Bessel functions and Elementary Properties

In this section, we will study eigenfunctions of the Laplacian on H_q. First, we note that eigenfunctions of the Laplacians Δ_a and of the adjacency operators A_a of $X_q(\delta,a)$ are the same. This follows from the definition of Δ_a above. It thus suffices to study the eigenfunctions of the operators A_a. We first look at the finite power function, which we show to be an eigenfunction of A_a. We then look at the $k-$Bessel functions which are a generalization of the finite power function and show that they are a family of eigenfunctions of A_a that possess the additional property of transforming by an additive character of $\mathbb{F}_q$ under horizontal translations of the input. We then conclude by showing that the $k-$Bessel functions are orthogonal in $L^2(H_q)$, whenever their additive characters differ. For a more detailed discussion of orthogonality, we direct the reader to [Evans(1994), p. 33-50].

Definition 2. *The finite power function $p_\chi(z)$ for $z \in H_q$ and χ a character of $\mathbb{F}_q^*$ is given by $p_\chi(z) = \chi(\mathrm{Im}\ z)$.*

We now show that the finite power function is an eigenfunction of A_a.

Theorem 4.1. *Let A be the adjacency operator of $X_q(\delta, a)$. The finite power function is an eigenfunction of A, that is $Ap_\chi = R_\chi p_\chi$ where $R_\chi = \sum_{w \in S_q(\delta,a)} \chi(\mathrm{Im}\ w)$.*

Proof. We have

$$\begin{aligned} Ap_\chi \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \sqrt{\delta} \right\} &= \sum_{\left(\begin{smallmatrix} v & u \\ 0 & 1 \end{smallmatrix}\right) \in S_q(\delta,a)} \chi \left(\mathrm{Im} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} \sqrt{\delta} \right) \\ &= \sum_{\left(\begin{smallmatrix} v & u \\ 0 & 1 \end{smallmatrix}\right) \in S_q(\delta,a)} \chi(yv) = \chi(y) \sum_{\left(\begin{smallmatrix} v & u \\ 0 & 1 \end{smallmatrix}\right) \in S_q(\delta,a)} \chi(v) = R_\chi p_\chi(z). \end{aligned}$$

□

Building upon the power functions previously discussed, we now turn to the $k-$Bessel functions, a family of eigenfunctions of the adjacency operators A_a of $X_q(\delta, a)$. More specifically, $k-$Bessel functions are eigenfunctions of A_a that behave well under transformations by the abelian subgroup N of $GL(2, \mathbb{F}_q)$ defined by $N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{F}_q\}$. Note N is isomorphic to the additive group of $\mathbb{F}_q$ and that transformations by N are effectively horizontal translations of the input.

Definition 3. *A $k-$Bessel function $f : H_q \to \mathbb{C}$ is an eigenfunction of all adjacency operators A_a of $X_q(\delta, a)$ such that $f(z)$ transforms by N according to the nontrivial additive character $\psi(x)$ of $\mathbb{F}_q$:*
$A_a(f) = \lambda f$, for all $a \in \mathbb{F}_q$
$f(z + u) = \psi(u)f(z)$, for all $z \in H_q, u \in \mathbb{F}_q$

We will usually assume $\psi(x) = exp(2\pi i Tr(x)/p)$ where $Tr(x)$ is the trace of $x \in \mathbb{F}_q$ down to $\mathbb{F}_p$ given by $x + x^p + x^{p^2} + ... + x^{p^{r-1}}$ (remember $q = p^r$).

Given that a $k-$Bessel function must satisfy a lot of constraints, how do we know that they exist? We provide a construction of one class of $k-$Bessel functions below:

We define the $k-$Bessel function $k(z|\chi, \psi)$ for $z \in H_q$, χ a multiplicative character and ψ the above-mentioned additive character of $\mathbb{F}_q$, by

$$k(z|\chi, \psi) = \sum_{u \in \mathbb{F}_q} \chi \left[\mathrm{Im} \left(\frac{-1}{z + u} \right) \right] \psi(-u).$$

We first note that if χ is the trivial character, we have $k(z|1,\psi) = \sum_{u\in\mathbb{F}_q} \psi(-u) = 0$ (by exploiting symmetry and noting that the p^{th} roots of unity sum to 0) and thus we ignore this case.

We now prove some elementary properties of $k(z|\chi,\psi)$ and that it actually is a valid form of the desired eigenfunction.

Theorem 4.2. *Let $z = x + y\sqrt{\delta}$. Then*

$$k(z|\chi,\psi) = \chi(y)\psi(x)\sum_{u\in\mathbb{F}_q} \overline{\chi(u^2-\delta y^2)}\psi(-u).$$

Proof. This is a simple result that will help us later show that the $k-$Bessel function as defined is an eigenfunction of the adjacency operator. Writing $v = x - u$, we have

$$\begin{aligned}\chi(y)\psi(x)\sum_{u\in\mathbb{F}_q} \overline{\chi(u^2-\delta y^2)}\psi(-u) &= \chi(y)\sum_{u\in\mathbb{F}_q} \overline{\chi(u^2-\delta y^2)}\psi(x-u)\\ &= \chi(y)\sum_{v\in\mathbb{F}_q} \overline{\chi((x-v)^2-\delta y^2)}\psi(v).\end{aligned}$$

We can write the LHS as

$$\sum_{u\in\mathbb{F}_q} \chi\left[\mathrm{Im}\left(\frac{-1}{z+u}\right)\right]\psi(-u) = \sum_{v\in\mathbb{F}_q} \chi\left[\mathrm{Im}\left(\frac{-1}{z-v}\right)\right]\psi(v)$$

$$= \sum_{v\in\mathbb{F}_q} \chi\left[\mathrm{Im}\left(\frac{-1}{x-v+y\sqrt{\delta}}\right)\right]\psi(v) = \sum_{v\in\mathbb{F}_q} \chi\left(\frac{y}{(x-v)^2-\delta y^2}\right)\psi(v).$$

It thus suffices to show that

$$\overline{\chi((x-v)^2-\delta y^2)} = \chi\left(\frac{1}{(x-v)^2-\delta y^2}\right).$$

As χ is a multiplicative character and $a^q = 1$ for all $a \in \mathbb{F}_q$, we have $\overline{\chi(a)} = \chi(a^{-1})$ which gives the result. □

As a brief aside, we will now define the Gauss sum and prove an important property of it which will aid us in proving the non-degeneracy of the $k-$Bessel functions.

Definition 4. *Given a multiplicative character χ of $\mathbb{F}_q^*$ and an additive character ψ of $\mathbb{F}_q$, their Gauss sum $\Gamma(\chi,\psi)$ is $\sum_{z\in\mathbb{F}_q} \chi(z)\psi(z)$.*

Theorem 4.3. $|\Gamma(\chi,\psi)| = \sqrt{q}$ *provided neither χ nor ψ are trivial.*

Proof. The proof of this fact is not particularly interesting or relevant to our main result in this section, so we direct the reader to [Terras(1999), p. 142-147]. □

We now show that the $k-$Bessel functions are eigenfunctions of the adjacency operators A_a of $X_q(\delta, a)$.

Theorem 4.4. *Let $f(z) = k(z|\chi, \psi)$. Then $f(z)$ is an eigenfunction of the adjacency operator A for $X_q(\delta, a)$ with the same eigenvalue as that for the finite power function, that is, R_χ from Theorem 4.1. In particular, $f(z)$ is not identically zero, if χ is a nontrivial multiplicative character.*

Proof. We recall that $Af(g\sqrt{\delta}) = \sum_{s\in S_q(\delta,a)} f(gs\sqrt{\delta})$. Also note that $\chi(\mathrm{Im}(\frac{-1}{z+u})) = p_\chi\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) z\right)$ where the second equality follows from Theorem 4.1. Thus,

$$Ak(z|\chi,\psi) = \sum_{u\in\mathbb{F}_q} Ap_\chi\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} z\right\}\psi(-u)$$

$$= R_\chi \sum_{u\in\mathbb{F}_q} p_\chi\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} z\right\}\psi(-u) = R_\chi k(z|\chi,\psi).$$

We now need to show that $k(z|\chi, \psi)$ is nonzero for nontrivial multiplicative character χ. We multiply both sides of the expression from Theorem 4.2 by $\overline{\chi(y)\psi(x)}$ and sum over elements $y \in \mathbb{F}_q^*$ as follows:

$$\sum_{y\in\mathbb{F}_q^*} \overline{\chi(y)\psi(x)}k(x + y\sqrt{\delta}|\chi,\psi)$$

$$= \sum_{y\in\mathbb{F}_q^*, u\in\mathbb{F}_q} \overline{\chi(u^2 - \delta y^2)}\psi(-u) \quad (\chi(y)\overline{\chi(y)} = \psi(x)\overline{\psi(x)} = 1)$$

$$= \sum_{w\in H_q} \overline{\chi(Nw)}\psi(-Tr(w)/2) \quad (\textit{write } w = v + y\sqrt{\delta})$$

$$= \Gamma_{q^2}(\overline{\chi\circ N}, \overline{\psi\circ \frac{1}{2}Tr}) - \Gamma_q(\overline{\chi^2}, \overline{\psi}).$$

For the third equality, note that H_q is effectively a subset of the field extension $\mathbb{F}_q(\sqrt{\delta})$ with the elements $x + 0\sqrt{\delta}$ excluded; Γ_{q^2} sums over $\mathbb{F}_q(\sqrt{\delta})$ and $-\Gamma_q$ term subtracts off the contribution from terms of the form $w = x + 0\sqrt{\delta}$ and noting that $\chi(Nw) = \chi(w^2) = (\chi(w))^2$, $\psi(-Tr(w)/2) = \psi(-w) = \overline{\psi(w)}$ for such w.

By Theorem 4.3, we have $|\Gamma_q(\chi,\psi)| = \sqrt{q}$ assuming neither χ, ψ are trivial. Thus the difference $\Gamma_{q^2} - \Gamma_q$ is nonvanishing, and $k(z|\chi,\psi)$ is nonzero for nontrivial χ. $\square$

We now show the second property of the $k-$Bessel function, namely that $k(z|\chi,\psi)$ transforms by N (as defined at the beginning of this section) according to the nontrivial additive character $\psi(x)$.

Theorem 4.5. *For every $u \in \mathbb{F}_q, z \in H_q$, we have $k(z + u|\chi,\psi) = \psi(u)k(z|\chi,\psi)$*

Proof. From Theorem 4.2, we have

$$k(z|\chi,\psi) = \chi(y)\psi(x)\sum_{v\in\mathbb{F}_q} \overline{\chi(v^2 - \delta y^2)}\psi(-v)$$

where $z = x + y\sqrt{\delta}$. Thus,

$$k(z + u|\chi,\psi) = \chi(y)\psi(x + u)\sum_{v\in\mathbb{F}_q} \overline{\chi(v^2 - \delta y^2)}\psi(-v)$$

$$= \chi(y)\psi(x)\psi(u)\sum_{v\in\mathbb{F}_q} \overline{\chi(v^2 - \delta y^2)}\psi(-v) = \psi(u)k(z + u|\chi,\psi).$$

□

We now conclude with the orthogonality of the $k-$Bessel functions when the additive characters differ.

Theorem 4.6. *The $k-$Bessel functions $k(z|\chi, \psi)$ and $k(z|\chi, \psi')$ are orthogonal (with respect to the standard inner product on $L^2(H_q)$ if ψ, ψ' are distinct.*

Proof. Recall the standard inner product for functions f, g on $L^2(H_q)$ is given by $\sum_{z \in H_q} f(z)\overline{g(z)}$. We first note that for $z = x + y\sqrt{\delta}$, we can write $k(z|\chi, \psi) = \psi(x)k(y\sqrt{\delta}|\chi, \psi)$ due to Theorem 4.2. The desired inner product is thus

$$\sum_{z \in H_q} k(z|\chi, \psi)\overline{k(z|\chi, \psi')} = \left(\sum_{x \in \mathbb{F}_q} \psi(x)\overline{\psi'(x)}\right)\left(\sum_{y \in \mathbb{F}_q*} k(y\sqrt{\delta}|\chi, \psi)\overline{k(y\sqrt{\delta}|\chi, \psi')}\right).$$

The first term is 0 for $\psi \neq \psi'$ by the orthogonality relations on the additive group $\mathbb{F}_q$, which gives the result. □

References

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