THE POINCARÉ UPPER HALF-PLANE

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1. Introduction

The *Poincaré upper half-plane* is a model of the hyperbolic plane. It consists of the points in the complex upper half-plane, but it has a non-standard arc length element (which leads to a non-Euclidean distance metric).

Definition 1.1. The Poincaré upper half-plane is defined as

$$H = \{x + iy \in \mathbb{C} | y > 0\}.$$

In the Poincaré upper half-plane, we use a non-euclidean arc length element ds, defined by

$$ds^2 = y^{-2}(dx^2 + dy^2).$$

2. Fractional Linear Transformations

The class of *fractional linear transformations* act on H. As we will see later, fractional linear transformations preserve the arc length element of H, which makes them useful for eliciting properties of distances in H.

Definition 2.1. For any element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$, we define a fractional linear transformation $T(g) : H \to H$ as follows:

$$T(g)(z) = \frac{az + b}{cz + d}.$$

Lemma 2.2. *The fractional linear transformations are a group action of $SL(2, \mathbb{R})$ on H.*

Proof. Let $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$. Then for all $z \in H$,

$$\begin{aligned} T(x)T(y)(z) &= \frac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} \\ &= \frac{aez + af + bgz + bh}{cez + cf + dgz + dh} \\ &= \frac{(ae + bg)z + (af + bh)}{(ce + dg)z + (cf + dh)} \\ &= T(xy)z. \end{aligned}$$

And trivially, the identity matrix yields the identity fractional linear transformation. Therefore, since since fractional linear transformations satisfy the necessary identity and composition properties, then they constitute a group action of $SL(2, \mathbb{R})$ on H. $\square$

3. Arc Lengths in the Poincaré Upper Half-Plane

We will soon go on to prove properties about distances and paths in H. When we do so, it will come in handy that the non-euclidean arc length element of H is invariant under fractional linear transformations, so we can apply a transformation that sends a pair of points to another pair of points, and the distance between them remains the same.

Theorem 3.1. *If $g \in SL(2,\mathbb{R})$, then the arc length element is invariant under the fractional linear transformation $T(g)$. That is, given a point $z \in H$ and its image $w = T(g)(z) \in T(g)(H)$, the arc length element in H at z is equal to the arc length element in $T(g)(H)$ at w.*

Proof. Let $w = u + iv$, and let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Since $g \in SL(2,\mathbb{R})$, then $ad - bc = 1$. We can then compute that $v = y\,|cz+d|^{-2}$ and $\frac{d}{dz}T(g)(z) = (cz+d)^{-2}$.

The Jacobian of the transformation from z to w is

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

The fractional linear transformation $T(g)$ is holomorphic (that is, complex differentiable everywhere), so we can apply the Cauchy-Riemann equations to rewrite the Jacobian as

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}.$$

Then the determinant of J is $(\frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2) = \left|\frac{d}{dz}T(g)(z)\right|^2 = |cz+d|^{-4}$.

We can now derive the arc length element in $T(g)(H)$:

$$\begin{aligned} ds^2|_{(x+iy)\in H} &= y^{-2}(dx^2+dy^2) \\ &= y^{-2}|cz+d|^4|cz+d|^{-4}(dx^2+dy^2) \\ &= v^{-2}|J|(dx^2+dy^2) \\ &= v^{-2}(du^2+dv^2) \\ &= ds^2|_{(u+iv)\in T(g)(H)} \end{aligned}$$

Then the arc length is invariant under $T(g)$. □

Lemma 3.2. *For any $y_0 > 0$, the imaginary axis is the shortest path in H from i to iy_0.*

Proof. Let $z(t) = x(t) + iy(t)$, defined on $0 \le t \le 1$, be a curve in H, with $z(0) = i$ and $z(1) = iy_0$.

The Poincaré length of $z(t)$ is

$$\int_0^1 y^{-1}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{3.1}$$

If z is the path on the imaginary axis, then this length becomes

$$\int_1^{y_0} y^{-1}|dy| = |\log(y_0)| \tag{3.2}$$

It's evident that (3.2) is the minimum attainable value of (3.1), so the shortest path from i to iy_0 is the imaginary axis. $\square$

4. Geodesics in the Poincaré Upper Half-Plane

A *geodesic* is the shortest curve between two points. For example, on the Euclidean plane with the standard arc length function (not our special Poincaré arc length element), the geodesic between two points is a straight line. On the surface of a sphere, the geodesics are great circles.

We will characterize the geodesics in H. We saw in Lemma 3.2 that the imaginary axis is one geodesic of H. We can describe the complete set of geodesics in H as follows:

Theorem 4.1. *In the Poincaré upper half-plane, the geodesics are vertical lines or circles whose centers lie on the real axis.*

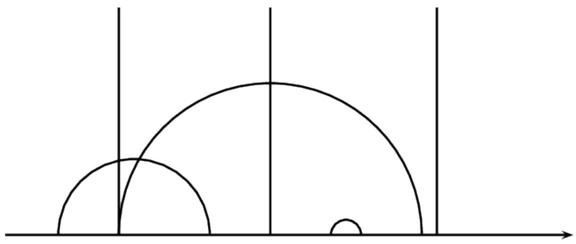

Figure 1. A diagram of some geodesics in the Poincaré upper half-plane [1]

Before we prove this, we'll first prove some lemmas:

Lemma 4.2. *Given any two points $p, q \in H$, there exists some matrix $g \in SL(2, \mathbb{R})$ and some $y_0 > 0$ such that the fractional linear transformation $T(g)$ maps $p \to i$ and $q \to iy_0$.*

Proof. Let $p = x + iy$ and $q = u + iv$. Let f be the matrix

$$f = \begin{pmatrix} \frac{1}{\sqrt{y}} & -\frac{x}{\sqrt{y}} \\ 0 & \sqrt{y} \end{pmatrix}$$

(Recall that the Poincaré upper half-plane restricts $y > 0$, so this is well-defined.) We see that f has determinant 1, so $f \in SL(2, \mathbb{R})$. Additionally,

$$\begin{aligned} T(f)(p) &= \frac{\frac{1}{\sqrt{y}}(x + iy) - \frac{x}{\sqrt{y}}}{\sqrt{y}} \\ &= \frac{iy}{y} \\ &= i \end{aligned}$$

Let g_θ be the matrix

$$g_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in SO(2, \mathbb{R}) \subset SL(2, \mathbb{R})$$

We can compute that

$$\begin{aligned}T(g_\theta)(i) &= \frac{\cos(\theta)i - \sin(\theta)}{\sin(\theta)i + \cos(\theta)}\\ &= i,\end{aligned}$$

so i is a fixed point of the transformation $T(g_\theta)$ for any θ.

We can proceed to compute that the real part of g_θ acting on an arbitrary point $x+iy$ is:

$$Re\left[T(g_\theta)(x+iy)\right] = \frac{x\cos(2\theta) + \frac{1}{2}\left(x^2+y^2-1\right)\sin(2\theta)}{(\cos(\theta))^2 + (x^2+y^2)(\sin(\theta))^2 + x\ \sin(2\theta)}$$

Note that if $\theta = \frac{1}{2}\arctan(\frac{-2x}{x^2+y^2-1})$, then $Re\left[T(g_\theta)(x+iy)\right] = 0$. Therefore, given any point $z = x+iy \in H$, there exists some θ such that $T(g_\theta)$ maps z to some point in H with zero real part; that is, $T(g_\theta)$ maps z to iy_0 for some positive y_0.

Then since f maps p to i, we can choose θ such that $T(g_\theta)$ maps $T(f)(q)$ to the imaginary axis. Therefore $T(g_\theta)T(f) = T(g_\theta f)$ continues to map p to i (since i is a fixed point of $T(g_\theta)$), and maps q to iy_0 for some positive y_0. □

Lemma 4.3. *For $g \in SL(2,\mathbb{R})$, the fractional linear transformation $T(g)$ maps the imaginary axis either to a circle whose center lies on the real axis, or to a vertical line.*

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. The imaginary axis in H consists of the points iy for $y > 0$. We consider several cases:

(1) $c = 0$. Then $T(g)(iy) = \frac{aiy+b}{d}$, which is a vertical line.
(2) $d = 0$. Then $T(g)(iy) = \frac{aiy+b}{ciy} = a - \frac{bi}{cy}$, which is a vertical line.
(3) c and d are both nonzero. Then I claim that $T(g)(iy)$ is a circle centered at $\frac{2bc+1}{2cd}$ with radius $\frac{1}{2cd}$. Since $|g| = 1$, we can substitute $a = \frac{bc+1}{d}$. Then:

$$\begin{aligned}\left|T(g)(iy) - \frac{2bc+1}{2dc}\right| &= \left|\frac{\frac{bc+1}{d}iy + b}{ciy+d} - \frac{2bc+1}{2dc}\right|\\ &= \left|\frac{bciy + iy + bd}{cdiy + d^2} - \frac{2bc+1}{2dc}\right|\\ &= \left|\frac{ciy - d}{2c^2diy + 2d^2c}\right|\\ &= \frac{1}{2cd}\left|\frac{ciy-d}{ciy+d}\right|\\ &= \frac{1}{2cd},\end{aligned}$$

confirming that $T(g)(iy)$ is a circle centered at a point on the real axis.

Therefore, in all cases, $T(g)(iy)$ is either a vertical line or a circle whose center lies on the real axis. □

Now we can prove Theorem 4.1:

Proof. Let p, q be two points in H. By Lemma 4.2, there exists some $g \in SL(2,\mathbb{R})$ such that $T(g)(p) = i$ and $T(g)(q) = iy_0$. By Lemma 3.2, the shortest path between i and iy_0 is the imaginary axis. Then by Theorem 3.1, $T(g)$ must map the shortest path between p and q must be mapped to the imaginary axis, so the shortest path is the image of the imaginary axis under the inverse map $T(g)^{-1}$, which by Lemma 2.2 is the fractional linear map $T(g^{-1})$. By Lemma 4.3, this image must either be a vertical line or a circle whose center lies on the real axis.

Therefore, the shortest path between p and q is either a vertical line or a circle whose center lies on the real axis. □

References

[1] L. Chekhov. Figure: "Some geodesics in the upper half plane." https://www.researchgate.net/figure/Some-geodesics-in-the-upper-half-plane_fig2_2217631

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