THE WAVELET TRANSFORM

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1. Introduction

In electrical engineering, signal processing is a technique used to decompose signals such as sound or images into sine waves, or other functions of a given type, so that these signals can be synthesized and interpreted by a human or computer program. Signal processing has applications in many technical fields, such as wireless communication, computer vision, and natural language processing (NLP).

In this paper, we will explore a method of signal processing known as the *wavelet transform*. The wavelet transform works with functions $\psi \in L^2(\mathbb{F}_p)$ such that

$$\langle \psi, 1 \rangle = \sum_{i \in \mathbb{F}_p} \psi(i) = 0 \tag{1.1}$$

which we call *wavelets*. Given a particular wavelet ψ, the wavelet transform allows one to express a function $f(x)$ as a linear combination of translated and dialated versions of ψ.

Computing the wavelet transform is computationally easier than using classical Fourier analysis, although it is often less sensitive, meaning that it is often less effective at distinguishing signals from noise. However, in applications where f is not composed of non-sinusoidal functions, the wavelet transform is both more efficient and more sensitive than other known methods for decomposing functions.

2. Background

We will be working closely with the definition of the discrete Fourier transform (DFT) in this paper.

Definition 2.1. Let $f \in L^2(\mathbb{F}_p)$. The discrete Fourier transform of f is the function

$$\hat{f}(x) = \sum_{y \in \mathbb{F}_p} f(y) e_x(-y) = \langle f, e_x \rangle,$$

where $e_x(y) = e^{2\pi i x y / p}$.

In addition, we have the following theorem which shows how the DFT interacts with the inner product of functions:

Theorem 2.2 (Plancherel)**.** *Let $f, g \in L^2(\mathbb{F}_p)$. Then $\langle \hat{f}, \hat{g} \rangle = p \langle f, g \rangle$.*

3. Affine Group

Recall the definition of the *affine group* on the field $\mathbb{F}_p$:

Definition 3.1. Let p be a prime. The affine group $\mathrm{Aff}(p)$ is the group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{F}_p \right\}$$

where the group operation is matrix multiplication. We denote the element $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ by (a, b) for short.

We can check that $\mathrm{Aff}(p)$ satisfies the properties of a group:

- $(1, 0)$ is the identity matrix, and thus the identity of $\mathrm{Aff}(p)$.
- Given two group elements $(a, b), (a', b')$, the product of the corresponding matrices is $(aa', ab' + b)$, another element of $\mathrm{Aff}(p)$.
- Given $(a, b) \in \mathrm{Aff}(p)$, one can see that $(\frac{1}{a}, -\frac{b}{a})$ is its unique inverse.

3.1. **Translation and Dialation.** We can express the group $\mathrm{Aff}(p)$ in a different way: specifically, as the group of affine transformations on $\mathbb{F}_p$. We first introduce notation to describe these affine transformations.

Definition 3.2. We define the operators $D_a, T_b : L^2(\mathbb{F}_p) \to L^2(\mathbb{F}_p)$ as dialation by a and translation by b, specifically:

$$D_a f(x) = f(a^{-1}x)$$
$$T_b f(x) = f(x - b)$$

Remark 3.3. In addition, we set $f_{a,b} = T_b D_a f$ for all $(a, b) \in \mathrm{Aff}(p)$.

Since we will be working a lot with both the translation and dialation operators and the discrete Fourier transform, it will help to know how they interact with each other.

Lemma 3.4. *Let $f \in L^2(\mathbb{F}_p)$. We have*

(1) $\widehat{D_a f} = D_{a^{-1}} \hat{f}$.
(2) $\widehat{T_b f} = \overline{e_b} \cdot \hat{f}$, *where $\cdot$ denotes pointwise multiplication: that is,* $\widehat{T_b f}(x) = \overline{e_b(x)} \cdot \hat{f}(x)$.
(3) $\widehat{T_b D_a f} = \overline{e_b} \cdot D_{a^{-1}} \hat{f}$.

Proof. (1) $\widehat{D_a f}(x) = \sum_{y \in \mathbb{F}_p} f(ay) e^{2\pi i x y/p}$. As y ranges over $\mathbb{F}_p$, so does ay (under our assumption that $(a, b) \in \mathrm{Aff}(p)$ and so $a \neq 0$), and so we can let $y' = ay$ to obtain $\sum_{y' \in \mathbb{F}_p} f(y') e^{2\pi i x y'/(ap)} = D_{a^{-1}} \hat{f}(x)$ as desired.

(2) $\widehat{T_b f}(x) = \sum_{y \in \mathbb{F}_p} f(y - b) e^{2\pi i x y/p}$. As y ranges over $\mathbb{F}_p$, so does $y - b$, and so we make the substitution $y' = y - b$ to obtain $\sum_{y' \in \mathbb{F}_p} f(y') e^{2\pi i x(y' - b)/p} = e^{-2\pi i b x/p} \hat{f}(x) = \overline{e_b(x)} \hat{f}(x)$, as desired.

(3) Applying part (2) we get $\widehat{T_b D_a f} = \overline{e_b} \cdot \widehat{D_a f}$, and applying part (1) we get that this is equal to $\overline{e_b} \cdot D_{a^{-1}} \hat{f}$.

□

3.2. **Representation of the Affine Group.** There is another way to represent of the affine group besides with matrices: instead, we can think of each element of the affine group as an affine transformation of $\mathbb{F}_p$. To that end, we can construct a map π as follows:

$$\pi : \text{Aff}(p) \to \text{GL}(L^2(\mathbb{F}_p))$$
$$(a, b) \mapsto T_b D_a$$

Here, every element of Aff(p) is mapped to an operator which takes functions on $\mathbb{F}_p$ to other functions on $\mathbb{F}_p$. Since these operators are elements of GL($L^2(\mathbb{F}_p)$), this map defines a representation of Aff(p) over the space $L^2(\mathbb{F}_p)$. To check this, we need to check that for any $(a, b), (c, d) \in \text{Aff}(p)$,

$$\pi(a, b)\pi(c, d) = \pi((a, b) \cdot (c, d)) = \pi(ac, ad + b)$$

where $\cdot$ denotes multiplication in Aff(p). Indeed we have that

$$\begin{aligned} \pi(a, b)\pi(c, d)f(x) &= \pi(a, b)f(c^{-1}(x - d)) \\ &= f(c^{-1}((a^{-1}(x - b)) - d) \\ &= f((ac)^{-1}(x - b - ad)) \\ &= \pi(ac, ad + b)f(x) = \pi((a, b) \cdot (c, d))f(x) \end{aligned}$$

which was what we wanted to show.

3.3. **Properties of π.** The representation $\pi(a, b) : \text{Aff}(p) \to GL(L^2(\mathbb{F}_p))$ has a few key properties. We will state and prove them here, eventually showing that π is a $(p - 1)$-dimensional *induced representation* of Aff(p).

Lemma 3.5. *$\pi(a, b)$ is a unitary representation: that is, its image consists only of matrices U such that $UU^* = I$, where U^* is the adjoint matrix.*

Proof. Recall that the set of functions $\{e_d = e^{2\pi i d x/p} | d \in \mathbb{F}_p\}$ is a basis of $L^2(\mathbb{F}_p)$. We write matrices in the image of $\pi(a, b)$ with respect to this basis.

Consider a function e_d for $d \in \mathbb{F}_p$. We have that $T_b e_d(x) = e^{2\pi i d(x-b)/p} = e^{-2\pi i b d/p} \cdot e_d(x) = \overline{e_d(b)} \cdot e_d(x)$. Thus, the matrix of T_b is a diagonal matrix, where the entry corresponding to e_d is $\overline{e_d(b)}$. The adjoint matrix will thus be a diagonal matrix where the entry corresponding to e_d is the conjugate $e_d(b)$.

Since $e_d(b)$ is on the unit circle, multiplying it by its conjugate gives 1, and so multiplying the matrix of T_b with its adjoint gives the identity.

Similarly, for any e_d, we have $D_a e_d(x) = e^{2\pi i d x/(ap)} = e_{a^{-1}d}$, and so the matrix of D_a is a permutation matrix, where the permutation is the action of multiplication by a on $\mathbb{F}_p$. The adjoint of a permutation matrix is a matrix corresponding to that permutation's inverse, and thus all permutation matrices are unitary. Therefore, the matrix of D_a is unitary.

Since $\pi(a, b)$ is a representation, the matrix of $T_b D_a$ is the product of the matrices of T_b and D_a, which is the product of two unitary matrices. Since the product of two unitary matrices is unitary, we have that $\pi(a, b)$ is a unitary representation, as desired.

□

Lemma 3.6. *If we restrict π to the space*

$$E = \{f \in L^2(\mathbb{F}_p) | \langle f, 1\rangle = 0\}$$

then we get an irreducible representation.

Proof. It turns out that E will be a very important space later, which we will define as the *wavelet space*. In section 4 we will show that $\{e_d | d \in \mathbb{F}_p, d \neq 0\}$ is a basis for E, and here we will use this fact without proof.

In order to check that π is irreducible on E, we check that

$$\langle \chi_\pi, \chi_\pi \rangle = |\mathrm{Aff}(p)| = p(p-1)$$

The left hand side is equal to the sum of the squared norms of traces of matrices of $T_b D_a$ for all $(a, b) \in \mathrm{Aff}(p)$. As we showed in the previous exercise, the matrix of D_a is a permutation matrix corresponding to multiplication by a in $\mathbb{F}_p$. This permutation will have no fixed points (and thus trace 0) unless $a = 1$. Thus, it suffices to find the trace of $T_b D_1 = T_b$ for $b \in \mathbb{F}_p$.

In the previous exercise, we showed that the matrix of T_b is diagonal with the entry corresponding to e_d equal to $\overline{e_b(d)}$ for all $d \in \mathbb{F}_p$. If $b = 0$, then $\overline{e_d(b)} = 1$ for all $d \in \mathbb{F}_p$. Therefore, the matrix of T_0 is the identity, which has trace $p-1$ since we have only $p-1$ basis elements when we restrict to E. If $b \neq 0$, then the trace of the matrix of T_b is $\sum_{\mathbb{F}_p \setminus \{0\}} \overline{e_d(b)}$. By symmetry, this is the sum of p-th roots of unity except for 1, and so their sum is -1.

In summary, over all $(a, b) \in \mathrm{Aff}(p)$, the trace of the matrix of $T_b D_a$ is $p-1$ for $a = 1, b = 0$, -1 for $a = 1, b \neq 0$, and 0 otherwise. Therefore,

$$\langle \chi_\pi, \chi_\pi \rangle = (p-1)^2 + (p-1) \cdot 1^2 = p(p-1) = |\mathrm{Aff}(p)|$$

and so π is irreducible.

□

Finally, we will show that π restricted to E is an *induced representation* of $\mathrm{Aff}(p)$:

Definition 3.7. Let H be a subgroup of G, and let $\sigma : H \to GL(W)$ be a representation of H. The induced representation, denoted $\mathrm{Ind}_H^G(\sigma)$, is a representation $G \to GL(V)$, where

$$V = \{f : G \to W | f(hg) = \sigma(h) f(g) \forall g \in G, h \in H\}.$$

In other words, V is the set of homomorphisms f which extend σ to G. For any $g \in G$, $\mathrm{Ind}_H^G(\sigma)(g)$ is an operator which maps each homomorphism in V to another homomorphism in V. The induced representation is then defined by $\mathrm{Ind}_H^G(\sigma)(g) f(x) = f(xg)$ for all $x, g \in G, f \in V$.

Lemma 3.8. *Let N be the subgroup of Aff(p) of matrices of the form $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, and let σ be the representation $N \to GL(\mathbb{C})$ defined by $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto e^{2\pi i x/p}$. Then $\pi|_E = \mathrm{Ind}_N^G(\sigma)$.*

Proof. Here, we have W as the subgroup of translations in $\mathrm{Aff}(p)$, $G = \mathrm{Aff}(p)$, and $W = \mathbb{C}$. Therefore, V is the space of functions

$$\{f : \mathrm{Aff}(p) \to \mathbb{C} | f((\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix})) = e^{2\pi i x/p} \forall x\}$$

Note that since every element of $\mathrm{Aff}(p)$ is the product of a translation and dialation, in order to define such a function $f : G \to W$ in V, it suffices to set $f((\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix})) = F_d$ for some choices of $F_d, d \neq 0$.

Suppose $f \in V$ is defined as $f((\begin{smallmatrix} A & B \\ 0 & 1 \end{smallmatrix})) = e^{2\pi i B/p} F_A$. To show that $\mathrm{Ind}_N^G(\sigma)$ is the same representation as $\pi|_E$, we will show that the map $\phi : V \to L^2(\mathbb{F}_p)$ defined as

$$\phi(f) = \sum_{d \neq 0} F_d e_d$$

is an isomorphism of G-reps: that is, that ϕ is an isomorphism, and that it respects the G-actions of the representations. We can see that ϕ is an isomorphism because functions in V can be uniquely defined by a $(p-1)$-tuple of complex values $F_1, ..., F_{p-1}$, and the same is true of functions in E, since they are generated by the basis $\{e_d, d \neq 0\}$.

To show that ϕ represents the G-actions, we want to show that this diagram commutes for any $g \in \mathrm{Aff}(p)$:

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{Ind}_N^G(\sigma)(g)} & V \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ L^2(\mathbb{F}_p) & \xrightarrow{\pi(g)} & L^2(\mathbb{F}_p) \end{array}$$

Let $g = (\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix})$ for some $a, b \in \mathbb{F}_p$. Starting from the top-left corner and going clockwise, let $f \in V$ be defined as $f((\begin{smallmatrix} A & B \\ 0 & 1 \end{smallmatrix})) = e^{2\pi i B/p} F_A$. By the definition of Ind_N^G, we have

$$\mathrm{Ind}_N^G(\sigma)(g) f((\begin{smallmatrix} A & B \\ 0 & 1 \end{smallmatrix})) = f((\begin{smallmatrix} A & B \\ 0 & 1 \end{smallmatrix}) g) = f((\begin{smallmatrix} Aa & Ab+B \\ 0 & 1 \end{smallmatrix}))$$

and by the definition of f this becomes

$$e^{2\pi i (Ab+B)/p} F_{Aa} = e^{2\pi i B/p} e^{2\pi i Ab/p} F_{Aa}$$

Thus, the image of f under the induced representation is a function with "new" F_A coefficient equal to $e^{2\pi i Ab/p} F_{Aa}$ for all A. Therefore, applying ϕ gives

$$\sum_{A \neq 0} F_A e_A = \sum_{A \neq 0} e^{2\pi i Ab/p} F_{Aa} e_A$$

Now starting with the same f and going counterclockwise, we first apply ϕ to f to obtain

$$\sum_{d \neq 0} F_d e_d$$

Since $g = (\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix})$, applying $\pi(g)$ takes a function $f(x)$ to $f(a^{-1}(x-b))$. Therefore, we get

$$\sum_{d \neq 0} F_d e^{2\pi i a^{-1}(x-b)d/p}$$

Now we substitute $A = da^{-1}$ and simplify:

$$\sum_{A\neq 0} F_{Aa} e^{2\pi i Ab/p} e^{2\pi i Ax/p} = \sum_{A\neq 0} F_{Aa} e^{2\pi i Ab/p} e_A$$

and so the diagram commutes. Therefore, ϕ is an isomorphism of representations, and thus, we have shown that $\mathrm{Ind}_N^G(\sigma)(g)$ and $\pi|_E$ are the same representation.

□

4. Wavelets

We now introduce the main object of our paper, the wavelet.

Definition 4.1. Let the *wavelet space* E be the subspace of $L^2(\mathbb{F}_p)$ defined as

$$\{f \in L^2(\mathbb{F}_p) | \langle f, 1\rangle = 0\}$$

We call $\psi \in E$ a *wavelet.* Note that $\langle f, 1\rangle = \hat{f}(0)$, and so it is equivalent to say that ψ is a wavelet if and only if $\hat{\psi}(0) = 0$.

Recall that the space $L^2(\mathbb{F}_p)$ is generated by the elements e_d for $d \in \mathbb{F}_p$. Since E is a strict subspace of $L^2(\mathbb{F}_p)$, only a subset of these generators are in E. However, one can check that the condition that $\langle e_d, 1\rangle = 0$ unless $d = 0$, and so all but one of the generators of $L^2(\mathbb{F}_p)$ except for $e_0 = 1$ are in the wavelet space E. Thus, E is the space generated by e_d for $d \in \mathbb{F}_p \setminus \{0\}$.

Given a particular choice of ψ, our goal is to express another function $f \in L^2(\mathbb{F}_p)$ as a linear combination of $\psi_{a,b}$'s. To this end, we define the *wavelet transform* of ψ on f.

Definition 4.2. Let ψ be a wavelet, and let $f \in L^2(\mathbb{F}_p)$. Then the wavelet transform of ψ on f is

$$\mathcal{W}f(a,b) = \sum_{x\in\mathbb{F}_p} f(x)\overline{\psi(a^{-1}(x-b))} = \langle f, \psi_{a,b}\rangle.$$

4.1. Inversion Formula.

Now we will show how to use the wavelet transform to decompose f, first with the additional condition that f must also be a wavelet.

Theorem 4.3. *Let f, ψ be wavelets, $\psi \neq 0$. Then we have*

$$f(x) = \frac{1}{p\langle\psi,\psi\rangle} \sum_{(a,b)\in Aff(p)} \mathcal{W}f(a,b)\psi_{a,b}(x).$$

Proof. Note that this formula is additive: since $\mathcal{W}$ respects addition, one can check that if this formula is true for f it is also true for g. Thus, it suffices to show this for a basis of E, using the condition that f is a wavelet. Thus, we will assume that $f = e_d$ for $d \neq 0$.

Denote the sum in our expression as Σ. Then we have

$$\Sigma = \sum_{(a,b)\in \mathrm{Aff}(p)} \langle e_d, \psi_{a,b}\rangle \psi_{a,b}(x)$$

By Theorem 2.2, we can change the inner product to incorporate the DFT:

$$\Sigma = \frac{1}{p} \sum_{(a,b)\in \mathrm{Aff}(p)} \langle \widehat{e_d}, \widehat{\psi_{a,b}} \rangle \psi_{a,b}(x)$$

By Lemma 3.4, we know that $\psi_{a,b} = \overline{e_b} \cdot D_{a^{-1}} \hat{\psi}$. In addition, we employ the fact that $\widehat{e_d} = p\delta_d$, which is the function which is p at d and 0 everywhere else. Then the expression becomes

$$\Sigma = \sum_{(a,b)\in \mathrm{Aff}(p)} \langle \delta_d, \overline{e_b} \cdot D_{a^{-1}} \hat{\psi} \rangle \psi_{a,b}(x)$$

To evaluate this inner product, we note that δ_d is only nonzero at d, and thus, the entire inner product will be the conjugate of the right multiplicand at d, which is equal to $\overline{\overline{e_b}(d) \cdot D_{a^{-1}}\hat{\psi}(d)} = e_b(d) \cdot \overline{D_{a^{-1}}\hat{\psi}(d)} = e_b(d)\overline{\hat{\psi}(ad)}$. So we have

$$\begin{aligned} \Sigma &= \sum_{(a,b)\in \mathrm{Aff}(p)} e_b(d)\overline{\hat{\psi}(ad)}\psi_{a,b}(x) \\ &= \sum_{(a,b)\in \mathrm{Aff}(p)} e^{2\pi i b d/p}\overline{\hat{\psi}(ad)}\psi(a^{-1}(x-b)) \end{aligned} \tag{4.1}$$

In order to make the $\psi(a^{-1}(x-b))$ term easier to work with, we make the substitution $c = a^{-1}(x-b)$, and it follows that $b = x - ac$. Now we have

$$\Sigma = \sum_{(a,c)\in \mathrm{Aff}(p)} e^{2\pi i (x-ac)d/p}\overline{\hat{\psi}(ad)}\psi(c)$$

The pair of (a,c) which is in $\mathrm{Aff}(p)$ must satisfy $a \in \mathbb{F}_p \setminus \{0\}$ and $c \in \mathbb{F}_p$, and so we now separate this sum:

$$\begin{aligned} \Sigma &= e^{2\pi i d x/p} \sum_{a\in \mathbb{F}_p\setminus\{0\}} \overline{\hat{\psi}(ad)} \sum_{c\in\mathbb{F}_p} e^{-2\pi i d a c/p}\psi(c) \\ &= e^{2\pi i d x/p} \sum_{a\in \mathbb{F}_p\setminus\{0\}} \overline{\hat{\psi}(ad)}\hat{\psi}(ad) \end{aligned} \tag{4.2}$$

Now we use the fact that ψ is a wavelet, and so $\hat{\psi}(0) = 0$. If we were to take the sum of a over $\mathbb{F}_p$ instead of $\mathbb{F}_p \setminus \{0\}$, the extra term would evaluate to $\hat{\psi}(0) = 0$, and so the sum would have the same value. Now as a ranges over $\mathbb{F}_p$, so does ad, since we assumed d to be nonzero. Thus, the above sum becomes $\langle \hat{\psi}, \hat{\psi} \rangle$ and so we have that $\Sigma = e^{2\pi i d x/p}\langle \hat{\psi}, \hat{\psi} \rangle = e_d(x) \cdot p\langle \psi, \psi \rangle$. Plugging this value of Σ into our original expression, we see that this is exactly what we wanted to show.

□

4.2. **Second Inversion Formula.** In the previous theorem, we assumed that ψ is a wavelet. Now we change that assumption slightly, and prove a different result on decomposition.

Theorem 4.4. *Let $f \in L^2(\mathbb{F}_p)$ and let ψ satisfy*

$$(p-1)|\hat{\psi}(0)|^2 = \sum_{k=1}^{p-1} |\hat{\psi}(k)|^2$$

Then the map $f \mapsto \mathcal{W}f$ is an isometry on $L^2(\mathbb{F}_p)$ up to a constant, and we have

$$f(x) = \frac{1}{(p-1)|\hat{\psi}(0)|^2} \sum_{(a,b)\in G} \mathcal{W}f(a,b)\psi_{a,b}(x).$$

Proof. Since $f \in L^2(\mathbb{F}_p)$, it suffices to show this on a basis of $L^2(\mathbb{F}_p)$. So we assume $f = e_d$ for $d \in \mathbb{F}_p$. (Note that since we dropped the assumption that f is a wavelet, now d can equal 0.)

Although we have also changed the assumption that ψ is a wavelet, in proving Theorem 4.3, we did not use this assumption until late into the proof. Thus, we can repeat the argument until that point to get

$$\Sigma = e_d(x) \sum_{a\in\mathbb{F}_p\setminus\{0\}} \hat{\psi}(ad)\overline{\hat{\psi}(ad)}$$

where Σ represents the sum of the wavelet transforms in the theorem statement.

Now we have two cases: either $d = 0$ or $d \neq 0$. In the case that $d = 0$, we have

$$\Sigma = e_d(x)(p-1)\hat{\psi}(0)\overline{\hat{\psi}(0)} = e_d(x)(p-1)|\hat{\psi}(0)|^2$$

and dividing both sides by $(p-1)|\hat{\psi}(0)|^2$ yields the result. On the other hand, in the case that $d \neq 0$, then as a ranges over $\mathbb{F}_p \setminus \{0\}$, so does ad, so we have

$$\Sigma = e_d(x) \sum_{k=1}^{p-1} |\hat{\psi}(k)|^2 = e_d(x)(p-1)|\hat{\psi}(0)|^2$$

by the theorem statement, and dividing both sides by $(p-1)|\hat{\psi}(0)|^2$ yields the result. □

References

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