CONJUGACY CLASSES OF $\mathrm{GL}(2, \mathbb{F}_q)$

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In this note, we discuss the conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_q)$ where q is odd. They play an essential role in obtaining the irreducible representations of $\mathrm{GL}(2, \mathbb{F}_q)$, which has applications in, *e.g.,* the spectral lines of the C_{60} molecule and bounding the spherical functions on the finite upper half plane [T].

The main goal is to build the following table:

Representative	No. Elements in Class	No. Classes
$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	1	$q-1$
$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	q^2-1	$q-1$
$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	q^2+q	$\frac{(q-1)(q-2)}{2}$
$d_{x,y} = \begin{pmatrix} x & \delta y \\ y & x \end{pmatrix}$	q^2-q	$\frac{q(q-1)}{2}$

TABLE 1. Four types of conjugacy classes in each row. First column: the representative of the conjugacy class. Second column: the number of elements in each class. Third column: the number of classes of each type.

The strategy is the following. We will find that there are four types of representatives, with the help of certain subgroups of $\mathrm{GL}(2, \mathbb{F}_q)$, the *maximal tori.* These conjugacy classes are all distinct since they have different minimal polynomials, and that they are complete since the total number of elements in the table equals to $|G|$. As a byproduct, we also obtain the centralizer of each representative in $\mathrm{GL}(2, \mathbb{F}_q)$.

The total number of conjugacy classes is $q^2 - 1$ (I believe there is a typo in [T] that says this number is $q^2 - q$.) This implies that there should be $q^2 - 1$ many irreducible representations of $\mathrm{GL}(2, \mathbb{F}_q)$ as well. Remarkably, all of them can be constructed explicitly, even though we will not do it in this piece. See, for example, Ch.21 of [T], Ch.5 of [FH], and [PS].

1. Classification from Jordan normal forms

First, we explain the interpretation of the four types of representatives: they are associated with the Jordan normal forms. However, the Jordan normal form (JNF) is a concept for the groups over the algebraic closure of the field. More precisely,

Theorem 1.1. *In* $\mathrm{GL}(n, \overline{\mathbb{F}})$, *where* $\overline{\mathbb{F}}$ *is the algebraic closure of* $\mathbb{F}$, *each conjugacy classes is labeled by an unordered set of Jordan blocks.*

For 2×2 matrices, there are two types:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}. \tag{1.1}$$

The former corresponds to the diagonalizable matrices, and the latter corrsponds to the non-diagonalizable ones.

In our case, the group over the algebraic closure is the group $\mathrm{GL}(2, \mathbb{F}_q(\sqrt{\delta}))$. It suffices to take the quadratic extension since the characteristic polynomial has degree at most 2. How is this applicable to $\mathrm{GL}(2, \mathbb{F}_q)$? Here we quote another theorem

Theorem 1.2. *If* $A, B \in \mathrm{GL}(2, \mathbb{F}_q)$ *are conjugate to each other in* $\mathrm{GL}(2, \mathbb{F}_q(\sqrt{\delta})$, *then they are already conjugate in* $\mathrm{GL}(2, \mathbb{F}_q)$.

Notice that the converse is obvious, since $\mathrm{GL}(2, \mathbb{F}_q)$ is the smaller group. This implies that we can analyze the conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_q)$ as if we have access to the quadratic extension; in other words, each Jordan normal form is still associated with a conjuagacy class.

There is an instructive way to organize the conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_q)$, which involves maximal tori. Roughly speaking the diagonalizable matrices split into two types, depending on whether the eigenvalues are in $\mathbb{F}_q$ or not. We will discuss this in the next section.

In addition, recall that to distinguish conjugacy classes, it is often useful to compare their minimal polynomials, since they are invariant under conjugation:

Theorem 1.3. *If* m *is a minimal polynomial for* g, *then it is also a minimal polynomial for* $x \cdot g \cdot x^{-1}$ *for every* $x \in \mathrm{GL}(V)$.

The proof is trivial. Therefore, having distinct minimum polynomials is a sufficient condition for two conjugacy classes to be disjoint.

Finally, there is an elementary and useful formula for the number of elements in the conjugacy class containing $g \in G$:

Theorem 1.4. *Let* $\{g\}$ *denote the conjugacy class containing* $g \in G$ *in* G. *The size of the class is*

$$|\{g\}| = |G|/|G_g|,$$

where G_g *is the centralizer of* g *in* G, *i.e.,* $G_g = \{x \in G| \quad gxg^{-1} = x\}$.

This follows from the orbit-stablizer theorem. Momentarily, we will construct some conjugacy classes. And then we will use Theorem 1.3 to show that they are disjoint and Theorem 1.4 to show that they are complete.

2. Subgroups of $\mathrm{GL}(2, \mathbb{F}_q)$

Let's survey some subgroups of $G = \mathrm{GL}(2, \mathbb{F}_q)$, where q is odd.

The first important subgroup is the *Borel* subgroup B of $G = \mathrm{GL}(2,\mathbb{F}_q)$ defined by

$$B \equiv \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \right\}. \tag{2.1}$$

Notice that a and d have to be non-zero to ensure that the determinant does not vanish. Hence, the order of the Borel subgroup is clearly $|B| = (q-1)^2 q$. One can convince themselves the following statement:

Proposition 2.1. *G acts transitively on the projective line $\mathbb{P}^1(\mathbb{F}_q)$, with B the isotropy group fixing a point, i.e.,*

$$G/B \cong \mathbb{P}^1(\mathbb{F}_q) \equiv \left(\mathbb{F}_q^2 - \{0\}\right)/\mathbb{F}_q^*.$$

Here the star sign indicates that $\mathbb{F}_q^*$ is the multiplicative group of the field.

Exercise 2.2. Prove Proposition 2.1 and try to "visualize" it.

This therefore provides a straightforward calculation for the order of $G = \mathrm{GL}(2,\mathbb{F}_q)$:

Lemma 2.3.

$$|G| = |B| \cdot |\mathbb{P}^1(\mathbb{F}_q)| = (q-1)^2 q(q+1)$$

This will be important for us when computing the number of elements in each conjugacy class.

Another even simpler class of subgroups is called the *maximal torus*,

Definition 2.4. A subgroup T is a *torus* if it is isomorphic to a product of multiplicative groups of finite fields. A torus is called *maximal* if it cannot be properly contained in any other tori.

For $\mathrm{GL}(2,\mathbb{F}_q)$ where q is odd, there are two maximal tori[1]:

- the diagonal matrices $\cong \mathbb{F}_q^* \times \mathbb{F}_q^*$ and
- the subgroup K of elements that fix $\sqrt{\delta}$, where δ is a non-sqare in $\mathbb{F}_q$, namely

$$K = \left\{ \begin{pmatrix} x & \delta y \\ y & x \end{pmatrix} \mid \quad x, y \in \mathbb{F}_q, \quad x^2 - \delta y^2 \neq 0 \right\}. \tag{2.2}$$

Notice that K is isomorphic to the multiplicative group of the quadratic extension $\mathbb{F}_q(\sqrt{\delta})$ $(= \mathbb{F}_{q^2})$ of $\mathbb{F}_q$. Thus we can make the following identification:

Proposition 2.5. *1 and $\sqrt{\delta}$ form a basis for the field extension $\mathbb{F}_{q^2}^*$ over $\mathbb{F}_q$.*

$$\begin{pmatrix} x & \delta y \\ y & x \end{pmatrix} \in K \Longleftrightarrow x + y\sqrt{\delta} \tag{2.3}$$

[1]There are also two maximal tori when the field is taken to be the real numbers $\mathbb{R}$.

As a side note, it is clear that $|K| = q^2 - 1$. Recall that the order of the finite upper half plane $H_q = q(q-1)$ and that it is a symmetric space $H_q = G/K$. This provides another way to compute the order of G, consistent with Lemma 2.3.

As mentioned, two maximal tori correspond to whether the eigenvalues are in $\mathbb{F}_q$ or not. In the next section, we analyze the eigenvalues altogether systematically.

3. Conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_q)$

First of all, we still have the non-diagonalizable matrices, namely

$$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} \tag{3.1}$$

in $\mathrm{GL}(2, \mathbb{F}_q)$, since x must be in $\mathbb{F}_q^*$. For the diagonalizable matrices, the simplest case is when two eigenvalues are the same,

$$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \tag{3.2}$$

so that, again, x has to be in $\mathbb{F}_q^*$. Although this may be obvious, one can check that a_x and b_x, albeit having the same characteristic polynomial, have different minimal polynomials: for a_x, the minimal polynomial (degree $= 1$) is different from the characteristic polynomial (degree $= 2$). By Lemma 1.3, they are indeed in distinct conjugacy classes.

When two eigenvalues are different, there are two possibilities. The first possibility is when two eigenvalues are in $\mathbb{F}_q$, namely the first maximal tori:

$$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad x \neq y. \tag{3.3}$$

x and y are clearly non-zero so that $c_{x,y}$ is invertible.

The second possibility, associated with elements in the second tori but not in the first one, is the case when eigenvalues do not sit in $\mathbb{F}_q^*$. As already shown, they take the following form and contain most of the elements in K, *i.e.*,

$$d_{x,y} = \begin{pmatrix} x & \delta y \\ y & x \end{pmatrix}, \quad y \neq 0. \tag{3.4}$$

As a side note, I think it is good to remind ourselves that this set cannot be the whole subgroup K, since the only conjugacy class that is a subgroup is the trivial one. Indeed, we need y to be non-zero for $d_{x,y}$, or otherwise it overlaps with the case of a_x.

Now we will count the number of conjugacy classes for each type. If the table is complete, then the sum will tell us the number of irreducible representations of $\mathrm{GL}(2, \mathbb{F}_q)$.

For a_x and b_x, it is obvious that the number of classes equals $q-1$ for both, since there are these many choices of x that are inequivalent. For $c_{x,y}$, an naive answer would be $(q-1)(q-2)$. Nevertheless, $c_{x,y}$ and $c_{y,x}$ are conjugate by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, *i.e.*,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}.$$

This is due to the fact that JNFs are defined as an *unordered* set of Jordan blocks, so we are free to permute two eigenvalues and still stay in the same conjugacy class. Hence, the number of classes (for inequivalent $c_{x,y}$) is $\frac{(q-1)(q-2)}{2}$.

Similarly, for $d_{x,y}$, an naive answer would be $q(q-1)$, but $d_{x,y}$ and $d_{x,-y}$ are conjugate by $\left(\begin{smallmatrix} a & -\delta c \\ c & -a \end{smallmatrix}\right)$ for any a and c. Thus the number of classes is $\frac{q(q-1)}{2}$.

Exercise 3.1. Check this.

At this point, you may have the concern: how do we make sure these four conjugacy classes are all of them? This may be obvious from Theorem 1.2 and our analysis on eigenvalues above. Yet, as a sanity check, we can straightforwardly compute the total number of elements in these 4 conjugacy classes and show that it equals $|G|$, which guarantees that we did not miss anyone in the group. To compute the number of elements in each class, we identify the centralizer and then resort to Theorem 1.4.

Let's do this for b_x. Compute both sides of the commutator:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} = \begin{pmatrix} ax & bx+a \\ cx & dx+c \end{pmatrix}$$

and

$$\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax+c & bx+d \\ cx & dx \end{pmatrix}.$$

Demanding that two sides are equal, we find the relation:

$$c = 0, \quad a = d. \tag{3.5}$$

In other words, the centralizer of b_x is $G_{b_x} = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in G\}$. Since a has to be be non-zero to ensure the invertibility, $|G_{b_x}|$ equals to $(q-1)q$. Plugging it into Lemma 1.4, we obtain the number of elements in each b_x's class:

$$|\{b_x\}| = \frac{(q-1)^2 q(q+1)}{(q-1)q} = q^2 - 1, \tag{3.6}$$

agreeing with Table 1.

We invite the readers to repeat a similar procedure in the following exercise:

Exercise 3.2. $G = \mathrm{GL}(2, \mathbb{F}_q)$. Check that

(1) The centralizer of $c_{x,y}$ in G is $\{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\}$.
(2) The centralizer of $d_{x,y}$ in G is K.
(3) Check the number of elements in each class and compare to the second column of Table 1.

We can then add up the total number of elements:

$$\begin{aligned} &1 \cdot (q-1) + (q^2-1) \cdot (q-1) + (q^2+q) \cdot (q-1) + (q^2+q) \cdot \frac{(q-1)(q-2)}{2} + (q^2-q) \cdot \frac{q(q-1)}{2} \\ &= (q-1)^2 q(q+1) \\ &= |G|. \end{aligned}$$

This shows that the conjugacy classes in Table 1 are complete. Adding up the numbers in the last column, we see that there are $q^2 - 1$ conjugacy classes. Hence, there are $q^2 - 1$ irreducible representations of $\mathrm{GL}(2, \mathbb{F}_q)$.

References

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