THE VOLUME OF $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$

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1. Introduction

In this paper, we'll aim to find the volume of a certain double coset: namely, $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$. This space naturally comes up as $SL(2,\mathbb{Z})\backslash\mathcal{H}$, where $\mathcal{H}$ is the upper half plane, and therefore is the fundamental domain for the $SL(2,\mathbb{Z})$-action on the upper half plane. This space is of utmost importance historically, as holomorphic functions on this space (with some extra conditions) are known as modular forms, which have played a pivotal role in number theory in the last century. There are many generalizations of the volume of fundamental domains of various modular groups obtained by Siegel and Langlands, but in this paper we will just be concerned with the most historically important case.

To do so, we will first define and explain the notion of Haar measures, which will allow us to form reasonable differential forms on certain topological groups (namely real Lie groups). We will then require the use of slight generalizations of Poisson summation, and finally apply Poisson summation to computing the integral of the Haar measure over the space $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$.

2. Haar measure

We will always denote by G, a locally compact Hausdorff topological group. In this paper, by locally compact Hausdorff topological groups, we really mean real Lie groups:

Example 2.1. The usual suspects $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $SO(n,\mathbb{R})$, etc. are all examples of locally compact Hausdorff topological groups.

Any such G comes with a maximal compact subgroup K, which is defined to be a maximal element among the compact subgroups. Although K is not unique, it is unique *up to conjugation*: we call this essentially unique. Much of the discussion in this paper is far more general, but we'll be primarily concerned with the $SL(2,\mathbb{R})$ case. Therefore, we'll only describe the maximal compact subgroup in this case.

Example 2.2. The maximal compact subgroup of $SL(2,\mathbb{R})$ is $SO(2,\mathbb{R})$, up to conjugation.

We are aiming to define a sort of integration on G, and so the first important step is defining a measure which is compatible with the group structure. However, we'll actually define it more generally, so that it can be extended to coset spaces of G in addition to G itself.

Definition 2.3. A **left Haar measure** on a locally compact Hausdorff topological space M with a continuous action of G is a Borel measure μ on M which is left G-invariant. i.e. $\mu(gX) = \mu(X)$ for all Borel sets $X \subset M$ and all $g \in G$.

Most of our early applications of a Haar measure will be on G itself, with the natural action of G given by left multiplication. When we say that μ is a Haar measure on G, we implicitly mean that μ is a Haar measure on G treated as a topological space, with the natural group action of G by multiplication. Defining it in this generality allows us to also extend the notion of Haar measure to quotients of G, such as G/K.

With every (Haar) measure μ, there is an associated (translation-invariant) differential top-form $d\mu$, which is defined in the following manner. First, the differential form must satisfy

$$\int_{h\in H} d\mu(h) = \mu(H)$$

for any measurable subset $H \subset M$. From there, we can define integration on compactly supported functions $f : M \to \mathbb{C}$ in the usual way. For the remainder of the paper, we will consider μ and $d\mu$ to be the same.

Example 2.4. [Bum04, Exercise 1.1] Let $\mu_M(X)$ be the usual Euclidean measure on $\mathrm{Mat}_n(\mathbb{R})$, viewed as $\mathbb{R}^{n^2}$. Let us view $\mathrm{Mat}_n(\mathbb{R})$ as a Lie group under addition. Concretely, letting $(x_{ij})_{1\le i,j\le n}$ be the coordinates of $\mathrm{Mat}_n(\mathbb{R})$, we can express the differential top-form associated to $d\mu_M$ as

$$\prod_{1\le i,j\le n} dx_{ij}$$

(where the product is taken to be the wedge product). Let us show that this is both left and right invariant. Take any $X \in \mathrm{Mat}_n(\mathbb{R})$, where $X = (X_{ij})_{i,j}$. Let us denote L_X to be the function $\mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R})$ which is left translation, i.e. $Y \mapsto X + Y$ (equivalently, right translation, because $\mathrm{Mat}_n(\mathbb{R})$ is abelian). Then since $d(\text{constant}) = 0$, we have

$$L_X^* \, d\mu_M = \prod_{1\le i,j\le n} d(x_{ij} - X_{ij}) = \prod_{1\le i,j\le n} dx_{ij} = d\mu_M,$$

hence $d\mu_M$ indeed defines a Haar measure on $\mathrm{Mat}_n(\mathbb{R})$.

Example 2.5. [Bum04, Exercise 1.1] Let $\mu_M(X)$ be the usual Euclidean measure on $\mathrm{Mat}_n(\mathbb{R})$ as in the previous example. Then $\frac{d\mu_M}{|\det X|^n}$ is both a left and right Haar measure on $GL(n,\mathbb{R}) \subset \mathrm{Mat}_n(\mathbb{R})$, which is a Lie group under multiplication. Now let $L_g : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ denote left multiplication $x \mapsto gx$; the case of right multiplication is completely analogous. We have

$$\begin{aligned}
L_g^* \frac{d\mu_M(x)}{|\det(x)|^n} &= \frac{d\mu_M(g^{-1}x)}{|\det(g^{-1}x)|^n}, \\
&= |\det g|^n |\det x|^{-n} \prod_{j=1}^{n} d(gx_j), \\
&= \prod_{j=1}^{n} |\det g^{-1}| \prod_{i=1}^{n} dx_{ij}, \\
&= \frac{d\mu_M(x)}{|\det x|^n}.
\end{aligned}$$

The important fact which allows us to sweep certain details under the rug is the following theorem.

Theorem 2.6. [Gol06, Theorem 1.4.2, 1.5.1] *There exists a unique left-invariant Haar measure μ on G, up to scalar. If μ is normalized so that $\mu(K) = 1$ for some subgroup $K \subseteq G$, then there is a unique left-invariant Haar measure on G/K (as a locally compact Hausdorff topological space with G-action), up to scalar.*

Armed with these technical assurances, we will construct the left Haar measure explicitly for $SL(2,\mathbb{R})$.

3. Left Haar measure on $SL(2,\mathbb{R})$

In this paper, our main interest is the group $SL(2,\mathbb{R})$. In order to construct an explicit Haar measure, we first want a simpler description of elements of $SL(2,\mathbb{R})$.

Proposition 3.1 (Iwasawa Decomposition)**.** *Any element in $SL(2,\mathbb{R})$ can be written uniquely as*

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}_{>0}$, and $\theta \in [0, 2\pi)$.

The proof is not particularly enlightening or important for us, so we omit it. However, the interested reader may find a proof in such sources as [Lan05, VI §4]. Note that in the Iwasawa decomposition, the K is precisely the K we mentioned above as the maximal compact subgroup $SO(2,\mathbb{R}) \subset SL(2,\mathbb{R})$. In particular, this gives us simple coordinates (x, y, θ) with which to express the Haar measure of $SL(2,\mathbb{R})$. In fact, the Haar measure is given as follows.

Proposition 3.2. *A left Haar measure for $SL(2,\mathbb{R})$ is given by*

$$\frac{dx\,dy}{y^2}\,d\theta.$$

The proof is again unenlightening, so we point the reader to [Lan12, III §1]. However, we make two remarks: one (which the reader can find in the aforementioned source), that such a Haar measure is (fairly) easily constructed under the general assumptions of an Iwasawa-type NAK decomposition from the Haar measures on N, A, and K; and two, that to check this manually, one needs only check invariance under matrices of the form $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, since these generate $SL(2,\mathbb{R})$.

But notice the very nice interpretation of the coordinates (x, y) in (x, y, θ): we have $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$, which we can view as the upper-half plane $\mathcal{H} \subset \mathbb{R}^2$. In fact, $SL(2,\mathbb{R})/SO(2,\mathbb{R}) \cong \mathcal{H}$ as manifolds, and this is a consequence of the following theorem applied to $M = SL(2,\mathbb{R})$ and $G = SO(2,\mathbb{R})$ (with $SO(2,\mathbb{R})$ acting in the obvious way):

Theorem 3.3. [Lee13, Theorem 21.10] *Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimension equal to $\dim M - \dim G$, and has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

In order to plausibly integrate on $SL(2,\mathbb{R})/SO(2,\mathbb{R})$, we need a left-invariant Haar measure on $SL(2,\mathbb{R})/SO(2,\mathbb{R})$. In order to apply Theorem 2.6, we need to scale the Haar measure given in Proposition 3.2 by

$$\int_{SO(2)} \frac{dx\,dy}{y^2}\,d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

We then obtain the Haar measure

$$\frac{1}{2\pi}\frac{dx\,dy}{y^2}\,d\theta,$$

which becomes a Haar measure on $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ by omitting the $d\theta$ term. Now rescaling, we can define the following differential form on $\mathcal{H}$.

Proposition 3.4. *Let $\mathcal{H} \coloneqq \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. Then we define*

$$d^*z \coloneqq \frac{dx\,dy}{y^2}.$$

(Note that this is essentially the image of the Haar measure on $SL(2,\mathbb{R})$, now onto $SL(2,\mathbb{R})/SO(2,\mathbb{R})$.)

Remark 3.5. Although we could have easily defined d^*z on $\mathcal{H}$ beforehand, the advantage of going through this theory is that we know that d^*z is $SL(2,\mathbb{R})$-invariant as well.

4. Setup and preliminary tools

The main result of our paper will be computing the volume of

$$SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R}),$$

which is more precisely defined as

$$\mathrm{Vol}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})) \coloneqq \int_{SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})} d^*z.$$

Before we do so, however, we need some preliminary results which will be discussed in this section. For convenience, let us set up some notation to fix for the remainder of the paper.

Notation 4.1. We will denote $\Gamma \coloneqq SL(2,\mathbb{Z})$, as a discrete subgroup of $SL(2,\mathbb{R})$.

Notation 4.2. As from before, we will identify $\mathcal{H}$, the upper-half plane, with $SL(2,\mathbb{R})/SO(2,\mathbb{R})$.

In this notation, our goal will be to compute

$$\mathrm{Vol}(\Gamma\backslash\mathcal{H}) \coloneqq \int_{\Gamma\backslash\mathcal{H}} d^*z.$$

Let us give a brief outline of the argument. The basic idea is to study integrals of test functions in order to extract information about the volume of $\Gamma\backslash\mathcal{H}$. Our chosen test function $f : SL(2,\mathbb{R}) \to \mathbb{C}$ will be a compactly supported right-$SO(2,\mathbb{R})$ invariant function. In order to make it Γ-invariant, we sum over all $\mathbb{Z}^2$-shifts to produce F, a function which is now right-$SO(2,\mathbb{R})$ invariant and left-Γ invariant, and we will take an integral of F over $\Gamma\backslash\mathcal{H}$ to relate it to the volume. Finally, summation over $\mathbb{Z}^2$-shifts of f lead us to naturally use Poisson summation to relate this to $\mathbb{Z}^2$-shifts of $\widehat{f}$, the Fourier transform of f, giving us enough information to compute the volume.

We'll begin with the following Lemma 4.3, which describes the orbits of $\mathbb{Z}^2$ under the action of $SL(2,\mathbb{Z})$ acting by right multiplication on row vectors.

Lemma 4.3. *The action of $SL(2,\mathbb{Z})$ on $\mathbb{Z}^2$ has orbits indexed by the nonnegative integers: the elements $(0,n)$ are representatives of each orbit, i.e.*

$$\mathbb{Z}^2 = \bigsqcup_{n\geq 0} (0,n)SL(2,\mathbb{Z}).$$

Furthermore, each orbit can be identified with $\Gamma_\infty\backslash SL(2,\mathbb{Z})$, where $\Gamma_\infty := \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$; therefore, we may identify

$$(0,n)SL(2,\mathbb{Z}) \leftrightarrow \{(0,n)\gamma \mid \gamma \in \Gamma_\infty\backslash SL(2,\mathbb{Z})\}.$$

Proof. It's clear that one orbit is $(0,0)$. We will demonstrate that the orbits are essentially the gcd of the two entries: let $(a',b') = n(a,b)$ with $\gcd(a,b) = 1$. Then by the Eulidean algorithm, there exist $x, y \in \mathbb{Z}$ with

$$ax + by = 1,$$

and therefore we have the matrix

$$(a,b)\begin{pmatrix} b & x \\ -a & y \end{pmatrix} = (0,1).$$

Furthermore, we have

$$\det\begin{pmatrix} b & x \\ -a & y \end{pmatrix} = ax + by = 1 \implies \begin{pmatrix} b & x \\ -a & y \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Therefore we have shown that $(a',b') \in (0, \gcd(a',b')) \cdot SL(2,\mathbb{Z})$. It suffices to show that any two orbits $(0,n_1)SL(2,\mathbb{Z})$ and $(0,n_2)SL(2,\mathbb{Z})$ are disjoint. We can easily rule out the $n = 0$ orbit, so let's assume $n_1, n_2 > 0$. But we can explicitly shown this: if

$$(0,n_1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (0,n_2),$$

then clearly $c = 0$, so $\det = ad$. Since this matrix must have determinant 1, we have $a = d = \pm 1$. However, multiplying out the second entry, we have that $n_2 = dn_1$, and since $n_1, n_2 > 0$, we must have $d = 1$, hence $a = 1$. Then the condition that $1 = \frac{1\cdot n_2}{n_1}$ implies that $n_1 = n_2$, thus we conclude that the orbits are disjoint.

To see the last statement, it suffices to see that for any $(0,n)$, the stabilizer in $SL(2,\mathbb{Z})$ is precisely Γ_∞. □

Now we will introduce a test function f, which is essentially an arbitrary function that we will conveniently specify later. From f, we will construct a left-$SL(2,\mathbb{Z})$

invariant function F on $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ in the usual way: by summation over all $\mathbb{Z}^2$-shifts.

Lemma 4.4. *Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a smooth, compactly supported function which is right-$SO(2,\mathbb{R})$ invariant. Define $F : SL(2,\mathbb{R}) \to \mathbb{C}$ to be the function*

$$F(z) = \sum_{(m,n)\in\mathbb{Z}^2} f((m,n)\cdot z).$$

Then F is left-$SL(2,\mathbb{Z})$ invariant and right-$SO(2,\mathbb{R})$ invariant, and hence defines a function $F : SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R}) \to \mathbb{C}$. By identifying $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ with the upper-half plane $\mathcal{H}$, F defines a function $\Gamma\backslash\mathcal{H} \to \mathbb{C}$.

Proof. The right-$SO(2,\mathbb{R})$ invariance is clear because f is right-$SO(2,\mathbb{R})$ invariant. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Using the fact that $SL(2,\mathbb{Z}) : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a bijection, we have that

$$F(\gamma z) = \sum_{(m,n)\in\mathbb{Z}^2} f((m,n)\begin{pmatrix} a & b \\ c & d \end{pmatrix} z) = \sum_{(m',n')\in\mathbb{Z}^2} f((m',n')z) = F(z),$$

where $(m',n') = (m,n)\gamma$. Note that the rearrangement is allowed because f is compactly supported, hence only finitely many terms are nonzero, so this is allowed. ☐

The important of Lemma 4.4 is that we can now produce many functions acting on our coset space $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$, and furthermore we can group the terms together due to Lemma 4.3. The last tool we will need is Poisson summation, which will relate the constructed function F to both its original test function f and the Fourier transform.

4.1. Poisson summation.

We first introduce the notion of Fourier transforms over more general vector spaces. Although we will only need the case $n = 2$, let us define it more generally for $\mathbb{R}^n$.

Definition 4.5. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a compactly supported function. Then we define the **Fourier transform** of f to be

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i\langle x,y\rangle}\, dx,$$

where $\langle , \rangle$ is the dot product on two n-vectors.

Proposition 4.6 (Poisson summation)**.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ to be compactly supported function. Then*

$$\sum_{x\in\mathbb{Z}^n} f(x) = \sum_{x\in\mathbb{Z}^n} \widehat{f}(x).$$

Proof. Define

$$F(x) := \sum_{t\in\mathbb{Z}^n} f(x+t).$$

Note that F is periodic, i.e. $F(x+t) = F(x)$ for $t \in \mathbb{Z}^n$. Therefore F has a Fourier transform

$$F(x) = \sum_{t\in\mathbb{Z}^n} \widehat{f}(t) e^{i\langle -,x\rangle},$$

and we can see that the Fourier coefficient of $t \in \mathbb{Z}^n$ is given by

$$\begin{aligned}\int_{[0,1]^n} F(x)e^{-2\pi i\langle t,x\rangle}\,dx &= \int_{[0,1]^n}\sum_{s\in\mathbb{Z}^n} f(x+s)e^{-2\pi i\langle t,x\rangle}\,dx,\\ &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle t,x\rangle}\,dx,\\ &= \widehat{f}(t).\end{aligned}$$

Now let us plug in $x = 0$: we find that

$$\sum_{t\in\mathbb{Z}^n} f(t) = \sum_{t\in\mathbb{Z}^n}\widehat{f}(t).$$

□

Remark 4.7. The hypothesis that f is compactly supported ensures that the sum makes sense. However, we don't actually need such a strong hypothesis; "rapidly decreasing functions" will also ensure that the sum converges. For convenience in this paper, however, we'll stick to the compactly supported hypothesis.

Corollary 4.8. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a compactly supported function. For $g \in SL(n,\mathbb{R})$, we have that*

$$\sum_{t\in\mathbb{Z}^n} f(tg) = \sum_{t\in\mathbb{Z}^n}\widehat{f}(t(g^{-1})^\intercal).$$

Proof. Define the function $h(x) = f(x\cdot g)$. Then clearly h is still compactly supported, so by Poisson summation (4.6), we have

$$\sum_{t\in\mathbb{Z}^n} h(t) = \sum_{t\in\mathbb{Z}^n}\widehat{h}(n).$$

But then note that

$$\begin{aligned}\widehat{h}(y) &= \int_{\mathbb{R}^n} f(xg)e^{-2\pi i\langle x,y\rangle}\,dx,\\ &= \int_{\mathbb{R}^n} f(z)e^{-2\pi i\langle zg^{-1},y\rangle}\,dz,\\ &= \int_{\mathbb{R}^n} f(z)e^{-2\pi i\langle z,y(g^{-1})^\intercal\rangle}\,dz,\\ &= \widehat{f}(y\cdot(g^{-1})^\intercal).\end{aligned}$$

Note that we crucially used that the Euclidean measure dx (defined to be $d\mu_M$ in Example 2.4) is $SL(n,\mathbb{R})$-invariant, and also that

$$xMy^\intercal = x(My^\intercal) = x(yM^\intercal)^\intercal \implies \langle xM,y\rangle = \langle x,My\rangle = \langle x,yM^\intercal\rangle.$$

□

5. Computation of volume

In this section, we will finally compute $\mathrm{Vol}(\Gamma\backslash\mathcal{H})$ using the results from the last section. In Lemma 4.4, we chose a smooth, compactly supported function $f : \mathbb{R}^2 \to \mathbb{C}$ which is right-$SO(2,\mathbb{R})$ invariant, and produced the auxiliary function

$F : SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R}) \to \mathbb{C}$ by summation over $\mathbb{Z}^2$-shifts. It turns out that integrating F is enough to produce a relation with $\mathrm{Vol}(\Gamma\backslash\mathcal{H})$, due to our description of $SL(2,\mathbb{Z})$-orbits of $\mathbb{Z}^2$ (Lemma 4.3). In this section, we will frequently identify $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ with the upper-half plane $\mathcal{H}$.

Proposition 5.1. *Let f and F be as in Lemma 4.4; the lemma shows that F defines a function $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R}) \to \mathbb{C}$. Taking the integral of F with respect to the Haar measure d^*z, we have the equality*

$$\int_{SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})} F(z)\, d^*z = \int_{\Gamma\backslash\mathcal{H}} F(z)\, d^*z = f((0,0))\cdot \mathrm{Vol}(\Gamma\backslash\mathcal{H}) + \frac{\pi}{3}\cdot \widehat{f}((0,0)).$$

Proof. Our strategy will be to break the integral into a sum of integrals over all of the orbits, then change variables until the integral is independent of the orbit. From Lemma 4.3, the decomposition of $\mathbb{Z}^2$ into right $SL(2,\mathbb{Z})$-orbits shows that

$$\int_{\Gamma\backslash\mathcal{H}} F(z)\, d^*z = f((0,0))\cdot \mathrm{Vol}(\Gamma\backslash\mathcal{H}) + \sum_{n>0} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash\mathcal{H}} f((0,n)\gamma z)\, d^*z.$$

We may change the second summation with the integral to

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash\mathcal{H}} f((0,n)\gamma z)\, d^*z = 2\int_{\Gamma_\infty\backslash\mathcal{H}} f((0,n)z)\, dz,$$

where the factor of 2 comes from the trivial action of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Intuitively, this is due to the fact that the orbits are indexed by $\mathbb{Z}_{\geq 0}$, when the natural action gives "natural" orderings by $\mathbb{Z}$, hence we have the trivial action which sends $n \mapsto -n$. We can view $\Gamma_\infty\backslash\mathcal{H}$ as $\int_0^\infty$ with respect to y, as from the Iwasawa decomposition, we have

$$\mathcal{H} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R}) \cong \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \right\} = \Gamma_\infty \cdot \left\{ \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \right\}.$$

Now writing

$$z = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix},$$

we have that

$$f((0,n)z) = f((0, ny^{-1/2})). \tag{5.1}$$

Then since

$$d^*z = \frac{dx\, dy}{y^2},$$

we can make the change of variables $y = n^2t^{-2}$, which yields $dy = -2n^2t^{-3}\,dt$, and hence we have

$$\begin{aligned}
2\sum_{n>0}\int_{\Gamma_\infty\backslash\mathcal{H}} f((0,ny^{-1/2}))\frac{dx\,dy}{y^2} &= 2\sum_{n>0}\int_0^\infty f((0,ny^{-1/2}))\frac{dx\,dy}{y^2},\\
&= 2\sum_{n>0}\int_\infty^0 f((0,t))\cdot\frac{-2}{n^2}t\,dt,\\
&= 4\sum_{n>0}\frac{1}{n^2}\int_0^\infty f((0,t))t\,dt,\\
&= \frac{2\pi^2}{3}\int_0^\infty f((0,t))t\,dt.
\end{aligned}$$

Now we may reformulate $\int_0^\infty f((0,t))t\,dt$ using right-$SO(2,\mathbb{R})$-invariance of f. Let $0 \le \theta \le 2\pi$; then since $f((0,t)) = f((0,t)\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix}) = f((t\cos\theta, t\sin\theta))$, we have

$$\begin{aligned}
\int_0^\infty f((0,t))t\,dt &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty f((t\cos\theta,t\sin\theta))t\,dt\,d\theta,\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} f((u,v))\,du\,dv,\\
&= \frac{1}{2\pi}\widehat{f}((0,0)).
\end{aligned}$$

Putting these together yields the result. □

The final step is using Poisson summation: this will give us another relation, allowing us to compute $\mathrm{Vol}(\Gamma\backslash\mathcal{H})$.

Theorem 5.2. [Gol06, Theorem 1.6.1] *We have the equality*

$$\int_{SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})} d^*z = \frac{\pi}{3}.$$

Proof. From Poisson summation (4.8), we have that

$$F(z) = \sum_{\mathbb{Z}^2} f((m,n)z) = \sum_{\mathbb{Z}^2}\widehat{f}((m,n)(z^{-1})^\intercal),$$

since $z \in SL(2,\mathbb{R})$. Note that Γ is stable under $((\cdot)^{-1})^\intercal$. However, after applying $((\cdot)^{-1})^\intercal$, the analogue of Lemma 4.3 is that

$$\mathbb{Z}^2 = \bigsqcup_{n\ge 0}(n,0)\cdot SL(2,\mathbb{Z})'$$

where we denote $SL(2,\mathbb{Z})'$ to indicate the "twisted" action of $(t,\gamma)\mapsto t\cdot(\gamma^{-1})^\intercal$. (This is in particular to ensure that Γ_∞ still acts trivially.) One important observation is that this is still a group action: we have $(\gamma^{-1})^\intercal(\psi^{-1})^\intercal = (\psi^{-1}\gamma^{-1})^\intercal = ((\gamma\psi)^{-1})^\intercal$. Therefore, we have that

$$\begin{aligned}
\int_{\Gamma\backslash\mathcal{H}} F(z)\,d^*z &= \widehat{f}((0,0))\cdot\mathrm{Vol}(\Gamma\backslash\mathcal{H}) + \sum_{n>0}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\Gamma\backslash\mathcal{H}}\widehat{f}((n,0)(\gamma^{-1})^\intercal(z^{-1})^\intercal d^*z,\\
&= \widehat{f}((0,0))\cdot\mathrm{Vol}(\Gamma\backslash\mathcal{H}) + \sum_{n>0}\int_{\Gamma_\infty\backslash\mathcal{H}}\widehat{f}((n,0)(z^{-1})^\intercal d^*z.
\end{aligned}$$

Now we again write $z = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$, and we find that

$$(n, 0) \cdot (z^{-1})^{\intercal} = (y^{-1/2} n, 0),$$

which (essentially) matches Equation 5.1. Now we may proceed identically to the proof of Proposition 5.1, from which we obtain the same result, but with $\widehat{f}$ instead of f. In other words, we see that

$$\begin{aligned} \int_{\Gamma \setminus \mathcal{H}} F(z)\, d^*z &= \widehat{f}((0,0)) \cdot \mathrm{Vol}(\Gamma \setminus \mathcal{H}) + \frac{\pi}{3} \cdot \hat{\hat{f}}((0,0)), \\ &= \widehat{f}((0,0)) \cdot \mathrm{Vol}(\Gamma \setminus \mathcal{H}) + \frac{\pi}{3} \cdot f((0,0)), \end{aligned}$$

since $\hat{\hat{f}}(x) = f(-x)$. Having computed $\int_{\Gamma \setminus \mathcal{H}} F(z)\, d^*z$ in two ways, we set them equal and find that

$$\Big(f((0,0)) - \widehat{f}((0,0))\Big) \left(\mathrm{Vol}(\Gamma \backslash \mathcal{H}) - \frac{\pi}{3}\right) = 0.$$

Now choosing f such that $f((0,0)) \neq \widehat{f}((0,0))$, we find that

$$\mathrm{Vol}(\Gamma \backslash \mathcal{H}) = \int_{SL(2,\mathbb{Z}) \backslash SL(2,\mathbb{R}) / SO(2,\mathbb{R})} d^*z = \frac{\pi}{3}.$$

□

Remark 5.3. We can actually calculate

$$\int_{SL(n,\mathbb{Z}) \backslash SL(n,\mathbb{R}) / SO(n,\mathbb{R})} d^*z$$

by induction from the $n = 2$ case (where d^*z is defined analogously), and amazingly, we will find factors of $\zeta(n)$ in the answer (see [Gol06, §1.6] for more details). The reader may have noticed that this work was quite intensive when we could have just integrated d^*z over the fundamental domain for $\Gamma \backslash \mathcal{H}$, which is well-known: one notes that Γ is generated by elements which act on $\mathcal{H}$ by translations and inversions, which immediately gives that the (closure of the) fundamental domain is bounded below by the unit circle and to the left and right by $-1/2 \leq x \leq 1/2$. Therefore, the volume $\mathrm{Vol}(\Gamma \backslash \mathcal{H})$ is also equal to

$$\mathrm{Vol}(\Gamma \backslash \mathcal{H}) = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2}\, dx = \int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) \Big|_{-1/2}^{1/2} = \frac{\pi}{3}.$$

However, the advantage of the method outlined in this paper is that it generalizes and allows us to compute the volume of $SL(n,\mathbb{Z}) \backslash SL(n,\mathbb{R}) / SO(n,\mathbb{R})$ for $n > 2$ as well.

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