RADAR CROSS-AMBIGUITY AND THE HEISENBERG GROUP

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Abstract. We explain the connection between the Schrödinger representation of the continuous Heisenberg group and the radar cross-ambiguity function, as well as an analogue for finite Heisenberg groups.

1. Introduction

The continuous Heisenberg group and its Schrödinger representations have played an important role in the study of quantum mechanics since the early twentieth century. In comparison, radar technology was developed many years later during World War II in the mid-twentieth century, and only entered mainstream use in the second half of the twentieth century. It turns out that an important function in the study of radars, called the radar cross-ambiguity function, is closely related to the Schrödinger representation, allowing many of its properties to be proven using the already developed theory surrounding the Schrödinger representation.

This paper focuses on explaining this relation between the Schrödinger representations and the radar cross-ambiguity function, and is organized as follows. Sections 2 and 3, respectively, introduce the Heisenberg group and its Schrödinger representations. Section 4 defines and motivates the radar cross-ambiguity function, leading to Section 5, which discusses the aforementioned connection to the Schrödinger representation. Finally, Section 6 discusses an analogue of this connection for finite Heisenberg groups. Our exposition mainly draws from [T] and [S].

2. The Heisenberg group and its properties

Definition 2.1. The *continuous Heisenberg group*, denoted $\mathrm{Heis}(\mathbb{R})$, is the group of 3×3 matrices of the following form:

$$\mathrm{Heis}(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in R \right\}.$$

A matrix in the Heisenberg group $\mathrm{Heis}(\mathbb{R})$ is determined only by the three entries x, y, z. It will be redundant to write out the full 3×3 matrix, so we will use the notation $[x, y, z]$ to denote the element $\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Let us now list some properties of $\mathrm{Heis}(\mathbb{R})$.

- By carrying out matrix multiplications, we see that the **group law** in $\mathrm{Heis}(\mathbb{R})$ is given by

$$[x, y, z]\,[x', y', z'] = [x + x', y + y', z + z' + xy']\,, \tag{2.1}$$

- The xy' term in the equation above calls attention to the fact that $\mathrm{Heis}(\mathbb{R})$ is **non-abelian.**

- The **inverse** of an element $[x, y, z]$ is given by

$$[x, y, z]^{-1} = [-x, -y, -z + xy] .$$

- The **conjugate** of $[x, y, z]$ by $[x', y', z']$ is given by

$$[x', y', z'] [x, y, z] [x', y', z']^{-1} = [x, y, z + x'y - xy'] .$$

- From the conjugation formula, an element $[x, y, z]$ is in the **center** Z of $\mathrm{Heis}(\mathbb{R})$ if and only if it is of the form $[0, 0, z]$. The quotient $\mathrm{Heis}(\mathbb{R})/Z$ consists of elements of the form $[x, y, 0]$, and this quotient group is isomorphic to $\mathbb{R}^2$.
- In general, zeroing out a nonempty proper subset of the three coordinates of $[x, y, z]$ gives us a subgroup of $\mathrm{Heis}(\mathbb{R})$ isomorphic to either $\mathbb{R}$ or $\mathbb{R}^2$.

3. The Schrödinger representations

The continuous Heisenberg group $\mathrm{Heis}(\mathbb{R})$ admits a family of representations called the Schrödinger representations.

Definition 3.1. The *Schrödinger representation* π_s, with parameter $s \in \mathbb{R}$, is the representation of $\mathrm{Heis}(\mathbb{R})$ on $L^2(\mathbb{R})$ given by

$$[\pi_s [x, y, z] f] (t) := f(t + x) \exp(2\pi i s(z + ty)). \tag{3.1}$$

Although the Schrödinger representations may look complicated at first glance, they are rather natural objects! First, we can actually think of it intuitively as follows. Suppose $f \in L^2(\mathbb{R})$ is a signal pulse, and t represents the time. Then,

- $[x, 0, 0]$ acts by shifting the **time** of the signal,
- $[0, y, 0]$ acts by shifting the **frequency** of the signal, and
- $[0, 0, z]$ acts by shifting the **phase** of the signal.

Furthermore, in some sense, the Schrödinger representations are the only 'interesting' representations of $\mathrm{Heis}(\mathbb{R})$, as shown by the following two theorems, whose proofs are beyond the scope of this paper.

Theorem 3.2. *When $s \neq 0$, π_s is an infinite-dimensional irreducible unitary representation.*

Theorem 3.3 (Stone-von Neumann)**.** *Every irreducible unitary representation of* $\mathrm{Heis}(\mathbb{R})$ *in which the action of the center Z is nontrivial is unitarily equivalent to π_s for a unique $s \neq 0$.*

Here, by a nontrivial action we simply mean an action that is not the identity. In π_s, an element $[0, 0, z]$ in the center Z acts by scalar multiplication by $\exp(2\pi i s z)$, and therefore the action of the center is trivial if and only if $s = 0$. Furthermore, if the action of the center is trivial, then our representation just becomes a representation of $\mathrm{Heis}(\mathbb{R})/Z \simeq \mathbb{R}^2$.

We have not yet verified that π_s is a representation. It is not hard to do so by directly checking that it is a group homomorphism, but we offer an alternate proof by showing that it is an induced representation. Recall that induced representations provide a way to induce a representation of G up from a representation of a subgroup $H \subset G$.

Definition 3.4. Let $H \subset G$ be groups, and let $\sigma : H \to GL(W)$ be a representation of H. The *induced representation* $\pi = \mathrm{Ind}_H^G(\sigma)$ from H up to G is a representation of G on the space of functions

$$V := \{f : G \to W \mid f(hg) = \sigma(h)f(g) \text{ for all } g \in G, h \in H\}$$

whose G-action is defined by $[\pi(g)f](x) = f(xg)$ for all $x, g \in G$.

In our case, the subgroup H of $\mathrm{Heis}(\mathbb{R})$ is given by zeroing out the x-coordinate.

Proposition 3.5. *Let $H := \{[0, y, z] \mid y, z \in \mathbb{F}_q\} \subset \mathrm{Heis}(\mathbb{R})$, and let $\sigma_s : H \to \mathrm{GL}(\mathbb{C})$ be the one-dimensional representation of H given by*

$$\sigma_s\,[0, y, z] = \exp\left(2\pi i s z\right).$$

Then, $\pi_s \simeq \mathrm{Ind}_H^G\, \sigma_s$.

Proof. The representation $\pi'_s = \mathrm{Ind}_H^G\, \sigma_s$ acts on the vector space

$$V = \{f : \mathrm{Heis}(\mathbb{F}_q) \to \mathbb{C} \mid f(hg) = \sigma_s(h)f(g) \text{ for all } h \in H, g \in \mathrm{Heis}(\mathbb{F}_q)\}.$$

The functions in V are uniquely determined by its values on the set $N = \{[x, 0, 0] \mid x \in \mathbb{R}\}$, because this set contains exactly one representative of each coset Hg. (In fact, N is a normal subgroup of G, and we have $G = N \rtimes H$.)

It follows that $V \simeq L^2(N) \simeq L^2(\mathbb{R})$. We now identify $[x, 0, 0] \in N$ with $x \in \mathbb{R}$, and the action of π'_s on a function $f \in V$ is then given by

$$\begin{aligned}
[\pi'_s\,[x, y, z]\, f](t) &= f([t, 0, 0]\,[x, y, z]) \\
&= f([t + x, y, z + ty]) \\
&= f([0, y, z + ty]\,[t + x, 0, 0]) \\
&= \sigma_s([0, y, z + ty])f([t + x, 0, 0]) \\
&= \exp\left(2\pi i s(z + ty)\right) f(t + x),
\end{aligned}$$

which agrees with Eq. (3.1). □

4. The radar cross-ambiguity function

In this section, we introduce the radar cross-ambiguity function, following Chapter 21 of [B]. Radar technology uses echoes of an electromagnetic pulse to determine the distance and radial velocity of a target. In an ideal scenario, when a signal $\psi(t)$, modeled as a continuous function on $\mathbb{R}$, is sent by an antenna to a moving target, the resulting echo will take the form

$$f_{x,y}(t) := f(t - x)\exp(-2\pi i y t),$$

where x and y are some fixed constants. The sign choice is not too important here, but we chose it to simplify later computations. This already bears a lot of resemblance with Eq. (3.1)! Indeed, again,

- x can be thought of as the **time** delay, depending on the distance of the target, and
- y is the **frequency** shift, which depends on the radial velocity of the target, due to the Doppler effect.

In the real world, however, due to noise and various imperfections, we will never receive an echo that is perfectly equal to $f_{x,y}(t)$ for some x, y. Rather, when we receive an echo $g(t)$, we try to match it up to an $f_{x,y}(t)$ that is closest to $g(t)$ in some sense. To be able to do this, we would like to have the following two conditions.

- First, we want there to be **high correlation** between g and $f_{x,y}$ for some x, y.
- However, to avoid ambiguity between various choices of x, y, we also want there to be **low correlation** between $f_{x,y}$ and $f_{x',y'}$.

The common use of correlation here motivates the definition of the radar cross-ambiguity function, which is essentially the correlation between functions f and $g_{x,y}$ (note the slightly reversed order).

Definition 4.1. The *radar cross-ambiguity function* $H_{f,g}$, defined for functions $f, g \in L^2(\mathbb{R})$, is a function $H_{f,g} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by

$$H_{f,g}(x, y) := \langle f, g_{x,y} \rangle \,, \tag{4.1}$$

where $\langle f, g \rangle := \int_{\mathbb{R}} f(t)\overline{g(t)}\,\mathrm{d}t$ is the usual inner product on $L^2(\mathbb{R})$.

As mentioned before, we use the cross-ambiguity function with distinct f, g to find the best-fit x, y, and use it with $f = g$ (in which case it is called the *auto-ambiguity function*) to compute the correlation between $f_{x,y}, f_{x',y'}$, noting that by a change of variables,

$$\langle f_{x,y}, f_{x',y'} \rangle = \langle f, f_{x'-x,y'-x} \rangle \,.$$

It is also worth noting that, in practice, what we care most about $H_{f,g}(x, y)$ is its absolute value, and therefore in literature there are a number of slightly different definitions of $H_{f,g}$. For example, in [T], the symmetrized definition

$$H_{f,g}(x, y) = \int_R f(t + x/2)\overline{g(t - x/2)}\exp(2\pi i y t)\,\mathrm{d}t \tag{4.2}$$

is used, and this differs from the definition in Eq. (4.1) by a factor of $\exp(\pi i x y)$.

5. Connection to the Schrödinger representation

In this section, we discuss the connection between the radar cross-ambiguity function, defined in the previous section, and the Schrödinger representation, defined in Section 3. The main goal of this section is to explain the following claim in [T].

Claim 5.1. *$H_{f,g}(x, y)$ is a matrix entry for the Schrödinger representation.*

We will try to unpack this claim step-by-step. First, what is a matrix entry of an infinite-dimensional representation? Recall that, given a matrix representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$, the *matrix entry* $\pi_{i,j} : G \to \mathbb{C}$ is the function given by the (i, j)-th entry of π. It turns out that for (abstract) representations $\pi : G \to \mathrm{GL}(V)$, we can generalize the notion of matrix entries into what will we call *matrix coefficients* (to avoid confusion with matrix entries).

Definition 5.2. Let $\pi : G \to \mathrm{GL}(V)$ be a representation of G on a Hilbert space V equipped with an inner product $\langle \cdot, \cdot \rangle$. For vectors $\vec{v}, \vec{w} \in V$, the *matrix coefficient* $\pi_{\vec{v},\vec{w}} : G \to \mathbb{C}$ is then given by $\pi_{\vec{v},\vec{w}}(g) := \langle \pi(g)\vec{v}, \vec{w} \rangle$.

Note that when $V = \mathbb{C}^n$ and $\vec{v} = \vec{e}_i, \vec{w} = \vec{e}_j$ are standard basis vectors, the matrix coefficient $\pi_{\vec{e}_i,\vec{e}_j}$ is simply the matrix entry $\pi_{i,j}$.

At this point, we now observe that a matrix entry for the Schrödinger representation is a function from $\mathrm{Heis}(\mathbb{R})$ to $\mathbb{C}$, and therefore takes a tuple of three real numbers as input. However, $H_{f,g}$ only takes in two real numbers—why the discrepancy? If we recall from previous sections, we really only care about the magnitude of $H_{f,g}(x, y)$, and the z-coordinate of the Heisenberg group acts by scalar multiplication by a complex number with absolute value 1, therefore we can simply 'ignore' the value of z, and let it be whatever.

Finally, we note that *the* Schrödinger representation here just means π_s with $s = 1$, and therefore we can now rigorously restate the claim as follows.

Proposition 5.3. *For all $x, y \in \mathbb{R}$, there is a $z \in \mathbb{R}$ such that*

$$H_{f,g}(x, y) = (\pi_1)_{f,g}[x, y, z],$$

where π_1 here denotes the Schrödinger representation with parameter $s = 1$.

Note that the subscript f, g in the left-hand-side refers to the cross-ambiguity function, while the subscripts in the right-hand-side gives a matrix coefficient—f and g are vectors in $L^2(\mathbb{R})$. After formulating in this way, the proof of the proposition becomes straightforward.

Proof. Since π_1 is unitary (Theorem 3.2), we have

$$(\pi_1)_{f,g}[x, y, z] = \langle \pi_1 [x, y, z] f, g \rangle = \left\langle f, \pi_1 [x, y, z]^{-1} g \right\rangle = \langle f, g_{x,y} \rangle$$

as long as z is chosen so that $[x, y, z]^{-1} = [-x, -y, 0]$, that is, $z = xy$. $\square$

Now it may feel like this is an inelegant result, and it turns out that by defining everything slightly differently it is possible to make $z = 0$ work, for example, by using Eq. (4.2) and a different presentation of the Heisenberg group, using the group law

$$[x, y, z][x', y', z'] = \left[x + x', y + y', z + z + \frac{1}{2}(xy' - x'y) \right].$$

This group law is more closely related to the Heisenberg Lie algebra, which is beyond the scope of this paper, and in some sense is more natural (but a bit more complicated) than our group law Eq. (2.1). For example, [F] refers to this group law as the (standard) group law for $\mathrm{Heis}(\mathbb{R})$, with our group law Eq. (2.1) being called instead the *polarized* group action.

6. The finite analogue

In this section, we discuss an analogue of Proposition 5.3 for finite Heisenberg groups, which are defined similarly to the continuous Heisenberg group, but with $\mathbb{R}$ replaced by either a finite field $\mathbb{F}_q$ or a finite ring $\mathbb{Z}/n\mathbb{Z}$. (Heisenberg groups can be defined over any commutative ring R, just by replacing $\mathbb{R}$ in Definition 2.1 with the ring R.)

First, we define representations analogous to the Schrödinger represenations. Let $q = p^r$ be a prime power. The Heisenberg groups over finite fields $\mathrm{Heis}(\mathbb{F}_q)$ admit representations ρ_s defined as follows.

Definition 6.1. The representation $\rho_s : \mathrm{Heis}(\mathbb{F}_q) \to \mathrm{GL}(L^2(\mathbb{F}_q))$, for nonzero $s \in \mathbb{F}_q$, is defined by

$$[\pi_s [x, y, z] f](t) := f(t + x) \exp\left(\frac{2\pi i \operatorname{tr}(s(z + ty))}{p}\right),$$

where tr here is the trace of the field extension $\mathbb{F}_q/\mathbb{F}_p$, i.e. $\operatorname{tr}(x) = x + x^p + \cdots + x^{p^r}$.

Similar to Proposition 3.5, the representation ρ_s is also an induced representation. We omit the proof of this fact, since it is essentially the same as before.

Proposition 6.2. *Let $H := \{[0, y, z] \mid y, z \in \mathbb{F}_q\} \subset \mathrm{Heis}(\mathbb{F}_q)$, and let $\sigma_s : H \to \mathrm{GL}(\mathbb{C})$ be the one-dimensional representation of H given by*

$$\sigma_s [0, y, z] = \exp\left(\frac{2\pi i \operatorname{tr}(sz)}{p}\right).$$

Then, $\rho_s \simeq \mathrm{Ind}_H^G \sigma_s$.

We can also define a cross-ambiguity function in the finite case, where again, the sign choices are somewhat arbitrary.

Definition 6.3. The *finite cross-ambiguity function* H over $\mathbb{F}_q$ is an operator that takes two functions $f, g \in L^2(\mathbb{F}_q)$ and outputs a function $H(f, g) : (\mathbb{F}_q)^2 \to \mathbb{C}$, defined by

$$H_{f,g}(x, y) := \sum_{t \in \mathbb{F}_q} f(t) g(t - x) \exp\left(\frac{2\pi i \operatorname{tr}(ty)}{p}\right).$$

Furthermore, we can also prove an analogous version of Proposition 5.3 in exactly the same way (and so the proof is once again omitted).

Proposition 6.4. *For all $x, y \in \mathbb{F}_q$, there is a $z \in \mathbb{F}_q$ such that*

$$H_{f,g}(x, y) = (\rho_1)_{f,g} [x, y, z],$$

where ρ_1 here denotes the representation ρ_s with parameter $s = 1$.

Finally, we can likewise define everything for $\mathrm{Heis}(\mathbb{Z}/n\mathbb{Z})$ instead of $\mathrm{Heis}(\mathbb{F}_q)$ by replacing the denominator p with n (and ignoring the trace) in the formulae above.

References

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