ON THE SPHERE AND CYLINDER.

BOOK I.

"ARCHIMEDES to Dositheus greeting.

On a former occasion I sent you the investigations which I had up to that time completed, including the proofs, showing that any segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base with the segment and equal height. Since then certain theorems not hitherto demonstrated (ἀνελέγκτων) have occurred to me, and I have worked out the proofs of them. They are these: first,[1] that the surface of any sphere is four times its greatest circle (τοῦ μεγίστου κύκλου); next,[2] that the surface of any segment of a sphere is equal to a circle whose radius (ἡ ἐκ τοῦ κέντρου) is equal to the straight line drawn from the vertex (κορυφή) of the segment to the circumference of the circle which is the base of the segment;[3] and, further, that any cylinder having its base equal to the greatest circle of those in the sphere, and height equal to the diameter of the sphere, is itself [*i.e.* in content] half as large again as the sphere, and its surface also [including its bases] is half as large again as the surface of the sphere. Now these properties were all along naturally inherent in the figures referred to (αὐτῇ τῇ φύσει προυπῆρχεν περὶ τὰ εἰρημένα σχήματα), but remained unknown to those who were before my time engaged in the study of geometry. Having, however, now discovered that the properties are true of these figures, I cannot feel any hesitation in setting them side by side both with my former investigations and with those of the theorems of Eudoxus on solids which are held to be most irrefragably established, namely, that any pyramid is one third part of the prism which has the same base with the pyramid and equal height, and that any cone is one third part of the cylinder which has the same base with the cone and equal height. For, though these properties also were naturally inherent in the figures all along, yet they were in fact unknown to all the many able geometers who lived before Eudoxus, and had not been observed by any one. Now, however, it will be open to those who possess the requisite ability to examine these discoveries of mine. They ought to have been published while Conon was still alive, for I should conceive that he would best have been able to grasp them and to pronounce upon them the appropriate verdict; but, as I judge it well to communicate them to those who are conversant with mathematics, I send them to you with the proofs written out, which it will be open to mathematicians to examine. Farewell.

I first set out the axioms* and the assumptions which I have used for the proofs of my propositions.

DEFINITIONS.

1. There are in a plane certain terminated bent lines (καμπύλαι γραμμαὶ πεπερασμέναι)†, which either lie wholly on the same side of the straight lines joining their extremities, or have no part of them on the other side.

2. I apply the term **concave in the same direction** to a line such that, if any two points on it are taken, either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself, but none on the other side.

* Though the word used is ἀξιώματα, the "axioms" are more of the nature of definitions; and in fact Eutocius in his notes speaks of them as such (ὅροι).

† Under the term *bent line* Archimedes includes not only curved lines of continuous curvature, but lines made up of any number of lines which may be either straight or curved.

3. Similarly also there are certain terminated surfaces, not themselves being in a plane but having their extremities in a plane, and such that they will either be wholly on the same side of the plane containing their extremities, or have no part of them on the other side.

4. I apply the term **concave in the same direction** to surfaces such that, if any two points on them are taken, the straight lines connecting the points either all fall on the same side of the surface, or some fall on one and the same side of it while some fall upon it, but none on the other side.

5. I use the term **solid sector,** when a cone cuts a sphere, and has its apex at the centre of the sphere, to denote the figure comprehended by the surface of the cone and the surface of the sphere included within the cone.

6. I apply the term **solid rhombus,** when two cones with the same base have their apices on opposite sides of the plane of the base in such a position that their axes lie in a straight line, to denote the solid figure made up of both the cones.

ASSUMPTIONS.

1. *Of all lines which have the same extremities the straight line is the least**.

* This well-known Archimedean assumption is scarcely, as it stands, a *definition* of a straight line, though Proclus says [p. 110 ed. Friedlein] "Archimedes defined (*ὡρίσατο*) the straight line as the least of those [lines] which have the same extremities. For because, as Euclid's definition says, *ἐξ ἴσου κεῖται τοῖς ἐφ' ἑαυτῆς σημείοις*, it is in consequence the least of those which have the same extremities." Proclus had just before [p. 109] explained Euclid's definition, which, as will be seen, is different from the ordinary version given in our text-books; a straight line is not "that which lies evenly between its extreme points," but "that which *ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται*." The words of Proclus are, "He [Euclid] shows by means of this that the straight line alone [of all lines] occupies a distance (*κατέχειν διάστημα*) equal to that between the points on it. For, as far as one of its points is removed from another, so great is the length (*μέγεθος*) of the straight line of which the points are the extremities; and this is the meaning of *τὸ ἐξ ἴσου κεῖσθαι τοῖς ἐφ' ἑαυτῆς σημείοις*. But, if you take two points on a circumference or any other line, the distance cut off between them along the line is greater than the interval separating them; and this is the case with every line except the straight line." It appears then from this that Euclid's definition should be understood in a sense very like that of

2. Of other lines in a plane and having the same extremities, [any two] such are unequal whenever both are concave in the same direction and one of them is either wholly included between the other and the straight line which has the same extremities with it, or is partly included by, and is partly common with, the other; and that [line] which is included is the lesser [of the two].

3. Similarly, of surfaces which have the same extremities, if those extremities are in a plane, the plane is the least [in area].

4. Of other surfaces with the same extremities, the extremities being in a plane, [any two] such are unequal whenever both are concave in the same direction and one surface is either wholly included between the other and the plane which has the same extremities with it, or is partly included by, and partly common with, the other; and that [surface] which is included is the lesser [of the two in area].

5. Further, of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with [it and with] one another*.

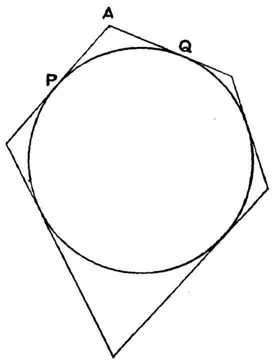
These things being premised, *if a polygon be inscribed in a circle, it is plain that the perimeter of the inscribed polygon is less than the circumference of the circle;* for each of the sides of the polygon is less than that part of the circumference of the circle which is cut off by it."

Archimedes' assumption, and we might perhaps translate as follows, "A straight line is that which extends equally (ἐξ ἴσου κεῖται) with the points on it," or, to follow Proclus' interpretation more closely, "A straight line is that which represents equal extension with [the distances separating] the points on it."

* With regard to this assumption compare the Introduction, chapter III. § 2.

Proposition 1.

If a polygon be circumscribed about a circle, the perimeter of the circumscribed polygon is greater than the perimeter of the circle.



Let any two adjacent sides, meeting in A, touch the circle at P, Q respectively.

Then [*Assumptions*, 2]

$$PA + AQ > (\text{arc } PQ).$$

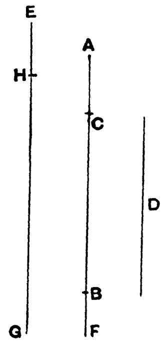
A similar inequality holds for each angle of the polygon; and, by addition, the required result follows.

Proposition 2.

Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less.

Let AB, D represent the two unequal magnitudes, AB being the greater.

Suppose BC measured along BA equal to D, and let GH be any straight line.



Then, if CA be added to itself a sufficient number of times, the sum will exceed D. Let AF be this sum, and take E on GH produced such that GH is the same multiple of HE that AF is of AC.

Thus $\qquad EH : HG = AC : AF.$

But, since $AF > D$ (or CB),

$$AC : AF < AC : CB.$$

Therefore, *componendo*,

$$EG : GH < AB : D.$$

Hence EG, GH are two lines satisfying the given condition.

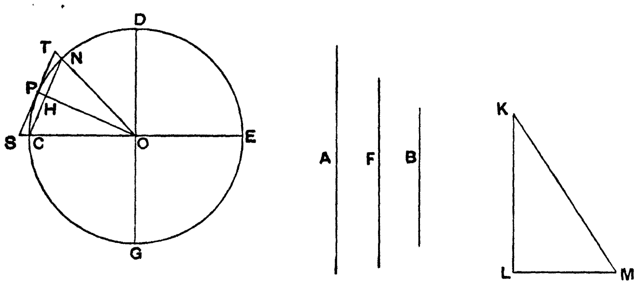
Proposition 3.

Given two unequal magnitudes and a circle, it is possible to inscribe a polygon in the circle and to describe another about it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than that of the greater magnitude to the less.

Let A, B represent the given magnitudes, A being the greater.

Find [Prop. 2] two straight lines F, KL, of which F is the greater, such that

$$F : KL < A : B \qquad \text{.......................(1).}$$



Draw LM perpendicular to LK and of such length that $KM = F$.

In the given circle let CE, DG be two diameters at right angles. Then, bisecting the angle DOC, bisecting the half again, and so on, we shall arrive ultimately at an angle (as NOC) less than twice the angle LKM.

Join NC, which (by the construction) will be the side of a regular polygon inscribed in the circle. Let OP be the radius of the circle bisecting the angle NOC (and therefore bisecting NC at right angles, in H, say), and let the tangent at P meet OC, ON produced in S, T respectively.

Now, since $\quad \angle CON < 2 \angle LKM,$

$$\angle HOC < \angle LKM,$$

and the angles at H, L are right;

$$\text{therefore } MK : LK > OC : OH$$
$$> OP : OH.$$

Hence $$ST : CN < MK : LK$$
$$< F : LK;$$

therefore, *a fortiori*, by (1),

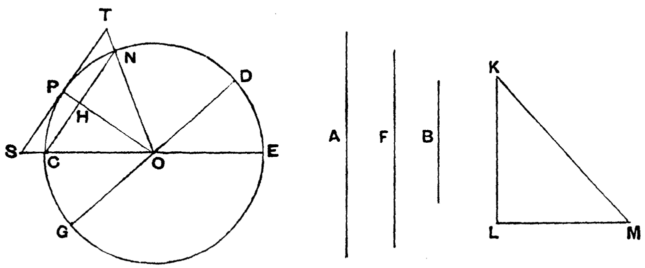
$$ST : CN < A : B.$$

Thus two polygons are found satisfying the given condition.

Proposition 4.

Again, given two unequal magnitudes and a sector, it is possible to describe a polygon about the sector and to inscribe another in it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than the greater magnitude has to the less.

[The "inscribed polygon" found in this proposition is one which has for two sides the two radii bounding the sector, while the remaining sides (the number of which is, by construction, some power of 2) subtend equal parts of the arc of the sector; the "circumscribed polygon" is formed by the tangents parallel to the sides of the inscribed polygon and by the two bounding radii produced.]

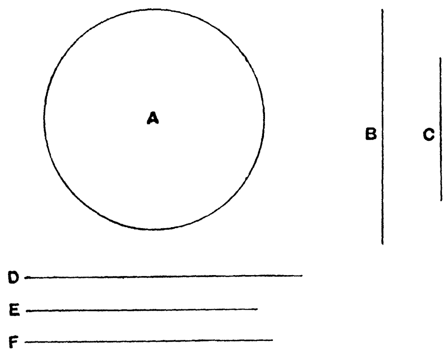


In this case we make the same construction as in the last proposition except that we bisect the angle COD of the sector, instead of the right angle between two diameters, then bisect the half again, and so on. The proof is exactly similar to the preceding one.

Proposition 5.

Given a circle and two unequal magnitudes, to describe a polygon about the circle and inscribe another in it, so that the circumscribed polygon may have to the inscribed a ratio less than the greater magnitude has to the less.

Let A be the given circle and B, C the given magnitudes, B being the greater.



Take two unequal straight lines D, E, of which D is the greater, such that $D : E < B : C$ [Prop. 2], and let F be a mean proportional between D, E, so that D is also greater than F.

Describe (in the manner of Prop. 3) one polygon about the circle, and inscribe another in it, so that the side of the former has to the side of the latter a ratio less than the ratio $D : F$.

Thus the duplicate ratio of the side of the former polygon to the side of the latter is less than the ratio $D^2 : F^2$.

But the said duplicate ratio of the sides is equal to the ratio of the areas of the polygons, since they are similar;

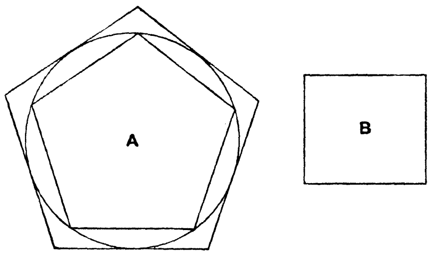
therefore the area of the circumscribed polygon has to the area of the inscribed polygon a ratio less than the ratio $D^2 : F^2$, or $D : E$, and *a fortiori* less than the ratio $B : C$.

Proposition 6.

"Similarly we can show that, *given two unequal magnitudes and a sector, it is possible to circumscribe a polygon about the sector and inscribe in it another similar one so that the circumscribed may have to the inscribed a ratio less than the greater magnitude has to the less.*

And it is likewise clear that, *if a circle or a sector, as well as a certain area, be given, it is possible, by inscribing regular polygons in the circle or sector, and by continually inscribing such in the remaining segments, to leave segments of the circle or sector which are* [*together*] *less than the given area.* For this is proved in the *Elements* [Eucl. XII. 2].

But it is yet to be proved that, *given a circle or sector and an area, it is possible to describe a polygon about the circle or sector, such that the area remaining between the circumference and the circumscribed figure is less than the given area.*"



The proof for the circle (which, as Archimedes says, can be equally applied to a sector) is as follows.

Let A be the given circle and B the given area.

Now, there being two unequal magnitudes $A + B$ and A, let a polygon (C) be circumscribed about the circle and a polygon (I) inscribed in it [as in Prop. 5], so that

$$C : I < A + B : A \text{ ....(1).}$$

The circumscribed polygon (C) shall be that required.

For the circle (A) is greater than the inscribed polygon (I).

Therefore, from (1), *a fortiori*,

$$C : A < A + B : A,$$

whence $$C < A + B,$$

or $$C - A < B.$$

Proposition 7.

If in an isosceles cone [i.e. *a right circular cone*] *a pyramid be inscribed having an equilateral base, the surface of the pyramid excluding the base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the perpendicular drawn from the apex on one side of the base.*

Since the sides of the base of the pyramid are equal, it follows that the perpendiculars from the apex to all the sides of the base are equal; and the proof of the proposition is obvious.

Proposition 8.

If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding its base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the side [i.e. *a generator*] *of the cone.*

The base of the pyramid is a polygon circumscribed about the circular base of the cone, and the line joining the apex of the cone or pyramid to the point of contact of any side of the polygon is perpendicular to that side. Also all these perpendiculars, being generators of the cone, are equal; whence the proposition follows immediately.

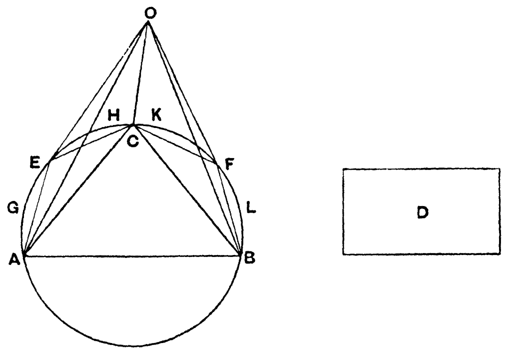
Proposition 9.

If in the circular base of an isosceles cone a chord be placed, and from its extremities straight lines be drawn to the apex of the cone, the triangle so formed will be less than the portion of the surface of the cone intercepted between the lines drawn to the apex.

Let ABC be the circular base of the cone, and O its apex.

Draw a chord AB in the circle, and join OA, OB. Bisect the arc ACB in C, and join AC, BC, OC.

Then $$\triangle OAC + \triangle OBC > \triangle OAB.$$



Let the excess of the sum of the first two triangles over the third be equal to the area D.

Then D is either less than the sum of the segments AEC, CFB, or not less.

I. Let D be not less than the sum of the segments referred to.

We have now two surfaces

(1) that consisting of the portion $OAEC$ of the surface of the cone together with the segment AEC, and

(2) the triangle OAC;

and, since the two surfaces have the same extremities (the perimeter of the triangle OAC), the former surface is greater than the latter, which is *included* by it [*Assumptions*, 3 or 4].

Hence (surface $OAEC$) + (segment AEC) > $\triangle\, OAC$.

Similarly (surface $OCFB$) + (segment CFB) > $\triangle\, OBC$.

Therefore, since D is not less than the sum of the segments, we have, by addition,

$$(\text{surface } OAECFB) + D > \triangle\, OAC + \triangle\, OBC$$
$$> \triangle\, OAB + D, \text{ by hypothesis.}$$

Taking away the common part D, we have the required result.

II. Let D be less than the sum of the segments AEC, CFB.

If now we bisect the arcs AC, CB, then bisect the halves, and so on, we shall ultimately leave segments which are together less than D. [Prop. 6]

Let AGE, EHC, CKF, FLB be those segments, and join OE, OF.

Then, as before,

$$(\text{surface } OAGE) + (\text{segment } AGE) > \triangle\, OAE$$

and $$(\text{surface } OEHC) + (\text{segment } EHC) > \triangle\, OEC.$$

Therefore $$(\text{surface } OAGHC) + (\text{segments } AGE,\, EHC)$$
$$> \triangle\, OAE + \triangle\, OEC$$
$$> \triangle\, OAC, \textit{ a fortiori}.$$

Similarly for the part of the surface of the cone bounded by OC, OB and the arc CFB.

Hence, by addition,

$$(\text{surface } OAGEHCKFLB) + (\text{segments } AGE,\, EHC,\, CKF,\, FLB)$$
$$> \triangle\, OAC + \triangle\, OBC$$
$$> \triangle\, OAB + D, \text{ by hypothesis.}$$

But the sum of the segments is less than D, and the required result follows.

Proposition 10.

If in the plane of the circular base of an isosceles cone two tangents be drawn to the circle meeting in a point, and the points of contact and the point of concourse of the tangents be respectively joined to the apex of the cone, the sum of the two triangles formed by the joining lines and the two tangents are together greater than the included portion of the surface of the cone.

Let ABC be the circular base of the cone, O its apex, AD, BD the two tangents to the circle meeting in D. Join OA, OB, OD.

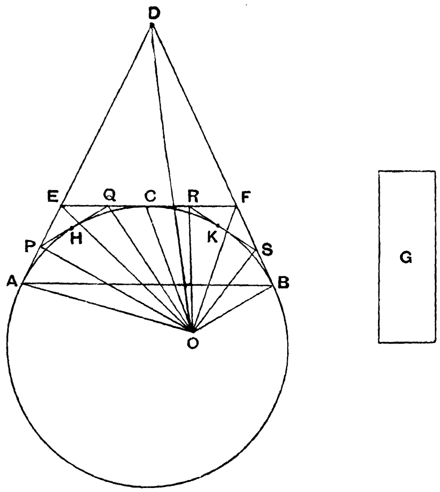
Let ECF be drawn touching the circle at C, the middle point of the arc ACB, and therefore parallel to AB. Join OE, OF.

Then $$ED + DF > EF,$$

and, adding $AE + FB$ to each side,

$$AD + DB > AE + EF + FB.$$

Now OA, OC, OB, being generators of the cone, are equal, and they are respectively perpendicular to the tangents at A, C, B.



It follows that

$$\triangle OAD + \triangle ODB > \triangle OAE + \triangle OEF + \triangle OFB.$$

Let the area G be equal to the excess of the first sum over the second.

G is then either less, or not less, than the sum of the spaces $EAHC$, $FCKB$ remaining between the circle and the tangents, which sum we will call L.

I. Let G be not less than L.

We have now two surfaces

(1) that of the pyramid with apex O and base $AEFB$, excluding the face OAB,

(2) that consisting of the part $OACB$ of the surface of the cone together with the segment ACB.

These two surfaces have the same extremities, viz. the perimeter of the triangle OAB, and, since the former *includes* the latter, the former is the greater [*Assumptions*, 4].

That is, the surface of the pyramid exclusive of the face OAB is greater than the sum of the surface $OACB$ and the segment ACB.

Taking away the segment from each sum, we have

$\triangle OAE + \triangle OEF + \triangle OFB + L >$ the surface $OAHCKB$.

And G is not less than L.

It follows that

$$\triangle OAE + \triangle OEF + \triangle OFB + G,$$

which is by hypothesis equal to $\triangle OAD + \triangle ODB$, is greater than the same surface.

II. Let G be less than L.

If we bisect the arcs AC, CB and draw tangents at their middle points, then bisect the halves and draw tangents, and so on, we shall lastly arrive at a polygon such that the sum of the parts remaining between the sides of the polygon and the circumference of the segment is less than G.

Let the remainders be those between the segment and the polygon $APQRSB$, and let their sum be M. Join OP, OQ, etc.

Then, as before,

$$\triangle OAE + \triangle OEF + \triangle OFB > \triangle OAP + \triangle OPQ + \ldots + \triangle OSB.$$

Also, as before,

(surface of pyramid $OAPQRSB$ excluding the face OAB) $>$ the part $OACB$ of the surface of the cone together with the segment ACB.

Taking away the segment from each sum,

$\triangle OAP + \triangle OPQ + \ldots + M >$ the part $OACB$ of the surface of the cone.

Hence, *a fortiori*,

$$\triangle OAE + \triangle OEF + \triangle OFB + G,$$

which is by hypothesis equal to

$$\triangle OAD + \triangle ODB,$$

is greater than the part $OACB$ of the surface of the cone.

Proposition 11.

If a plane parallel to the axis of a right cylinder cut the cylinder, the part of the surface of the cylinder cut off by the plane is greater than the area of the parallelogram in which the plane cuts it.

Proposition 12.

If at the extremities of two generators of any right cylinder tangents be drawn to the circular bases in the planes of those bases respectively, and if the pairs of tangents meet, the parallelograms formed by each generator and the two corresponding tangents respectively are together greater than the included portion of the surface of the cylinder between the two generators.

[The proofs of these two propositions follow exactly the methods of Props. 9, 10 respectively, and it is therefore unnecessary to reproduce them.]

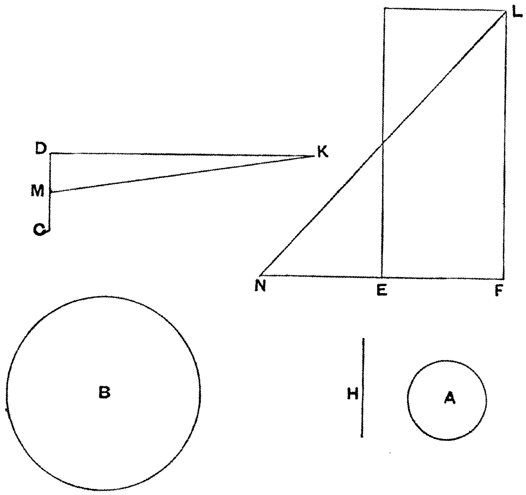
"From the properties thus proved it is clear (1) that, *if a pyramid be inscribed in an isosceles cone, the surface of the pyramid excluding the base is less than the surface of the cone* [*excluding the base*], and (2) that, *if a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding the base is greater than the surface of the cone excluding the base.*

"It is also clear from what has been proved both (1) that, *if a prism be inscribed in a right cylinder, the surface of the prism made up of its parallelograms* [i.e. *excluding its bases*] *is less than the surface of the cylinder excluding its bases,* and (2) that, *if a prism be circumscribed about a right cylinder, the surface of the prism made up of its parallelograms is greater than the surface of the cylinder excluding its bases.*"

Proposition 13.

The surface of any right cylinder excluding the bases is equal to a circle whose radius is a mean proportional between the side [i.e. *a generator*] *of the cylinder and the diameter of its base.*

Let the base of the cylinder be the circle A, and make CD equal to the diameter of this circle, and EF equal to the height of the cylinder.



Let H be a mean proportional between CD, EF, and B a circle with radius equal to H.

Then the circle B shall be equal to the surface of the cylinder (excluding the bases), which we will call S.

For, if not, B must be either greater or less than S.

I. Suppose $B < S$.

Then it is possible to circumscribe a regular polygon about B, and to inscribe another in it, such that the ratio of the former to the latter is less than the ratio $S : B$.

Suppose this done, and circumscribe about A a polygon similar to that described about B; then erect on the polygon about A a prism of the same height as the cylinder. The prism will therefore be circumscribed to the cylinder.

Let KD, perpendicular to CD, and FL, perpendicular to EF, be each equal to the perimeter of the polygon about A. Bisect CD in M, and join MK.

Then $\triangle KDM =$ the polygon about A.

Also $\square EL =$ surface of prism (excluding bases).

Produce FE to N so that $FE = EN$, and join NL.

Now the polygons about A, B, being similar, are in the duplicate ratio of the radii of A, B.

Thus

$$\begin{aligned}\triangle KDM : (\text{polygon about } B) &= MD^2 : H^2 \\ &= MD^2 : CD \cdot EF \\ &= MD : NF \\ &= \triangle KDM : \triangle LFN \\ &\qquad\qquad (\text{since } DK = FL).\end{aligned}$$

$$\begin{aligned}\text{Therefore (polygon about } B) &= \triangle LFN \\ &= \square EL \\ &= (\text{surface of prism about } A), \\ &\qquad\qquad \text{from above.}\end{aligned}$$

But $(\text{polygon about } B) : (\text{polygon in } B) < S : B$.

Therefore

$$(\text{surface of prism about } A) : (\text{polygon in } B) < S : B,$$

and, alternately,

$$(\text{surface of prism about } A) : S < (\text{polygon in } B) : B;$$

which is impossible, since the surface of the prism is greater than S, while the polygon inscribed in B is less than B.

Therefore $$B \nless S.$$

II. Suppose $B > S$.

Let a regular polygon be circumscribed about B and another inscribed in it so that

$$(\text{polygon about } B) : (\text{polygon in } B) < B : S.$$

Inscribe in A a polygon similar to that inscribed in B, and erect a prism on the polygon inscribed in A of the same height as the cylinder.

Again, let DK, FL, drawn as before, be each equal to the perimeter of the polygon inscribed in A.

Then, in this case,

$$\triangle KDM > (\text{polygon inscribed in } A)$$

(since the perpendicular from the centre on a side of the polygon is less than the radius of A).

Also $\triangle LFN = \square\, EL =$ surface of prism (excluding bases).

Now

$$(\text{polygon in } A) : (\text{polygon in } B) = MD^2 : H^2,$$
$$= \triangle KDM : \triangle LFN, \text{ as before.}$$

And $$\triangle KDM > (\text{polygon in } A).$$

Therefore

$$\triangle LFN, \text{ or } (\text{surface of prism}) > (\text{polygon in } B).$$

But this is impossible, because

$$(\text{polygon about } B) : (\text{polygon in } B) < B : S,$$
$$< (\text{polygon about } B) : S, \textit{ a fortiori},$$

so that $$(\text{polygon in } B) > S,$$
$$> (\text{surface of prism}), \textit{ a fortiori}.$$

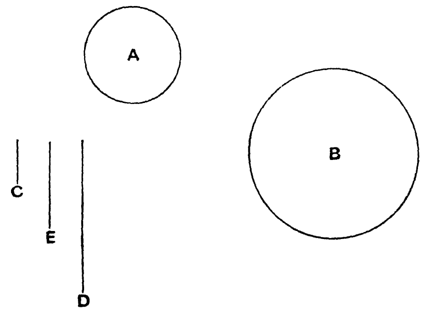
Hence B is neither greater nor less than S, and therefore

$$B = S.$$

Proposition 14.

The surface of any isosceles cone excluding the base is equal to a circle whose radius is a mean proportional between the side of the cone [a generator] and the radius of the circle which is the base of the cone.

Let the circle A be the base of the cone; draw C equal to the radius of the circle, and D equal to the side of the cone, and let E be a mean proportional between C, D.



Draw a circle B with radius equal to E.

Then shall B be equal to the surface of the cone (excluding the base), which we will call S.

If not, B must be either greater or less than S.

I. Suppose $B < S$.

Let a regular polygon be described about B and a similar one inscribed in it such that the former has to the latter a ratio less than the ratio $S : B$.

Describe about A another similar polygon, and on it set up a pyramid with apex the same as that of the cone.

Then (polygon about A) : (polygon about B)

$= C^2 : E^2$

$= C : D$

= (polygon about A) : (surface of pyramid excluding base).

Therefore

$$(\text{surface of pyramid}) = (\text{polygon about } B).$$

Now $(\text{polygon about } B) : (\text{polygon in } B) < S : B.$

Therefore

$$(\text{surface of pyramid}) : (\text{polygon in } B) < S : B,$$

which is impossible, (because the surface of the pyramid is greater than S, while the polygon in B is less than B).

Hence $B \not< S.$

II. Suppose $B > S$.

Take regular polygons circumscribed and inscribed to B such that the ratio of the former to the latter is less than the ratio $B : S$.

Inscribe in A a similar polygon to that inscribed in B, and erect a pyramid on the polygon inscribed in A with apex the same as that of the cone.

In this case

$$\begin{aligned}(\text{polygon in } A) : (\text{polygon in } B) &= C^2 : E^2 \\ &= C : D\end{aligned}$$

$$> (\text{polygon in } A) : (\text{surface of pyramid excluding base}).$$

This is clear because the ratio of C to D is greater than the ratio of the perpendicular from the centre of A on a side of the polygon to the perpendicular from the apex of the cone on the same side*.

Therefore

$$(\text{surface of pyramid}) > (\text{polygon in } B).$$

But $(\text{polygon about } B) : (\text{polygon in } B) < B : S.$

Therefore, *a fortiori*,

$$(\text{polygon about } B) : (\text{surface of pyramid}) < B : S;$$

which is impossible.

Since therefore B is neither greater nor less than S,

$$B = S.$$

* This is of course the geometrical equivalent of saying that, if α, β be two angles each less than a right angle, and $\alpha > \beta$, then $\sin\alpha > \sin\beta$.

Proposition 15.

The surface of any isosceles cone has the same ratio to its base as the side of the cone has to the radius of the base.

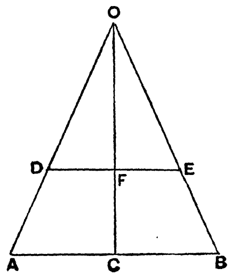
By Prop. 14, the surface of the cone is equal to a circle whose radius is a mean proportional between the side of the cone and the radius of the base.

Hence, since circles are to one another as the squares of their radii, the proposition follows.

Proposition 16.

If an isosceles cone be cut by a plane parallel to the base, the portion of the surface of the cone between the parallel planes is equal to a circle whose radius is a mean proportional between (1) *the portion of the side of the cone intercepted by the parallel planes and* (2) *the line which is equal to the sum of the radii of the circles in the parallel planes.*

Let OAB be a triangle through the axis of a cone, DE its intersection with the plane cutting off the frustum, and OFC the axis of the cone.



Then the surface of the cone OAB is equal to a circle whose radius is equal to $\sqrt{OA \cdot AC}$. [Prop. 14.]

Similarly the surface of the cone ODE is equal to a circle whose radius is equal to $\sqrt{OD \cdot DF}$.

And the surface of the frustum is equal to the difference between the two circles.

Now

$$OA \cdot AC - OD \cdot DF = DA \cdot AC + OD \cdot AC - OD \cdot DF.$$

But $$OD \cdot AC = OA \cdot DF,$$

since $$OA : AC = OD : DF.$$

Hence $OA \,.\, AC - OD \,.\, DF = DA \,.\, AC + DA \,.\, DF$

$$= DA \,.\, (AC + DF).$$

And, since circles are to one another as the squares of their radii, it follows that the difference between the circles whose radii are $\sqrt{OA \,.\, AC}$, $\sqrt{OD \,.\, DF}$ respectively is equal to a circle whose radius is $\sqrt{DA \,.\, (AC + DF)}$.

Therefore the surface of the frustum is equal to this circle.

Lemmas.

"1. *Cones having equal height have the same ratio as their bases; and those having equal bases have the same ratio as their heights**.

2. *If a cylinder be cut by a plane parallel to the base, then, as the cylinder is to the cylinder, so is the axis to the axis*†.

3. *The cones which have the same bases as the cylinders* [*and equal height*] *are in the same ratio as the cylinders.*

4. *Also the bases of equal cones are reciprocally proportional to their heights; and those cones whose bases are reciprocally proportional to their heights are equal*‡.

5. *Also the cones, the diameters of whose bases have the same ratio as their axes, are to one another in the triplicate ratio of the diameters of the bases* §.

And all these propositions have been proved by earlier geometers."

* Euclid XII. 11. "Cones and cylinders of equal height are to one another as their bases."

Euclid XII. 14. "Cones and cylinders on equal bases are to one another as their heights."

† Euclid XII. 13. "If a cylinder be cut by a plane parallel to the opposite planes [the bases], then, as the cylinder is to the cylinder, so will the axis be to the axis."

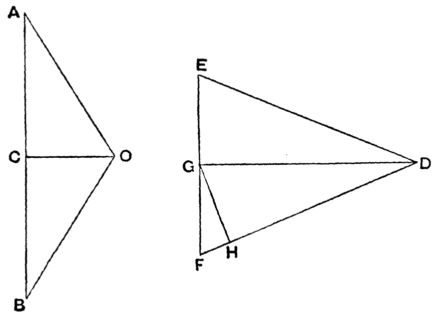
‡ Euclid XII. 15. "The bases of equal cones and cylinders are reciprocally proportional to their heights; and those cones and cylinders whose bases are reciprocally proportional to their heights are equal."

§ Euclid XII. 12. "Similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases."

Proposition 17.

If there be two isosceles cones, and the surface of one cone be equal to the base of the other, while the perpendicular from the centre of the base [of the first cone] on the side of that cone is equal to the height [of the second], the cones will be equal.

Let OAB, DEF be triangles through the axes of two cones respectively, C, G the centres of the respective bases, GH the



perpendicular from G on FD; and suppose that the base of the cone OAB is equal to the surface of the cone DEF, and that $OC = GH$.

Then, since the base of OAB is equal to the surface of DEF,

(base of cone OAB) : (base of cone DEF)

= (surface of DEF) : (base of DEF)

$= DF : FG$ [Prop. 15]

$= DG : GH$, by similar triangles,

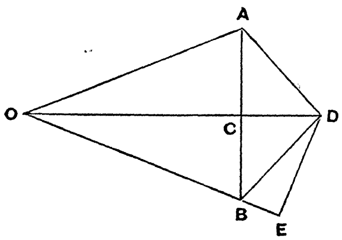
$= DG : OC$.

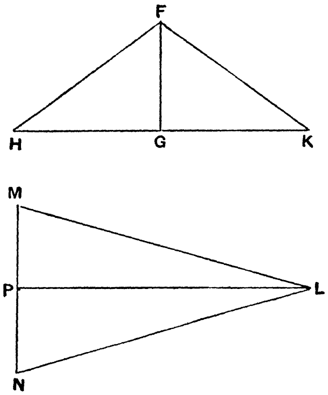
Therefore the bases of the cones are reciprocally proportional to their heights; whence the cones are equal. [*Lemma* 4.]

Proposition 18.

Any solid rhombus consisting of isosceles cones is equal to the cone which has its base equal to the surface of one of the cones composing the rhombus and its height equal to the perpendicular drawn from the apex of the second cone to one side of the first cone.

Let the rhombus be $OABD$ consisting of two cones with apices O, D and with a common base (the circle about AB as diameter).





Let FHK be another cone with base equal to the surface of the cone OAB and height FG equal to DE, the perpendicular from D on OB.

Then shall the cone FHK be equal to the rhombus.

Construct a third cone LMN with base (the circle about MN) equal to the base of OAB and height LP equal to OD.

Then, since $LP = OD$,

$$LP : CD = OD : CD.$$

But [*Lemma* 1] $OD : CD =$ (rhombus $OADB$) : (cone DAB),

and $LP : CD =$ (cone LMN) : (cone DAB).

It follows that

(rhombus $OADB$) = (cone LMN) .... (1).

Again, since $AB = MN$, and

(surface of OAB) = (base of FHK),

(base of FHK) : (base of LMN)

= (surface of OAB) : (base of OAB)

= $OB : BC$ [Prop. 15]

= $OD : DE$, by similar triangles,

= $LP : FG$, by hypothesis.

Thus, in the cones FHK, LMN, the bases are reciprocally proportional to the heights.

Therefore the cones FHK, LMN are equal,

and hence, by (1), the cone FHK is equal to the given solid rhombus.

Proposition 19.

If an isosceles cone be cut by a plane parallel to the base, and on the resulting circular section a cone be described having as its apex the centre of the base [of the first cone], and if the rhombus so formed be taken away from the whole cone, the part remaining will be equal to the cone with base equal to the surface of the portion of the first cone between the parallel planes and with height equal to the perpendicular drawn from the centre of the base of the first cone on one side of that cone.

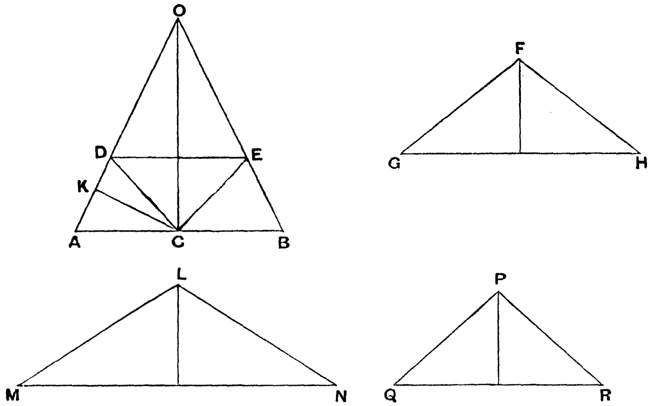
Let the cone OAB be cut by a plane parallel to the base in the circle on DE as diameter. Let C be the centre of the base of the cone, and with C as apex and the circle about DE as base describe a cone, making with the cone ODE the rhombus $ODCE$.

Take a cone FGH with base equal to the surface of the frustum $DABE$ and height equal to the perpendicular (CK) from C on AO.

Then shall the cone FGH be equal to the difference between the cone OAB and the rhombus $ODCE$.

Take (1) a cone LMN with base equal to the surface of the cone OAB, and height equal to CK,

(2) a cone PQR with base equal to the surface of the cone ODE and height equal to CK.



Now, since the surface of the cone OAB is equal to the surface of the cone ODE together with that of the frustum $DABE$, we have, by the construction,

$$(\text{base of } LMN) = (\text{base of } FGH) + (\text{base of } PQR)$$

and, since the heights of the three cones are equal,

$$(\text{cone } LMN) = (\text{cone } FGH) + (\text{cone } PQR).$$

But the cone LMN is equal to the cone OAB [Prop. 17], and the cone PQR is equal to the rhombus $ODCE$ [Prop. 18].

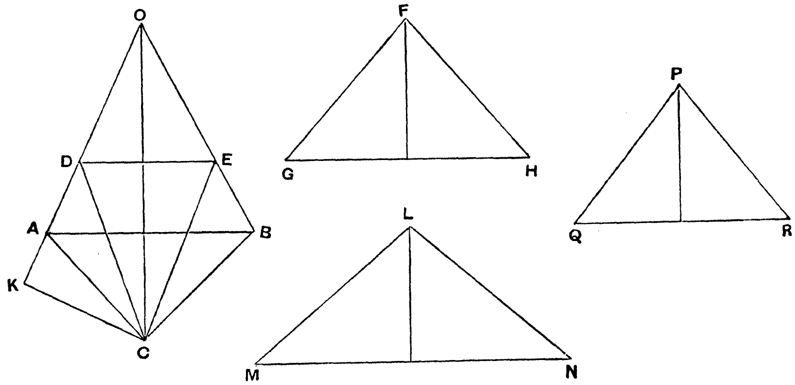
Therefore (cone OAB) = (cone FGH) + (rhombus $ODCE$), and the proposition is proved.

Proposition 20.

If one of the two isosceles cones forming a rhombus be cut by a plane parallel to the base and on the resulting circular section a cone be described having the same apex as the second cone, and if the resulting rhombus be taken from the whole rhombus, the remainder will be equal to the cone with base equal to the surface of the portion of the cone between the parallel planes and with height equal to the perpendicular drawn from the apex of the second cone to the side of the first cone.*

* There is a slight error in Heiberg's translation "prioris coni" and in the corresponding note, p. 93. The perpendicular is not drawn from the apex of the cone which is cut by the plane but from the apex of the other.

Let the rhombus be $OACB$, and let the cone OAB be cut by a plane parallel to its base in the circle about DE as diameter. With this circle as base and C as apex describe a cone, which therefore with ODE forms the rhombus $ODCE$.



Take a cone FGH with base equal to the surface of the frustum $DABE$ and height equal to the perpendicular (CK) from C on OA.

The cone FGH shall be equal to the difference between the rhombi $OACB$, $ODCE$.

For take (1) a cone LMN with base equal to the surface of OAB and height equal to CK,

(2) a cone PQR, with base equal to the surface of ODE, and height equal to CK.

Then, since the surface of OAB is equal to the surface of ODE together with that of the frustum $DABE$, we have, by construction,

$$(\text{base of } LMN) = (\text{base of } PQR) + (\text{base of } FGH),$$

and the three cones are of equal height;

therefore $(\text{cone } LMN) = (\text{cone } PQR) + (\text{cone } FGH)$.

But the cone LMN is equal to the rhombus $OACB$, and the cone PQR is equal to the rhombus $ODCE$ [Prop. 18].

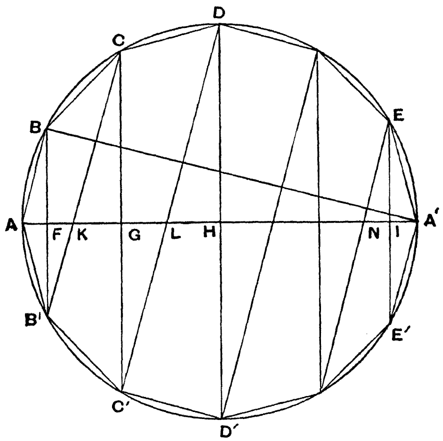
Hence the cone FGH is equal to the difference between the two rhombi $OACB$, $ODCE$.

Proposition 21.

A regular polygon of an even number of sides being inscribed in a circle, as $ABC...A'...C'B'A$, so that AA' is a diameter, if two angular points next but one to each other, as B, B', be joined, and the other lines parallel to BB' and joining pairs of angular points be drawn, as CC', DD'..., then

$$(BB' + CC' + ...) : AA' = A'B : BA.$$

Let BB', CC', DD',... meet AA' in F, G, H,...; and let CB', DC',... be joined meeting AA' in K, L,... respectively.



Then clearly CB', DC',... are parallel to one another and to AB.

Hence, by similar triangles,

$$\begin{aligned} BF : FA &= B'F : FK \\ &= CG : GK \\ &= C'G : GL \\ &\cdots\cdots\cdots\cdots \\ &= E'I : IA'; \end{aligned}$$

and, summing the antecedents and consequents respectively, we have

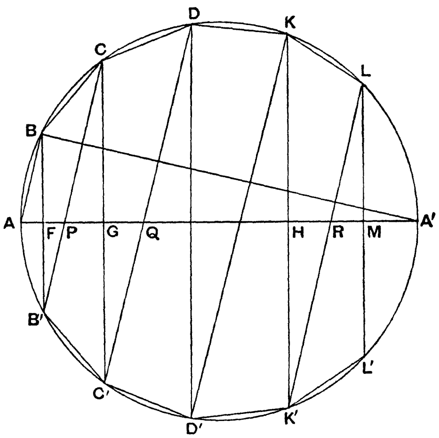
$$(BB' + CC' + \ldots) : AA' = BF : FA$$
$$= A'B : BA.$$

Proposition 22.

If a polygon be inscribed in a segment of a circle LAL' so that all its sides excluding the base are equal and their number even, as $LK\ldots A\ldots K'L'$, A being the middle point of the segment, and if the lines BB', CC',... parallel to the base LL' and joining pairs of angular points be drawn, then

$$(BB' + CC' + \ldots + LM) : AM = A'B : BA,$$

where M is the middle point of LL' and AA' is the diameter through M.



Joining CB', DC',...LK', as in the last proposition, and supposing that they meet AM in P, Q,...R, while BB', CC',..., KK' meet AM in F, G,... H, we have, by similar triangles,

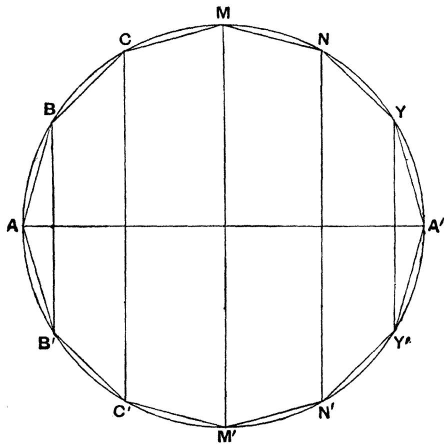
$$BF : FA = B'F : FP$$
$$= CG : PG$$
$$= C'G : GQ$$
$$\cdots\cdots\cdots\cdots$$
$$= LM : RM;$$

and, summing the antecedents and consequents, we obtain

$$(BB' + CC' + \ldots + LM) : AM = BF : FA$$
$$= A'B : BA.$$

Proposition 23.

Take a great circle $ABC\ldots$ of a sphere, and inscribe in it a regular polygon whose sides are a multiple of four in number. Let AA', MM' be diameters at right angles and joining opposite angular points of the polygon.



Then, if the polygon and great circle revolve together about the diameter AA', the angular points of the polygon, except A, A', will describe circles on the surface of the sphere at right angles to the diameter AA'. Also the sides of the polygon will describe portions of conical surfaces, e.g. BC will describe a surface forming part of a cone whose base is a circle about CC' as diameter and whose apex is the point in which CB, $C'B'$ produced meet each other and the diameter AA'.

Comparing the hemisphere MAM' and that half of the figure described by the revolution of the polygon which is included in the hemisphere, we see that the surface of the hemisphere and the surface of the inscribed figure have the same boundaries in one plane (viz. the circle on MM' as diameter), the former surface entirely includes the latter, and they are both concave in the same direction.

Therefore [*Assumptions*, 4] the surface of the hemisphere is greater than that of the inscribed figure; and the same is true of the other halves of the figures.

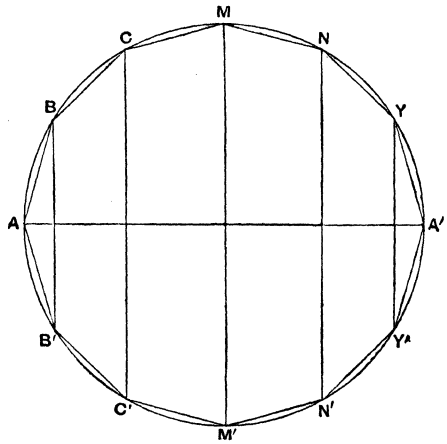
Hence *the surface of the sphere is greater than the surface described by the revolution of the polygon inscribed in the great circle about the diameter of the great circle.*

Proposition 24.

If a regular polygon $AB\ldots A'\ldots B'A$, the number of whose sides is a multiple of four, be inscribed in a great circle of a sphere, and if BB' subtending two sides be joined, and all the other lines parallel to BB' and joining pairs of angular points be drawn, then the surface of the figure inscribed in the sphere by the revolution of the polygon about the diameter AA' is equal to a circle the square of whose radius is equal to the rectangle

$$BA\,(BB' + CC' + \ldots).$$

The surface of the figure is made up of the surfaces of parts of different cones.



Now the surface of the cone ABB' is equal to a circle whose radius is $\sqrt{BA \,.\, \frac{1}{2}BB'}$. [Prop. 14]

The surface of the frustum $BB'C'C$ is equal to a circle of radius $\sqrt{BC \cdot \frac{1}{2}(BB' + CC')}$, [Prop. 16]
and so on.

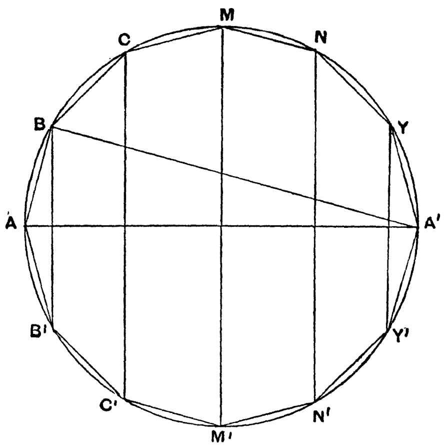
It follows, since $BA = BC = \ldots$, that the whole surface is equal to a circle whose radius is equal to

$$\sqrt{BA\,(BB' + CC' + \ldots + MM' + \ldots + YY')}.$$

Proposition 25.

The surface of the figure inscribed in a sphere as in the last propositions, consisting of portions of conical surfaces, is less than four times the greatest circle in the sphere.

Let $AB\ldots A'\ldots B'A$ be a regular polygon inscribed in a great circle, the number of its sides being a multiple of four.



As before, let BB' be drawn subtending two sides, and $CC', \ldots YY'$ parallel to BB'.

Let R be a circle such that the square of its radius is equal to

$$AB\,(BB' + CC' + \ldots + YY'),$$

so that the surface of the figure inscribed in the sphere is equal to R. [Prop. 24]

Now

$$(BB' + CC' + \ldots + YY') : AA' = A'B : AB, \quad \text{[Prop. 21]}$$

whence $$AB(BB' + CC' + \ldots + YY') = AA' . A'B.$$

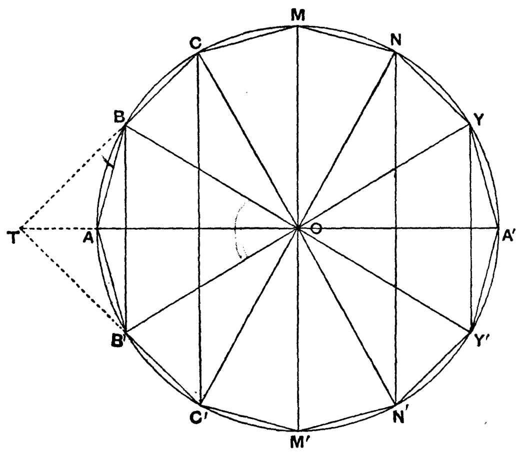
Hence $$(\text{radius of } R)^2 = AA' . A'B$$
$$< AA'^2.$$

Therefore the surface of the inscribed figure, or the circle R, is less than four times the circle $AMA'M'$.

Proposition 26.

The figure inscribed as above in a sphere is equal [in volume] to a cone whose base is a circle equal to the surface of the figure inscribed in the sphere and whose height is equal to the perpendicular drawn from the centre of the sphere to one side of the polygon.

Suppose, as before, that $AB \ldots A' \ldots B'A$ is the regular polygon inscribed in a great circle, and let BB', CC', ... be joined.



With apex O construct cones whose bases are the circles on BB', CC', ... as diameters in planes perpendicular to AA'.

Then $OBAB'$ is a solid rhombus, and its volume is equal to a cone whose base is equal to the surface of the cone ABB' and whose height is equal to the perpendicular from O on AB [Prop. 18]. Let the length of the perpendicular be p.

Again, if CB, $C'B'$ produced meet in T, the portion of the solid figure which is described by the revolution of the triangle BOC about AA' is equal to the difference between the rhombi $OCTC'$ and $OBTB'$, i.e. to a cone whose base is equal to the surface of the frustum $BB'C'C$ and whose height is p [Prop. 20].

Proceeding in this manner, and adding, we prove that, since cones of equal height are to one another as their bases, the volume of the solid of revolution is equal to a cone with height p and base equal to the sum of the surfaces of the cone BAB', the frustum $BB'C'C$, etc., i.e. a cone with height p and base equal to the surface of the solid.

Proposition 27.

The figure inscribed in the sphere as before is less than four times the cone whose base is equal to a great circle of the sphere and whose height is equal to the radius of the sphere.

By Prop. 26 the volume of the solid figure is equal to a cone whose base is equal to the surface of the solid and whose height is p, the perpendicular from O on any side of the polygon. Let R be such a cone.

Take also a cone S with base equal to the great circle, and height equal to the radius, of the sphere.

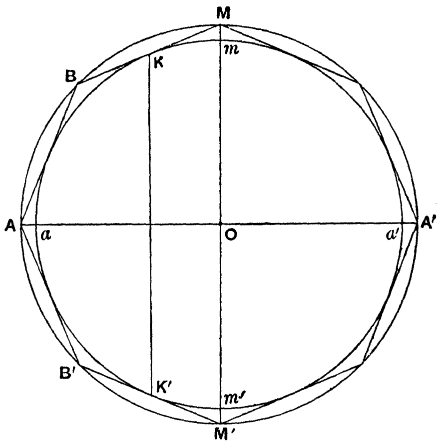
Now, since the surface of the inscribed solid is less than four times the great circle [Prop. 25], the base of the cone R is less than four times the base of the cone S.

Also the height (p) of R is less than the height of S.

Therefore the volume of R is less than four times that of S; and the proposition is proved.

Proposition 28.

Let a regular polygon, whose sides are a multiple of four in number, be circumscribed about a great circle of a given sphere, as $AB...A'...B'A$; and about the polygon describe another circle, which will therefore have the same centre as the great circle of the sphere. Let AA' bisect the polygon and cut the sphere in a, a'.



If the great circle and the circumscribed polygon revolve together about AA', the great circle will describe the surface of a sphere, the angular points of the polygon except A, A' will move round the surface of a larger sphere, the points of contact of the sides of the polygon with the great circle of the inner sphere will describe circles on that sphere in planes perpendicular to AA', and the sides of the polygon themselves will describe portions of conical surfaces. *The circumscribed figure will thus be greater than the sphere itself.*

Let any side, as BM, touch the inner circle in K, and let K' be the point of contact of the circle with $B'M'$.

Then the circle described by the revolution of KK' about AA' is the boundary in one plane of two surfaces

(1) the surface formed by the revolution of the circular segment KaK', and

(2) the surface formed by the revolution of the part $KB\ldots A\ldots B'K'$ of the polygon.

Now the second surface entirely includes the first, and they are both concave in the same direction;

therefore [*Assumptions*, 4] the second surface is greater than the first.

The same is true of the portion of the surface on the opposite side of the circle on KK' as diameter.

Hence, adding, we see that *the surface of the figure circumscribed to the given sphere is greater than that of the sphere itself.*

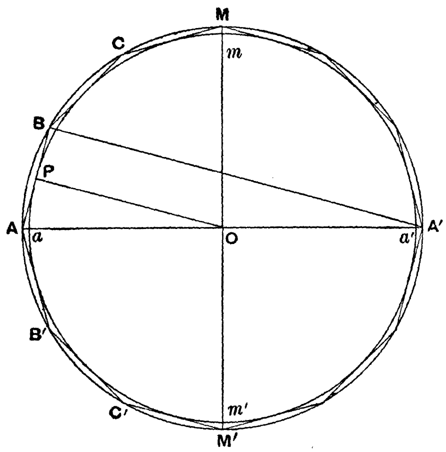
Proposition 29.

In a figure circumscribed to a sphere in the manner shown in the previous proposition the surface is equal to a circle the square on whose radius is equal to $AB(BB' + CC' + \ldots)$.

For the figure circumscribed to the sphere is inscribed in a larger sphere, and the proof of Prop. 24 applies.

Proposition 30.

The surface of a figure circumscribed as before about a sphere is greater than four times the great circle of the sphere.



Let $AB...A'...B'A$ be the regular polygon of $4n$ sides which by its revolution about AA' describes the figure circumscribing the sphere of which $ama'm'$ is a great circle. Suppose aa', AA' to be in one straight line.

Let R be a circle equal to the surface of the circumscribed solid.

Now $(BB' + CC' + \ldots) : AA' = A'B : BA$, [as in Prop. 21]

so that $AB\,(BB' + CC' + \ldots) = AA' \,.\, A'B.$

Hence (radius of R) $= \sqrt{AA' \,.\, A'B}$ [Prop. 29]

$$> A'B.$$

But $A'B = 2OP$, where P is the point in which AB touches the circle $ama'm'$.

Therefore (radius of R) > (diameter of circle $ama'm'$);

whence R, and therefore the surface of the circumscribed solid, is greater than four times the great circle of the given sphere.

Proposition 31.

The solid of revolution circumscribed as before about a sphere is equal to a cone whose base is equal to the surface of the solid and whose height is equal to the radius of the sphere.

The solid is, as before, a solid inscribed in a larger sphere; and, since the perpendicular on any side of the revolving polygon is equal to the radius of the inner sphere, the proposition is identical with Prop. 26.

COR. *The solid circumscribed about the smaller sphere is greater than four times the cone whose base is a great circle of the sphere and whose height is equal to the radius of the sphere.*

For, since the surface of the solid is greater than four times the great circle of the inner sphere [Prop. 30], the cone whose base is equal to the surface of the solid and whose height is the radius of the sphere is greater than four times the cone of the same height which has the great circle for base. [*Lemma* 1.]

Hence, by the proposition, the volume of the solid is greater than four times the latter cone.

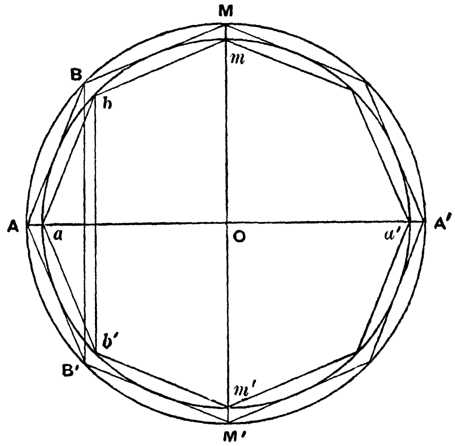
Proposition 32.

If a regular polygon with 4n sides be inscribed in a great circle of a sphere, as ab...a′...b′a, and a similar polygon AB...A′...B′A be described about the great circle, and if the polygons revolve with the great circle about the diameters aa′, AA′ respectively, so that they describe the surfaces of solid figures inscribed in and circumscribed to the sphere respectively, then

(1) *the surfaces of the circumscribed and inscribed figures are to one another in the duplicate ratio of their sides, and*

(2) *the figures themselves [i.e. their volumes] are in the triplicate ratio of their sides.*

(1) Let AA', aa' be in the same straight line, and let $MmOm'M'$ be a diameter at right angles to them.



Join BB', CC', ... and bb', cc', ... which will all be parallel to one another and MM'.

Suppose R, S to be circles such that

$$R = (\text{surface of circumscribed solid}),$$

$$S = (\text{surface of inscribed solid}).$$

Then $\quad$ (radius of $R)^2 = AB\,(BB' + CC' + \ldots)$ $\quad$ [Prop. 29]

(radius of $S)^2 = ab\,(bb' + cc' + \ldots)$. $\quad$ [Prop. 24]

And, since the polygons are similar, the rectangles in these two equations are similar, and are therefore in the ratio of

$$AB^2 : ab^2.$$

Hence

(surface of circumscribed solid) : (surface of inscribed solid)

$$= AB^2 : ab^2.$$

(2) Take a cone V whose base is the circle R and whose height is equal to Oa, and a cone W whose base is the circle S and whose height is equal to the perpendicular from O on ab, which we will call p.

Then V, W are respectively equal to the volumes of the circumscribed and inscribed figures. [Props. 31, 26]

Now, since the polygons are similar,

$$AB : ab = Oa : p$$

$$= \text{(height of cone } V) : \text{(height of cone } W);$$

and, as shown above, the bases of the cones (the circles R, S) are in the ratio of AB^2 to ab^2.

Therefore $\quad V : W = AB^3 : ab^3.$

Proposition 33.

The surface of any sphere is equal to four times the greatest circle in it.

Let C be a circle equal to four times the great circle.

Then, if C is not equal to the surface of the sphere, it must either be less or greater.

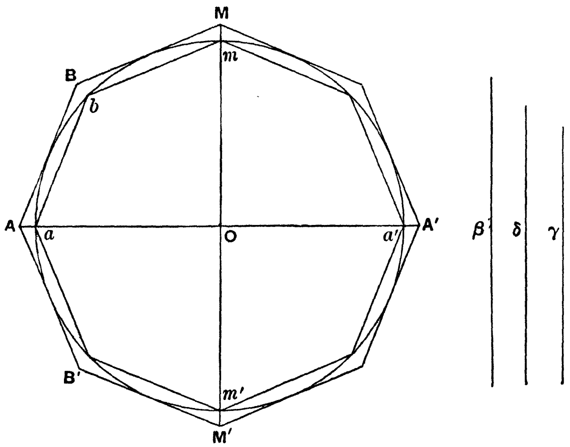
I. Suppose C less than the surface of the sphere.

It is then possible to find two lines β, γ, of which β is the greater, such that

$$\beta : \gamma < \text{(surface of sphere)} : C. \quad \text{[Prop. 2]}$$

Take such lines, and let δ be a mean proportional between them.

Suppose similar regular polygons with $4n$ sides circumscribed about and inscribed in a great circle such that the ratio of their sides is less than the ratio $\beta : \delta$. [Prop. 3]



Let the polygons with the circle revolve together about a diameter common to all, describing solids of revolution as before.

Then (surface of outer solid) : (surface of inner solid)

$= (\text{side of outer})^2 : (\text{side of inner})^2$ [Prop. 32]

$< \beta^2 : \delta^2$, or $\beta : \gamma$

$<$ (surface of sphere) : C, *a fortiori.*

But this is impossible, since the surface of the circumscribed solid is greater than that of the sphere [Prop. 28], while the surface of the inscribed solid is less than C [Prop. 25].

Therefore C is not less than the surface of the sphere.

II. Suppose C greater than the surface of the sphere.

Take lines β, γ, of which β is the greater, such that

$$\beta : \gamma < C : (\text{surface of sphere}).$$

Circumscribe and inscribe to the great circle similar regular polygons, as before, such that their sides are in a ratio less than that of β to δ, and suppose solids of revolution generated in the usual manner.

Then, in this case,

(surface of circumscribed solid) : (surface of inscribed solid)
$< C$: (surface of sphere).

But this is impossible, because the surface of the circumscribed solid is greater than C [Prop. 30], while the surface of the inscribed solid is less than that of the sphere [Prop. 23].

Thus C is not greater than the surface of the sphere.

Therefore, since it is neither greater nor less, C is equal to the surface of the sphere.

Proposition 34.

Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere.

Let the sphere be that of which $ama'm'$ is a great circle.

If now the sphere is not equal to four times the cone described, it is either greater or less.

I. If possible, let the sphere be greater than four times the cone.

Suppose V to be a cone whose base is equal to four times the great circle and whose height is equal to the radius of the sphere.

Then, by hypothesis, the sphere is greater than V; and two lines β, γ can be found (of which β is the greater) such that

$$\beta : \gamma < (\text{volume of sphere}) : V.$$

Between β and γ place two arithmetic means δ, ϵ.

As before, let similar regular polygons with sides $4n$ in number be circumscribed about and inscribed in the great circle, such that their sides are in a ratio less than $\beta : \delta$.

Imagine the diameter aa' of the circle to be in the same straight line with a diameter of both polygons, and imagine the latter to revolve with the circle about aa', describing the surfaces of two solids of revolution. The volumes of these solids are therefore in the triplicate ratio of their sides. [Prop. 32]

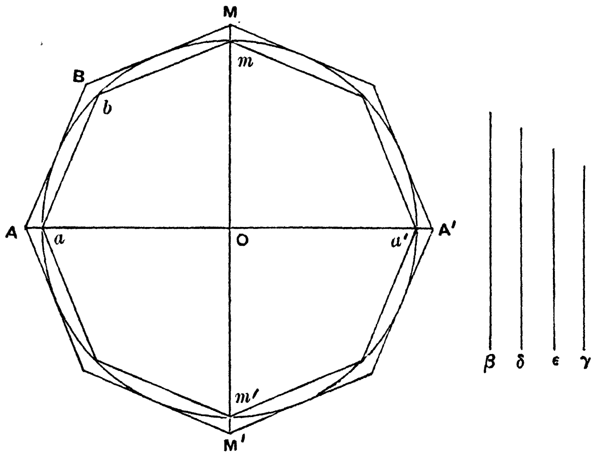
Thus (vol. of outer solid) : (vol. of inscribed solid)

$< \beta^3 : \delta^3$, by hypothesis,

$< \beta : \gamma$, *a fortiori* (since $\beta : \gamma > \beta^3 : \delta^3$)*,

< (volume of sphere) : V, *a fortiori*.

But this is impossible, since the volume of the circumscribed



* That $\beta : \gamma > \beta^3 : \delta^3$ is assumed by Archimedes. Eutocius proves the property in his commentary as follows.

Take x such that $\beta : \delta = \delta : x$.

Thus $\beta - \delta : \beta = \delta - x : \delta$

and, since $\beta > \delta$, $\beta - \delta > \delta - x$.

But, by hypothesis, $\beta - \delta \doteq \delta - \epsilon$.

Therefore $\delta - \epsilon > \delta - x$,

or $x > \epsilon$.

Again, suppose $\delta : x = x : y$,

and, as before, we have $\delta - x > x - y$,

so that, *a fortiori*, $\delta - \epsilon > x - y$.

Therefore $\epsilon - \gamma > x - y$;

and, since $x > \epsilon$, $y > \gamma$.

Now, by hypothesis, β, δ, x, y are in continued proportion;

therefore $\beta^3 : \delta^3 = \beta : y$

$< \beta : \gamma$.

solid is greater than that of the sphere [Prop. 28], while the volume of the inscribed solid is less than V [Prop. 27].

Hence the sphere is not greater than V, or four times the cone described in the enunciation.

II. If possible, let the sphere be less than V.

In this case we take β, γ (β being the greater) such that

$$\beta : \gamma < V : (\text{volume of sphere}).$$

The rest of the construction and proof proceeding as before, we have finally

$$(\text{volume of outer solid}) : (\text{volume of inscribed solid})$$
$$< V : (\text{volume of sphere}).$$

But this is impossible, because the volume of the outer solid is greater than V [Prop. 31, Cor.], and the volume of the inscribed solid is less than the volume of the sphere.

Hence the sphere is not less than V.

Since then the sphere is neither less nor greater than V, it is equal to V, or to four times the cone described in the enunciation.

Cor. From what has been proved it follows that *every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere is $\frac{3}{2}$ of the sphere, and its surface together with its bases is $\frac{3}{2}$ of the surface of the sphere.*

For the cylinder is three times the cone with the same base and height [Eucl. XII. 10], i.e. six times the cone with the same base and with height equal to the radius of the sphere.

But the sphere is four times the latter cone [Prop. 34]. Therefore the cylinder is $\frac{3}{2}$ of the sphere.

Again, the surface of a cylinder (excluding the bases) is equal to a circle whose radius is a mean proportional between the height of the cylinder and the diameter of its base [Prop. 13].

In this case the height is equal to the diameter of the base and therefore the circle is that whose radius is the diameter of the sphere, or a circle equal to four times the great circle of the sphere.

Therefore the surface of the cylinder with the bases is equal to six times the great circle.

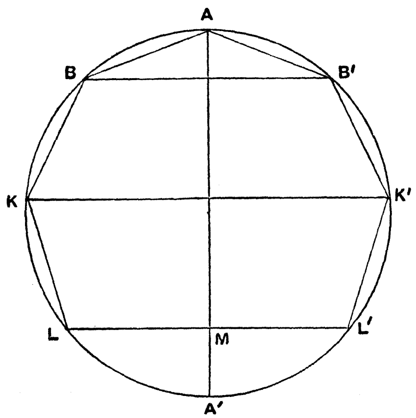
And the surface of the sphere is four times the great circle [Prop. 33]; whence

(surface of cylinder with bases) $= \frac{3}{2}$. (surface of sphere).

Proposition 35.

If in a segment of a circle LAL' (where A is the middle point of the arc) a polygon $LK...A...K'L'$ be inscribed of which LL' is one side, while the other sides are $2n$ in number and all equal, and if the polygon revolve with the segment about the diameter AM, generating a solid figure inscribed in a segment of a sphere, then the surface of the inscribed solid is equal to a circle the square on whose radius is equal to the rectangle

$$AB\left(BB' + CC' + \dots + KK' + \frac{LL'}{2}\right).$$



The surface of the inscribed figure is made up of portions of surfaces of cones.

If we take these successively, the surface of the cone BAB' is equal to a circle whose radius is

$$\sqrt{AB \cdot \tfrac{1}{2}BB'}. \qquad \text{[Prop. 14]}$$

The surface of the frustum of a cone $BCC'B'$ is equal to a circle whose radius is

$$\sqrt{AB \cdot \frac{BB' + CC'}{2}}; \qquad \text{[Prop. 16]}$$

and so on.

Proceeding in this way and adding, we find, since circles are to one another as the squares of their radii, that the surface of the inscribed figure is equal to a circle whose radius is

$$\sqrt{AB\left(BB' + CC' + \ldots + KK' + \frac{LL'}{2}\right)}.$$

Proposition 36.

The surface of the figure inscribed as before in the segment of a sphere is less than that of the segment of the sphere.

This is clear, because the circular base of the segment is a common boundary of each of two surfaces, of which one, the segment, includes the other, the solid, while both are concave in the same direction [*Assumptions*, 4].

Proposition 37.

The surface of the solid figure inscribed in the segment of the sphere by the revolution of $LK \ldots A \ldots K'L'$ about AM is less than a circle with radius equal to AL.

Let the diameter AM meet the circle of which LAL' is a segment again in A'. Join $A'B$.

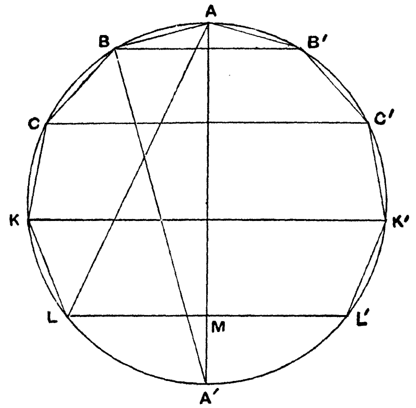
As in Prop. 35, the surface of the inscribed solid is equal to a circle the square on whose radius is

$$AB\,(BB' + CC' + \ldots + KK' + LM).$$

But this rectangle $= A'B \,.\, AM$ [Prop. 22]

$< A'A \,.\, AM$

$< AL^2$.



Hence the surface of the inscribed solid is less than the circle whose radius is AL.

Proposition 38.

The solid figure described as before in a segment of a sphere less than a hemisphere, together with the cone whose base is the base of the segment and whose apex is the centre of the sphere, is equal to a cone whose base is equal to the surface of the inscribed solid and whose height is equal to the perpendicular from the centre of the sphere on any side of the polygon.

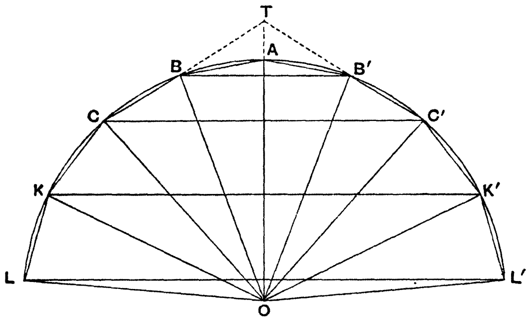
Let O be the centre of the sphere, and p the length of the perpendicular from O on AB.

Suppose cones described with O as apex, and with the circles on BB', CC', ... as diameters as bases.

Then the rhombus $OBAB'$ is equal to a cone whose base is equal to the surface of the cone BAB', and whose height is p. [Prop. 18]

Again, if CB, $C'B'$ meet in T, the solid described by the triangle BOC as the polygon revolves about AO is the difference between the rhombi $OCTC'$ and $OBTB'$, and is therefore equal to a cone whose base is equal to the surface of the frustum $BCC'B'$ and whose height is p. [Prop. 20]

Similarly for the part of the solid described by the triangle COD as the polygon revolves; and so on.



Hence, by addition, the solid figure inscribed in the segment together with the cone OLL' is equal to a cone whose base is the surface of the inscribed solid and whose height is p.

Cor. *The cone whose base is a circle with radius equal to AL and whose height is equal to the radius of the sphere is greater than the sum of the inscribed solid and the cone OLL'.*

For, by the proposition, the inscribed solid together with the cone OLL' is equal to a cone with base equal to the surface of the solid and with height p.

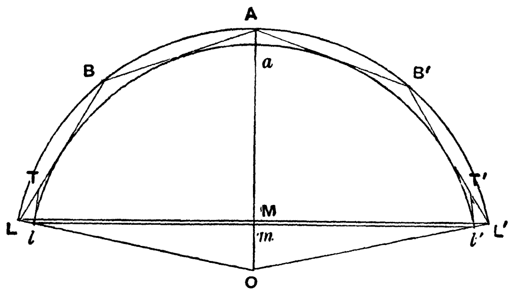
This latter cone is less than a cone with height equal to OA and with base equal to the circle whose radius is AL, because the height p is less than OA, while the surface of the solid is less than a circle with radius AL. [Prop. 37]

Proposition 39.

Let lal' be a segment of a great circle of a sphere, being less than a semicircle. Let O be the centre of the sphere, and join Ol, Ol'. Suppose a polygon circumscribed about the sector $Olal'$ such that its sides, excluding the two radii, are $2n$ in number and all equal, as $LK, \ldots BA, AB', \ldots K'L'$; and let OA be that radius of the great circle which bisects the segment lal'.

The circle circumscribing the polygon will then have the same centre O as the given great circle.

Now suppose the polygon and the two circles to revolve together about OA. The two circles will describe spheres, the



angular points except A will describe circles on the outer sphere, with diameters BB' etc., the points of contact of the sides with the inner segment will describe circles on the inner sphere, the sides themselves will describe the surfaces of cones or frusta of cones, and the whole figure circumscribed to the segment of the inner sphere by the revolution of the equal sides of the polygon will have for its base the circle on LL' as diameter.

The surface of the solid figure so circumscribed about the sector of the sphere [*excluding its base*] *will be greater than that of the segment of the sphere whose base is the circle on ll' as diameter.*

For draw the tangents lT, $l'T'$ to the inner segment at l, l'. These with the sides of the polygon will describe by their revolution a solid whose surface is greater than that of the segment [*Assumptions*, 4].

But the surface described by the revolution of lT is less than that described by the revolution of LT, since the angle TlL is a right angle, and therefore $LT > lT$.

Hence, *a fortiori*, the surface described by $LK \ldots A \ldots K'L'$ is greater than that of the segment.

COR. *The surface of the figure so described about the sector of the sphere is equal to a circle the square on whose radius is equal to the rectangle*

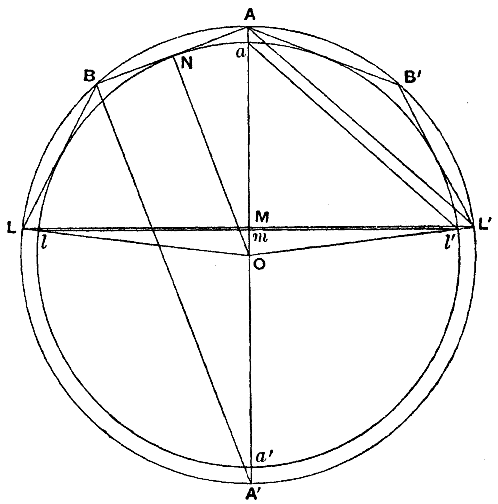
$$AB(BB' + CC' + \ldots + KK' + \tfrac{1}{2}LL').$$

For the circumscribed figure is inscribed in the outer sphere, and the proof of Prop. 35 therefore applies.

Proposition 40.

The surface of the figure circumscribed to the sector as before is greater than a circle whose radius is equal to al.

Let the diameter AaO meet the great circle and the circle circumscribing the revolving polygon again in a', A'. Join $A'B$, and let ON be drawn to N, the point of contact of AB with the inner circle.



Now, by Prop. 39, Cor., the surface of the solid figure circumscribed to the sector $OlAl'$ is equal to a circle the square on whose radius is equal to the rectangle

$$AB\left(BB' + CC' + \ldots + KK' + \frac{LL'}{2}\right).$$

But this rectangle is equal to $A'B \cdot AM$ [as in Prop. 22].

Next, since AL', al' are parallel, the triangles AML', aml' are similar. And $AL' > al'$; therefore $AM > am$.

Also $$A'B = 2ON = aa'.$$

Therefore $$A'B \,.\, AM > am \,.\, aa'$$
$$> al'^2.$$

Hence the surface of the solid figure circumscribed to the sector is greater than a circle whose radius is equal to al', or al.

Cor. 1. *The volume of the figure circumscribed about the sector together with the cone whose apex is O and base the circle on LL' as diameter, is equal to the volume of a cone whose base is equal to the surface of the circumscribed figure and whose height is ON.*

For the figure is inscribed in the outer sphere which has the same centre as the inner. Hence the proof of Prop. 38 applies.

Cor. 2. *The volume of the circumscribed figure with the cone OLL' is greater than the cone whose base is a circle with radius equal to al and whose height is equal to the radius (Oa) of the inner sphere.*

For the volume of the figure with the cone OLL' is equal to a cone whose base is equal to the surface of the figure and whose height is equal to ON.

And the surface of the figure is greater than a circle with radius equal to al [Prop. 40], while the heights Oa, ON are equal.

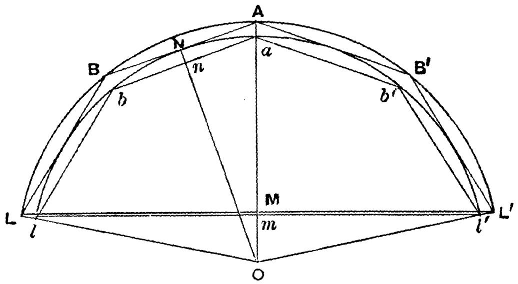
Proposition 41.

Let lal' be a segment of a great circle of a sphere which is less than a semicircle.

Suppose a polygon inscribed in the sector $Olal'$ such that the sides $lk, \ldots ba$, $ab', \ldots k'l'$ are $2n$ in number and all equal. Let a similar polygon be circumscribed about the sector so that its sides are parallel to those of the first polygon; and draw the circle circumscribing the outer polygon.

Now let the polygons and circles revolve together about OaA, the radius bisecting the segment lal'.

Then (1) *the surfaces of the outer and inner solids of revolution so described are in the ratio of* AB^2 *to* ab^2, *and* (2) *their volumes together with the corresponding cones with the same base and with apex* O *in each case are as* AB^3 *to* ab^3.



(1) For the surfaces are equal to circles the squares on whose radii are equal respectively to

$$AB\left(BB' + CC' + \ldots + KK' + \frac{LL'}{2}\right),$$

[Prop. 39, Cor.]

and $$ab\left(bb' + cc' + \ldots + kk' + \frac{ll'}{2}\right).$$ [Prop. 35]

But these rectangles are in the ratio of AB^2 to ab^2. Therefore so are the surfaces.

(2) Let OnN be drawn perpendicular to ab and AB; and suppose the circles which are equal to the surfaces of the outer and inner solids of revolution to be denoted by S, s respectively.

Now the volume of the circumscribed solid together with the cone OLL' is equal to a cone whose base is S and whose height is ON [Prop. 40, Cor. 1].

And the volume of the inscribed figure with the cone Oll' is equal to a cone with base s and height On [Prop. 38].

But $$S : s = AB^2 : ab^2,$$

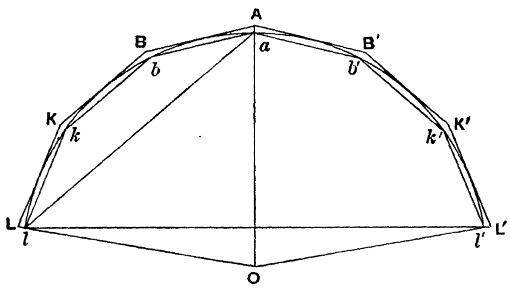
and $$ON : On = AB : ab.$$

Therefore the volume of the circumscribed solid together with the cone OLL' is to the volume of the inscribed solid together with the cone Oll' as AB^3 is to ab^3 [*Lemma* 5].

Proposition 42.

If lal′ be a segment of a sphere less than a hemisphere and Oa the radius perpendicular to the base of the segment, the surface of the segment is equal to a circle whose radius is equal to al.

Let R be a circle whose radius is equal to al. Then the surface of the segment, which we will call S, must, if it be not equal to R, be either greater or less than R.



I. Suppose, if possible, $S > R$.

Let lal' be a segment of a great circle which is less than a semicircle. Join Ol, Ol', and let similar polygons with $2n$ equal sides be circumscribed and inscribed to the sector, as in the previous propositions, but such that

(circumscribed polygon) : (inscribed polygon) $< S : R$. [Prop. 6]

Let the polygons now revolve with the segment about OaA, generating solids of revolution circumscribed and inscribed to the segment of the sphere.

Then

(surface of outer solid) : (surface of inner solid)

$= AB^2 : ab^2$ [Prop. 41]

= (circumscribed polygon) : (inscribed polygon)

$< S : R$, by hypothesis.

But the surface of the outer solid is greater than S [Prop. 39].

Therefore the surface of the inner solid is greater than R; which is impossible, by Prop. 37.

II. Suppose, if possible, $S < R$.

In this case we circumscribe and inscribe polygons such that their ratio is less than $R : S$; and we arrive at the result that

$$(\text{surface of outer solid}) : (\text{surface of inner solid}) < R : S.$$

But the surface of the outer solid is greater than R [Prop. 40]. Therefore the surface of the inner solid is greater than S: which is impossible [Prop. 36].

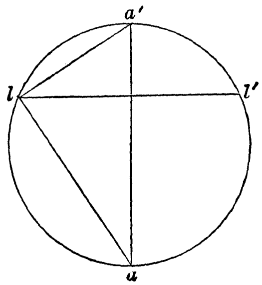
Hence, since S is neither greater nor less than R,

$$S = R.$$

Proposition 43.

Even if the segment of the sphere is greater than a hemisphere, its surface is still equal to a circle whose radius is equal to al.

For let $lal'a'$ be a great circle of the sphere, aa' being the diameter perpendicular to ll'; and let $la'l'$ be a segment less than a semicircle.



Then, by Prop. 42, the surface of the segment $la'l'$ of the sphere is equal to a circle with radius equal to $a'l$.

Also the surface of the whole sphere is equal to a circle with radius equal to aa' [Prop. 33].

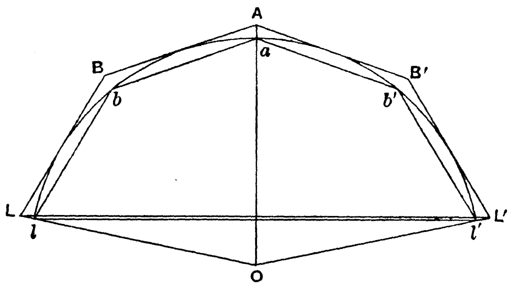
But $aa'^2 - a'l^2 = al^2$, and circles are to one another as the squares on their radii.

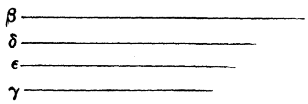
Therefore the surface of the segment lal', being the difference between the surfaces of the sphere and of $la'l'$, is equal to a circle with radius equal to al.

Proposition 44.

The volume of any sector of a sphere is equal to a cone whose base is equal to the surface of the segment of the sphere included in the sector, and whose height is equal to the radius of the sphere.

Let R be a cone whose base is equal to the surface of the segment lal' of a sphere and whose height is equal to the radius of the sphere; and let S be the volume of the sector $Olal'$.





Then, if S is not equal to R, it must be either greater or less.

I. Suppose, if possible, that $S > R$.

Find two straight lines β, γ, of which β is the greater, such that

$$\beta : \gamma < S : R;$$

and let δ, ϵ be two arithmetic means between β, γ.

Let lal' be a segment of a great circle of the sphere. Join Ol, Ol', and let similar polygons with $2n$ equal sides be circumscribed and inscribed to the sector of the circle as before, but such that their sides are in a ratio less than $\beta : \delta$. [Prop. 4].

Then let the two polygons revolve with the segment about OaA, generating two solids of revolution.

Denoting the volumes of these solids by V, v respectively, we have

$$(V + \text{cone } OLL') : (v + \text{cone } Oll') = AB^3 : ab^3 \qquad \text{[Prop. 41]}$$
$$< \beta^3 : \delta^3$$
$$< \beta : \gamma, \textit{ a fortiori*},$$
$$< S : R, \text{ by hypothesis.}$$

Now $\qquad (V + \text{cone } OLL') > S.$

Therefore also $\qquad (v + \text{cone } Oll') > R.$

But this is impossible, by Prop. 38, Cor. combined with Props. 42, 43.

Hence $\qquad S \not> R.$

II. Suppose, if possible, that $S < R$.

In this case we take β, γ such that

$$\beta : \gamma < R : S,$$

and the rest of the construction proceeds as before.

We thus obtain the relation

$$(V + \text{cone } OLL') : (v + \text{cone } Oll') < R : S.$$

Now $\qquad (v + \text{cone } Oll') < S.$

Therefore $\qquad (V + \text{cone } OLL') < R;$

which is impossible, by Prop. 40, Cor. 2 combined with Props. 42, 43.

Since then S is neither greater nor less than R,

$$S = R.$$

* Cf. note on Prop. 34, p. 42..