UNIPOTENT ELEMENTS AND TWISTING IN LINK HOMOLOGY

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ABSTRACT. Let \( \mathcal{U} \) be the unipotent variety of a complex reductive group \( G \). Fix opposed Borel subgroups \( B_+ \subseteq G \) with unipotent radicals \( U_+ \). The map that sends \( x_+x_- \mapsto x_+x_-x_+^{-1} \) for all \( x_+ \in U_+ \) restricts to a map from \( U_+U_- \cap gB_+ \) into \( \mathcal{U} \cap gB_+ \), for any \( g \). We conjecture that the restricted map forms half of a homotopy equivalence between these varieties, and thus, induces a weight-preserving isomorphism between their compactly-supported cohomologies. Noting that the map is equivariant with respect to certain actions of \( B_+ \cap gB_+g^{-1} \), we prove for type \( A \) that an equivariant analogue of this isomorphism exists. Curiously, this follows from a certain duality in Khovanov–Rozansky homology, a tool from knot theory.

1. Introduction

1.1. Let \( G \) be a complex, connected, reductive algebraic group. Let \( B_+ \) and \( B_- \) be opposed Borel subgroups of \( G \), and let \( U_\pm \) be the unipotent radical of \( B_\pm \). For instance, if \( G \) is the group of invertible \( n \times n \) matrices \( GL_n \), then we can choose \( U_+ \), resp. \( U_- \), to be the subgroup of upper-triangular, resp. lower-triangular, matrices with 1’s along the diagonal. Every element of \( U_+U_- \) can be written uniquely in the form \( x_+x_- \), where \( x_+ \in U_+ \).

Let \( \mathcal{U} \subseteq G \) be the closed subvariety of unipotent elements. With the notation above, there is a map \( \Phi : U_+U_- \to \mathcal{U} \) defined by

\[
\Phi(x_+x_-) = x_+x_-x_+^{-1}.
\]

Note that \( U_+U_- \) is an affine space, whereas \( \mathcal{U} \) is usually singular. Nonetheless, in the analytic topology, the sets \( U_+U_-(C) \) and \( \mathcal{U}(C) \) are homotopy equivalent, as they are both contractible.

For any \( g \in G(C) \), the map \( \Phi \) restricts to a map \( \Phi_g : \mathcal{U}_g \to \mathcal{U}_g \), where

\[
\mathcal{U}_g = \mathcal{U} \cap gB_+,
\]

\[
\mathcal{V}_g = U_+U_- \cap gB_+.
\]

Note that \( \mathcal{U}_g, \mathcal{V}_g, \Phi_g \) only depend on the coset \( gB_+ \). We endow \( \mathcal{U}_g(C) \) and \( \mathcal{V}_g(C) \) with the analytic topology. The new idea proposed in this note is:

**Conjecture 1.** The map \( \Phi_g \) defines half of a homotopy equivalence between \( \mathcal{U}_g(C) \) and \( \mathcal{V}_g(C) \).

The simplest case is \( g \in B_+(C) \). Here, \( \mathcal{U}_g(C) = U_+(C) = \mathcal{V}_g(C) \) and \( \Phi_g \) is a retract from the affine space \( U_+(C) \) onto the point corresponding to the identity of \( G(C) \). In general, neither \( \mathcal{U}_g(C) \) nor \( \mathcal{V}_g(C) \) will be contractible, as we will show in Section 3 through examples.
We emphasize that for an arbitrary map of varieties $\Phi : Y \to X$ that induces a homotopy equivalence on $\mathbb{C}$-points, there may not exist a nonconstant map of varieties $X \to S$ such that $\Phi$ induces a homotopy equivalence of fibers $Y_s(\mathbb{C}) \to X_s(\mathbb{C})$ for every $s \in S(\mathbb{C})$. For instance, take $\Phi$ to be the projection from a quadric cone onto its axis of symmetry. Both total spaces are contractible, so $\Phi$ is automatically a homotopy equivalence. For the map of fibers over $s$ to be a homotopy equivalence as well, $X_s(\mathbb{C})$ must contract onto the origin of $X(\mathbb{C}) = \mathbb{C}$.

Since $X \to S$ is nonconstant, some $s$ must violate this condition.

1.2. Some motivation for Conjecture 1 comes from classical results about finite groups of Lie type. To state them, let us implicitly replace $G$ with its split form over a finite field $\mathbb{F}$ of good characteristic.

In [S, Thm. 15.1], Steinberg showed the identity $|U(\mathbb{F})| = |U_+U_-(\mathbb{F})|$. In [Ka, §4], Kawanaka showed an identity equivalent to $|U_g(\mathbb{F})| = |V_g(\mathbb{F})|$; (1.1) see Remark 22. More recently, Lusztig has given a new proof of (1.1) in [L].

Conjecture 1 essentially implies (1.1). Indeed, once one checks that $U_g$ and $V_g$ have the same dimension, Conjecture 1 implies that $\Phi_g$ induces an isomorphism between the rational, compactly-supported cohomologies of $U_g$ and $V_g$. Since $\Phi_g$ is algebraic, this isomorphism must match their weight filtrations in the sense of mixed Hodge theory. One can further check that both sides of (1.1) are polynomial functions of $|\mathbb{F}|$, so by the results explained in [Kat], the virtual weight polynomials of $U_g$ and $V_g$ specialize to their $\mathbb{F}$-point counts.

Our main result is evidence for an equivariant analogue of Conjecture 1. Let $T = B_+ \cap B_-$, so that $B_+ = TU_+ \simeq T \times U_+$. The map $\Phi$ transports the $B_+$-action on $U_+ U_-$ defined by

$$tu \cdot x = (txu(t^{-1}))(txu(t^{-1}))$$

for all $(t, u) \in T \times U_+$ onto the $B_+$-action on $U$ by left conjugation. Setting

$$H_g = B_+ \cap gB_+ g^{-1},$$

we find that these $B_+$-actions restrict to $H_g$-actions on $V_g$ and $U_g$, respectively. The $B_+$-equivariance of $\Phi$ thus restricts to $H_g$-equivariance of $\Phi_g$. Though we do not have a general theorem about $\Phi_g$ itself, we prove:

**Theorem 2.** If $G$ is a split reductive group of type $A$ over $\mathbb{F}$, then for all $g \in G(\mathbb{F})$, there is an isomorphism of bigraded vector spaces

$$\text{gr}_W^* H^*_{c,H_g}(U_g, \mathbb{Q}_\ell) \simeq \text{gr}_W^* H^*_{c,H_g}(V_g, \mathbb{Q}_\ell),$$

where $H^*_{c,H_g}(-, \mathbb{Q}_\ell)$ denotes $H_g$-equivariant, compactly-supported $\ell$-adic cohomology and $W$ denotes its weight filtration.

The proof in Sections 4–5 uses ideas from the rather different world of low-dimensional topology, as we now explain.
Let $\mathcal{X}_g$ be the variety of Borel subgroups of $G$ in generic position with respect to both $B_+$ and $gB_+g^{-1}$. In Section 4, we will construct an isomorphism $\mathcal{X}_g \rightarrow \mathcal{Y}_g$ that transports the $H_g$-action on $\mathcal{X}_g$ by left conjugation to the $H_g$-action on $\mathcal{Y}_g$ described above.

The variety $\mathcal{X}_g$ is closely related to the so-called braid varieties that have been studied recently by several authors, including [M, SW, CGGS, GL]. To be more precise: Let $W$ be the Weyl group of $G$, and let $Br^+_W$ be the positive braid monoid of $W$. By definition, $Br^+_W$ is generated by elements $\sigma_w$ for each $w \in W$ modulo $\sigma_{ww'} = \sigma_w \sigma_{w'}$ whenever $\ell(ww') = \ell(w) + \ell(w')$, where $\ell$ is the Bruhat length function on $W$. A braid variety is a configuration space of tuples of Borels, where the relative positions of cyclically consecutive Borels have constraints determined by a fixed element $\beta \in Br^+_W$. Note that there is a central element $\pi \in Br^+_W$ known as the full twist and given by $\pi = \sigma_{w_0}^2$, where $w_0$ is the longest element of $W$. In the notation of [T, Appendix B], the variety $\mathcal{X}_g$ is isomorphic to the braid variety attached to $\sigma_w \pi$, where $w$ is the relative position of the pair $(B_+, gB_+g^{-1})$.

In [T], it was shown that the weight filtration on the equivariant, compactly-supported cohomology of the braid variety of $\beta$ encodes a certain summand of a certain triply-graded vector space attached to $\beta$, known as its HOMFLYPT or Khovanov–Rozansky (KR) homology. When $W$ is the symmetric group $S_n$, the braid $\beta$ represents the isotopy class of a topological braid on $n$ strands, and the KR homology of $\beta$ is an isotopy invariant of the link closure of $\beta$, up to grading shifts [KR, Kh].

At the same time, when $w$ is the relative position of $(B_+, gB_+g^{-1})$, it turns out that $\mathcal{U}_g$ is closely related to another variety attached to $\sigma_w$ in [T]. Just as $\mathcal{X}_g$ encodes the “highest $a$-degree” of the KR homology of $\sigma_w \pi$, so $\mathcal{U}_g$ encodes the “lowest $a$-degree” of the KR homology of $\sigma_w$. For $W = S_n$, Gorsky–Hogancamp–Mellit–Nagakane have established an isomorphism between these bigraded vector spaces for general braids $\beta$, not just $\sigma_w$, which they deduce from an analogue of Serre duality for the homotopy category of Soergel bimodules for $S_n$ [GHMN]. Our Theorem 2 follows from this isomorphism.

This proof suggests that we regard Conjecture 1 as a geometric realization of the (purely algebraic) Serre duality of [GHMN]. Moreover, it suggests an extension of Conjecture 1 with positive braids in place of elements of $W$. We state the extended conjecture in Section 5.

The isomorphism of ibid. categorifies an earlier identity of Kálmán, relating the bivariate HOMFLY series of the link closures of $\beta$ and $\beta \pi$. In [T], we generalized Kálmán’s theorem to the braids associated with arbitrary finite Coxeter groups. (We expect, but do not show, that [GHMN] admits a similar generalization.) For Weyl groups, we also interpreted our result as an identity of $\mathbb{F}$-point counts. In Section 6, we review these results, and explain how they recover (1.1).

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2. Reductions

In this section, we collect lemmas about easy reductions and special cases of the conjecture.

**Lemma 3.** If \( G = G_1 \times G_2 \), where \( G_1, G_2 \) are again reductive, then Conjecture 1 holds if and only if it holds with \( G_1 \) in place of \( G \) and with \( G_2 \) in place of \( G \).

**Proof.** We can factor \( U_{\pm} = U_{\pm,1} \times U_{\pm,2} \) and \( \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \) and \( \Phi = \Phi^{(1)} \times \Phi^{(2)} \), where \( U_{\pm,i}, \mathcal{U}_i, \Phi^{(i)} \) are the analogues of \( U_{\pm}, \mathcal{U}, \Phi \) with \( G_i \) in place of \( G \). \( \square \)

**Lemma 4.** Both \( U_g \) and \( V_g \) are contained within the derived subgroup \( G_{\text{der}} \subseteq G \). In particular, we can assume \( G \) is semisimple in Conjecture 1.

**Proof.** It is enough to show \( U \subseteq G_{\text{der}} \), because in that case, we also have \( U_+ U_- \subseteq G_{\text{der}} G_{\text{der}} = G_{\text{der}} \). Suppose instead that \( U \not\subseteq G_{\text{der}} \). Then \( G/G_{\text{der}} \) contains nontrivial unipotent elements, because homomorphisms of algebraic groups preserve Jordan decompositions [Mi, §9.21]. But \( G/G_{\text{der}} \) is isogeneous to the center of \( G \) by [Mi, Ex. 19.25], so it is a torus, in which the only unipotent element is the identity. \( \square \)

**Lemma 5.** The map \( \Phi_g \) is unchanged, up to composition with maps that induce homeomorphisms on \( \mathbb{C} \)-points, when we replace \( G \) with the adjoint group \( G_{\text{ad}} = G/Z(G) \) and \( g \) with its image in \( G_{\text{ad}} \). In particular, Conjecture 1 is equivalent to its analogue where we replace \( G \) with any quotient by a central isogeny, and \( g \) with its image in that quotient.

**Proof.** Let \( \bar{U}_{\pm} \) and \( \bar{\mathcal{U}} \) be the respective analogues of \( U_{\pm} \) and \( \mathcal{U} \) with \( G_{\text{ad}} \) in place of \( G \). Again using the preservation of Jordan decomposition, and the fact that central elements of \( G \) are semisimple, we can check that the quotient map \( G \to G_{\text{ad}} \) restricts to maps \( U_{\pm} \to \bar{U}_{\pm} \) and \( \mathcal{U} \to \bar{\mathcal{U}} \) that give rise to bijections on field-valued points. Writing \( \bar{\Phi} : \bar{U}_+ \bar{U}_- \to \bar{\mathcal{U}} \) for the analogue of \( \Phi \), we see that the diagram

\[
\begin{array}{ccc}
U_+ U_- & \xrightarrow{\Phi} & \mathcal{U} \\
\downarrow \quad & & \downarrow \\
\bar{U}_+ \bar{U}_- & \xrightarrow{\bar{\Phi}} & \bar{\mathcal{U}}
\end{array}
\]

commutes. Now the result follows. \( \square \)

The following lemma is motivated by the Bruhat decomposition

\[
G = \coprod_{w \in W} U_{\pm} \dot{w} B_+,
\]

where \( w \mapsto \dot{w} \) is any choice of set-theoretic lift from \( W \simeq N_G(T)/T \) into \( N_G(T) \). Note that \( \dot{w} B_+ \) only depends on \( w \) because \( T \subseteq B_+ \).

**Lemma 6.** If Conjecture 1 holds for some \( g \in G(\mathbb{C}) \), then it holds with \( u g \) in place of \( g \), for all \( u \in U_+(\mathbb{C}) \). In particular, if Conjecture 1 holds for all \( g \in N_G(T)(\mathbb{C}) \), then it holds in general.
Proof. We can factor $\Phi_{ug}$ as a composition

$$\gamma_{ug} \overset{\sim}{\rightarrow} \gamma_g \overset{\Phi_g}{\rightarrow} \mathcal{U}_g \overset{\sim}{\rightarrow} \mathcal{U}_{ug},$$

where the first arrow is left multiplication by $u^{-1}$, and the last arrow is left conjugation by $u$. Since these are both isomorphisms of varieties, we get the first assertion of the lemma. The second follows from the first via Bruhat. \hfill \square

If $P_+$ and $P_-$ are opposed parabolic subgroups of $G$ containing $B_+$ and $B_-$, respectively, and $L = P_+ \cap P_-$ is their common Levi subgroup, then we write $U_{\pm, P}$ for the unipotent radical of $P_{\pm}$, and write $B_{\pm, L}, U_{\pm, L}, \mathcal{U}_L$ for the analogues of $B_\pm, U_\pm, \mathcal{U}$ with $L$ in place of $G$. Thus we have

$$P_\pm = LU_{\pm, P} \simeq L \ltimes U_{\pm, P},$$
$$B_\pm = B_{\pm, L}U_{\pm, P} \simeq B_{\pm, L} \ltimes U_{\pm, P},$$
$$U_\pm = U_{\pm, L}U_{\pm, P} \simeq U_{\pm, L} \ltimes U_{\pm, P}. $$

If $g \in L$, then we write $\mathcal{U}_{L,g}, \mathcal{V}_{L,g}, \Phi_{L,g}$ for the analogues of $\mathcal{U}_g, \mathcal{V}_g, \Phi_g$ with $L$ in place of $G$.

Lemma 7. Let $P_\pm \supseteq B_\pm$ be opposed parabolic subgroups of $G$, and let $L = P_+ \cap P_-$. Then for all $g \in L(C)$, we have isomorphisms of algebraic varieties

$$\mathcal{U}_g \simeq \mathcal{U}_{L,g}U_{+, P} \simeq \mathcal{U}_{L,g} \times U_{+, P},$$
$$\mathcal{V}_g \simeq \mathcal{V}_{L,g}U_{+, P} \simeq \mathcal{V}_{L,g} \times U_{+, P}. $$

In particular, if $g \in G(C)$ belongs to a Levi subgroup of $G$, then in Conjecture 1, we can replace $G$ with that Levi subgroup.

Proof. Since the decomposition $P_+ \simeq L \ltimes U_{+, P}$ preserves Jordan decompositions, we have $U \cap P_+ = \mathcal{U}_{L}U_{+, P} \simeq \mathcal{U}_{L} \times U_{+, P}$. Intersecting with $gB_+$, we get

$$\mathcal{U} \cap gB_+ = (\mathcal{U}_L \cap gB_{+, L})U_{+, P} \simeq (\mathcal{U}_L \cap gB_{+, L}) \times U_{+, P}. $$

Next, $U_+U_- \cap gB_+ \subseteq gB_+ \subseteq P_+$ and $U_- \cap P_+ = U_{-, L} \cap P_+$ together imply $U_+U_- \cap gB_+ = U_+U_{-, L} \cap gB_+$, from which

$$U_+U_- \cap gB_+ = U_+U_{-, L}U_{+, P}U_{-, L} \cap gB_+U_{+, P}$$
$$= (U_{+, L}U_{-, L} \cap gB_{+, L})U_{+, P}$$
$$\simeq (U_{+, L}U_{-, L} \cap gB_{+, L}) \times U_{+, P}. $$

So it remains to prove the last assertion in the lemma. For this, it is more convenient to use the decomposition $\gamma_g = U_{+, P} \mathcal{V}_{L,g} \simeq U_{+, P} \times \mathcal{V}_{L,g}$. For all $(x, y_+, y_-) \in U_{+, P} \times U_{+, L} \times U_{-, L}$, observe that

$$\Phi_g(xy+y^{-1}x^{-1} = \text{Ad}_x(\Phi_{L,g}(y+y_-),$$

where $\text{Ad}_x(u) = xux^{-1}$. A choice of deformation retract from $U_{+, P}(C)$ onto $\{1\}$ induces a homotopy from $\text{Ad}_x : \mathcal{U}_g \rightarrow \mathcal{U}_g$ onto $\text{id} : \mathcal{U}_g \rightarrow \mathcal{U}_g$, which in turn
induces a homotopy from $\Phi_g : V_g \to U_g$ onto the composition

$$V_g \xrightarrow{p_{L,g}} V_{L,g} \xrightarrow{\Phi_{L,g}} U_{L,g} \xrightarrow{i_{L,g}} U_g,$$

where $p_{L,g} : V_g \to V_{L,g}$ is the retract induced by the projection $U_{+,p}(C) \to \{1\}$ and $i_{L,g} : U_{L,g} \to U_g$ is the section induced by the inclusion $\{1\} \to U_{+,p}(C)$. Therefore, $\Phi_{L,g}$ being half of a homotopy equivalence is equivalent to $\Phi_g$ being half of a homotopy equivalence. □

**Lemma 8.** Conjecture 1 holds for $g \in B_+(C)$.

As observed in the introduction, this can be proved by computing $U_g(C), V_g(C),$ and $\Phi_g$ directly. Alternatively:

**Proof.** Since $U_g, V_g, \Phi_g$ only depend on $gB_+$, we can assume $g = 1$. Then Lemma 7 reduces us to the case $G = T$, where $U_g = \{1\} = V_g$. □

### 3. Examples

#### 3.1. By Lemmas 3–5, it suffices to check Conjecture 1 for one representative $G$ from each central isogeny class of semisimple algebraic group with connected Dynkin diagram. Moreover, by Lemma 6, it suffices to fix a lift $w \mapsto \dot{w}$ from $W$ into $N_G(T)$ and check Conjecture 1 for cosets of the form $gB_+ = \dot{w}B_+$. In this section, we settle the conjecture completely for $G \in \{\text{SL}_2, \text{SL}_3\}$, and check one further case in which $G = \text{Sp}_4$.

Without loss of generality, we can always take $U_+, \text{resp.} U_-$, to be the group of unipotent upper-triangular, resp. unipotent lower-triangular, matrices in $G$. To produce defining equations for $U$, we use the coefficients of the map that sends $g \in G$ to its characteristic polynomial. We write $e$ for the identity element of $W$.

#### 3.2. The Group $\text{SL}_2$. We have $U = \{g \in \text{SL}_2 \mid \text{tr}(g) = 2\}$.

The map $\Phi : U_+ U_- \to U$ is

$$\Phi \left( \begin{pmatrix} 1 & b \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ b' \end{pmatrix} \right) = \begin{pmatrix} 1 + bb' & -b'^2 \\ b' & 1 - bb' \end{pmatrix}.$$

We can write $W = \{e, w_0\}$. By Lemmas 6 and 8, it suffices to check $gB_+ = \dot{w}_0B_+$. The varieties $U_g$ and $V_g$ are

$$U_g = \left\{ \begin{pmatrix} X \\ -\frac{1}{2} \end{pmatrix} \right\} \bigg| X \neq 0,$$

$$V_g = \left\{ \begin{pmatrix} b' \\ -\frac{1}{b'} \end{pmatrix} \right\} \bigg| b' \neq 0.$$

The coordinates define isomorphisms $b' : U_g \xrightarrow{\sim} \mathbb{G}_m$ and $X : V_g \xrightarrow{\sim} \mathbb{G}_m$. The map $\Phi_g$ is an isomorphism of varieties, corresponding to setting $X = b'$. 

3.3. The Group $\text{SL}_3$. Let $\Lambda^2(g)$ denote the exterior square of a matrix $g$. From the identity $2 \text{tr}(\Lambda^2(g)) = \text{tr}(g)^2 - \text{tr}(g^2)$, we have

$$\mathcal{U} = \left\{ g \in \text{SL}_3 \mid \begin{array}{l}
\text{tr}(g) = 3, \\
\text{tr}(\Lambda^2(g)) = 3
\end{array} \right\} = \left\{ g \in \text{SL}_3 \mid \begin{array}{l}
\text{tr}(g) = 3, \\
\text{tr}(g^2) = 3
\end{array} \right\}.$$

The map $\Phi : U_+ U_- \to \mathcal{U}$ is

$$\Phi \left( \begin{pmatrix} 1 & a & b \\ 1 & c & a' \\ 1 & b' & c' \end{pmatrix} \right) = \begin{pmatrix} 1 + aa' + bb' & ba' - a^2a' - abb' & -aca'b + a^2ca' + abc' \\
abla' + cb' & 1 - aa' + cc' - acb' & -ba' + ac'a - bcb' - c^2c + ac^2b' \\
b' & c' - ab' & 1 - bb' - cc' + abc' \end{pmatrix}.$$

We can write $W = \{ e, s, t, ts, st, w_0 \}$, where $s$ and $t$ are the simple reflections. The simple reflections lift to elements of $N_G(T)$ contained in proper Levi subgroups of $G$, so by the $\text{SL}_2$ case and Lemmas 4, 6, and 7, it remains to consider $gB_+ = \bar{w}B_+$ for $w \in \{ ts, st, w_0 \}$. In what follows, we choose $s, t$ so that

$$sB_+ \subseteq \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \right\} \text{ and } tB_+ \subseteq \left\{ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \right\}.$$

3.3.1. If $w = ts$, then the varieties $\mathcal{U}_g$ and $\mathcal{V}_g$ are

$$\mathcal{U}_g = \left\{ \begin{array}{l}
Y \\
\frac{1}{Y}Z \\
\frac{1}{Z} \left( 3 + \frac{C}{Y} \right)
\end{array} \bigg| \begin{array}{l}
Y, Z \neq 0 \\
Z > \frac{1}{Y} \end{array} \right\},$$

$$\mathcal{V}_g = \left\{ \begin{array}{l}
-\frac{1}{\alpha'} b \\
\frac{1}{\alpha'} (2 - \frac{b}{c}) c
\end{array} \bigg| \begin{array}{l}
\alpha', c \neq 0 \\
\alpha' > -\frac{1}{\alpha'} \end{array} \right\}.$$

The coordinates define isomorphisms $(Y, Z, C) : \mathcal{U}_g \xrightarrow{\sim} G^2_m \times A^1$ and $(c, \alpha', b) : \mathcal{V}_g \xrightarrow{\sim} G^2_m \times A^1$. The map

$$\Phi_g(x_+ x_-) = \begin{pmatrix} -\frac{1}{\alpha'} & b + \frac{c}{\alpha'} \\
-\frac{1}{\alpha'} & c
\end{pmatrix}$$

is an isomorphism of varieties, corresponding to $(Y, Z, C) = (-\frac{1}{\alpha'}, c, b + \frac{c}{\alpha'})$.

3.3.2. If $w = st$, then the varieties are

$$\mathcal{U}_g = \left\{ \begin{array}{l}
X \\
Y \\
\frac{1}{Y} \left( 3 - 3A + A^2 \right)
\end{array} \bigg| \begin{array}{l}
X, Y \neq 0 \\
X > Y \end{array} \right\},$$

$$\mathcal{V}_g = \left\{ \begin{array}{l}
\alpha' + \frac{c}{\alpha'} b \\
\frac{1}{\alpha'} c
\end{array} \bigg| \begin{array}{l}
b, \alpha' \neq 0 \\
\alpha' > -\frac{1}{\alpha'} \end{array} \right\}.$$
The coordinates define isomorphisms \((X,Y,A): \mathcal{U}_g \xrightarrow{\sim} G_m^2 \times \mathbb{A}^1\) and \((b,a',c): \mathcal{V}_g \xrightarrow{\sim} G_m^2 \times \mathbb{A}^1\). The map

\[
\Phi_g(x+x^-) = \begin{pmatrix}
a' & 2 + \frac{c}{b} \\
-ba' + \frac{c}{b} & 1 - \frac{c}{b}
\end{pmatrix}
\]

is an isomorphism of varieties, corresponding to \((X,Y,A) = (a', \frac{1}{b}, 1 + \frac{c}{b})\).

3.3.3. If \(w = w_0\), then the varieties are

\[
\mathcal{U}_g = \left\{ \begin{pmatrix} X & Z \\ A & C \end{pmatrix} \right| XZ \neq 0, (1 + \frac{A}{XZ})^3 + \frac{AC}{XZ} = 0 \right\},
\mathcal{V}_g = \left\{ \begin{pmatrix} b \\ 1 + cc' \\ c' \\
bb' + (bb')(cc') \right| b, b' \neq 0, b, b' \neq 0 \right\}.
\]

The coordinates define isomorphisms

\[
\mathcal{U}_g \xrightarrow{\sim} \{ (X, A, C) \in G_m^2 \times \mathbb{A}^2 | (1 + \frac{1}{XZ})^3 + \frac{AC}{XZ} = 0 \}
\]

\[
\mathcal{V}_g \xrightarrow{\sim} \{ (b, b', c, c') \in G_m^2 \times \mathbb{A}^2 | 1 + bb' + (bb')(cc') = 0 \}.
\]

The map

\[
\Phi_g(x+x^-) = \begin{pmatrix}
1 + cc' & b \\
(1 + bb')c' & 2 - cc'
\end{pmatrix}
\]

corresponds to setting \((X, Y, A, C) = (b, b, (1 + bb')c', (1 + \frac{1}{bb'})c)\). Note that \(\mathcal{U}_g\) and \(\mathcal{V}_g\) are not isomorphic as varieties.

**Proposition 9.** For \(G = SL_3\) and \(gG_+ = w_0B_+\), the map \(\Phi_g\) is neither injective nor surjective on \(\mathbb{C}\)-points, but does define half of a homotopy equivalence.

**Proof.** Let

\[
\mathcal{U}_g^1 = \{ (u, A, C) \in G_m \times \mathbb{A}^2 | AC = -(1 + u)(1 + \frac{1}{u})^2 \}
\]

\[
\mathcal{V}_g^1 = \{ (u, c, c') \in G_m \times \mathbb{A}^2 | cc' = -(1 + \frac{1}{u}) \}.
\]

Let \(\Phi_g^1: \mathcal{V}_g^1 \to \mathcal{U}_g^1\) be the map \(\Phi_g^1(u, c, c') = (u, (1 + u)c', (1 + \frac{1}{u})c)\). Then \(\Phi_g\) is a pullback of \(\Phi_g^1\), so it suffices to show the claim of the proposition with \(\mathcal{U}_g^1, \mathcal{V}_g^1, \Phi_g^1\) in place of \(\mathcal{U}_g, \mathcal{V}_g, \Phi_g\).

Observe that \(\Phi_g^1\) preserves \(u \in G_m\). Over the subvariety of \(G_m\) where \(u \neq -1\), the fibers of \(\mathcal{U}_g^1\) and \(\mathcal{V}_g^1\) are copies of \(G_m\): say, via the coordinates \(A\) and \(c\). In these coordinates, \(\Phi_g^1\) amounts to rotating \(G_m\) by \(1 + u\). Over the point \(u = -1\), the fibers are copies of the transverse intersection of two lines. Altogether, \(\mathcal{U}_g^1(C)\) and \(\mathcal{V}_g^1(C)\) are both homotopic to pinched tori, and \(\Phi_g^1\) induces a self-map of the pinched torus that preserves its longitude and top homology. Thus \(\Phi_g^1\) fits into a homotopy equivalence. It is neither injective nor surjective because \(\Phi_g^1(-1, c, c') = (-1, 0, 0)\). \(\square\)
3.4. The Group $\text{Sp}_4$. We set $\text{Sp}_4 = \{g \in \text{GL}_4 \mid g^t J g = J\}$, where

$$J = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $\text{Sp}_4$, the only nontrivial coefficients of the characteristic polynomial are $\text{tr}(g) = \text{tr}(A^3(g))$ and $\text{tr}(A^2(g))$, so similarly to $\text{SL}_3$, we have:

$$\mathcal{U} = \left\{ g \in \text{Sp}_4 \mid \begin{array}{l} \text{tr}(g) = 4, \\
\text{tr}(A^2(g)) = 4 \end{array} \right\} = \left\{ g \in \text{Sp}_4 \mid \text{tr}(g^2) = 4 \right\}.$$

The map $\Phi : U_4 U_4 \to \mathcal{U}$ is

$$\Phi = \begin{pmatrix} 1 + f_1 + g_1 & f_{1,2} + g_{1,2} & f_{1,3} + g_{1,3} & h_{1,4} \\ f_{2,1} + g_{2,1} & 1 - f_1 + g_2 & h_{2,3} & f_{1,3} - g_{1,3} \\ f_{3,1} + g_{3,1} & h_{3,2} & 1 - f_1 - g_2 & f_{1,2} - g_{1,2} \\ h_{4,1} & f_{3,1} - g_{3,1} & f_{2,1} - g_{2,1} & 1 + f_1 - g_1 \end{pmatrix},$$

where we set

$$f_1 = ba'd' + ada'd',
\quad g_1 = a'a' + bb' + cc' + adb',
\quad g_2 = -aa' + bb' + 4dd' - abc' - 3adb' + a^2 dc'$$

and

$$f_{1,2} = -(c + ab + a^2d)a'd',
\quad g_{1,2} = 2bd' + cb' - a(aa' + bb' + cc' - 2dd' + adb'),
\quad f_{1,3} = -ca' - b(aa' + bb' + cc' + 4dd') - 2cdb' + ad(aa' + cc' - 4dd' + adb'),
\quad g_{1,3} = -(b^2 - 2cd - a^2d^2)a'd',
\quad f_{2,1} = 2da'd',
\quad g_{2,1} = a' + be' + 2db' - ade',
\quad f_{3,1} = b' - ac',
\quad g_{3,1} = a'd'$$

and

$$h_{1,4} = -c(2aa' + 2bb' + cc') - 2b^2 d' - 2ad(cb' + 2bd' + add'),
\quad h_{2,3} = -2a'd' + 2d(aa' - 2bb' - 4dd') - b^2 c' + ad(2be' + 4db' - ade'),
\quad h_{3,2} = 2d' - 2ab' + a^2 c',
\quad h_{4,1} = c'.$$
We can write $W = \{ e, s, t, ts, st, sts, tst, w_0 \}$, where $s$ and $t$ are the simple reflections. By the SL$_2$ case and Lemmas 4, 6, and 7, it remains to consider $gB_+ = \check{w}B_+$ for $w \in \{ ts, st, sts, tst, w_0 \}$.

Below, we will only check $w = sts$. Without loss of generality, we can assume

$$stsB_+ = \left\{ \begin{pmatrix} Y & 2YD & \frac{1}{Y} \\ -X & -XA & -X(B + AD) \end{pmatrix} \right\} \quad X, Y \neq 0 \}.$$  

The varieties $\mathcal{U}_g$ and $\mathcal{V}_g$ are

$$\mathcal{U}_g = \left\{ \begin{pmatrix} Y & 2YD & \frac{1}{Y} \\ -X & -XA & -X(B + AD) \end{pmatrix} \right\} \quad X, Y \neq 0, \quad XA(Y(B - AD) - \frac{1}{Y}(B + AD)) = \frac{1}{Y}(1 - Y)^4,$$

$$\mathcal{V}_g = \left\{ \begin{pmatrix} c \ |
2(1 + aa')d - c(a')^2 & ca' - 2ad \\
1 + aa' & -a \\
-\frac{1}{c} & -\frac{a}{c(1 + aa')} \\
\end{pmatrix} \right\} \quad c, 1 + aa' \neq 0 \}.$$

The coordinates define isomorphisms

$$\mathcal{U}_g \xrightarrow{\phi} \left\{ (X, Y, A, B, D) \in G_m^2 \times A^3 \mid XA(Y(B - AD) - \frac{1}{Y}(B + AD)) = \frac{1}{Y}(1 - Y)^4 \right\},$$

$$\mathcal{V}_g \xrightarrow{\phi} \{ (c, a, d, a') \in G_m \times A^3 \mid 1 + aa' \neq 0 \}.$$

The map

$$\Phi_g(x_+x_-) = \begin{pmatrix} c \\
1 + aa' & 2a(a' + \frac{1}{1 + aa'}) - c(a')^2 \\
\frac{1}{c} & \frac{a^2a'}{c(1 + aa')} \\
\frac{1}{1 + aa'} & 2a^2d'a' - \frac{a^2c(a')^3}{1 + aa'} \\
\end{pmatrix}$$

corresponds to setting

$$\begin{pmatrix} X \\
Y \\
A \\
B \\
D \end{pmatrix} = \begin{pmatrix} \frac{1}{c} \\
\frac{1}{1 + aa'} \\
-\frac{a^2a'}{1 + aa'} \\
a^2d'(1 + aa' + \frac{1}{1 + aa'}) - \frac{1}{2}a^2c(a')^3 \\
\end{pmatrix}.$$  

**Proposition 10.** For $G = \text{Sp}_4$ and $gB_+ = stsB_+$, the map $\Phi_g$ is neither injective nor surjective on $\mathbb{C}$-points, but does define half of a homotopy equivalence.
Proof. The map $\Phi_g$ fits into a commutative diagram

\[ \begin{array}{ccc}
\mathcal{V}_g & \xrightarrow{\Phi_g} & \mathcal{U}_g \\
\Psi_v & \downarrow & \Psi_u \\
\mathcal{V}_g^\dagger & \xrightarrow{\Phi_g^\dagger} & \mathcal{U}_g^\dagger
\end{array} \]

where the new varieties are

\[ \mathcal{U}_g^1 = \{(X,Y,A,B_-,B_+) \in G_m^2 \times A^3 \mid XA(YB_- - \frac{B_+}{Y}) = \frac{1}{Y^2}(1 - Y)^4\}, \]

\[ \mathcal{V}_g^1 = \{(c,u,a_1,a_2,d') \in G_m^2 \times A^3 \mid (u - 1)^4 = a_1a_2\}, \]

and the new maps are

\[ \Phi_g^1(c,u,a_1,a_2,d') = (\frac{1}{u}, \frac{1}{u}, -\frac{a_2}{u}, 2ud' - ca_2, \frac{2d'}{u}), \]

\[ \Psi_{uv}(X,Y,A,B) = (X, Y, A, B - AD, B + AD), \]

\[ \Psi_v(c,a,d,a') = (c, 1 + aa', a^2a', a^2(a')^3, a^2da'). \]

The map $\Phi_g^1$ is algebraically invertible, so to show that $\Phi_g$ induces a homotopy equivalence, it remains to study the topology of the maps $\Psi_{uv}$ and $\Psi_v$.

To show that $\Psi_{uv}$ induces a homotopy equivalence, we first note that it preserves $(X, A) \in A^2$. Over the subvariety of $A^2$ where $A \neq 0$, it is invertible. Over the line $A = 0$, the defining equations of $\mathcal{U}_g$ and $\mathcal{U}_g^1$ both simplify to $(1 - Y)^4 = 0$, which has the unique solution $Y = 1$ over $C$, so over this line, the fibers of $\mathcal{U}_g(C)$ and $\mathcal{U}_g^1(C)$ are contractible, being copies of $C^2$.

To show that $\Psi_v$ induces a homotopy equivalence, it suffices to show the same for the map from $\{(a,a') \in A^2 \mid aa' \neq -1\}$ into $\{(u,a_1,a_2) \in G_m \times A^2 \mid (u - 1)^4 = a_1a_2\}$. Indeed, this map restricts to an isomorphism from the subvariety where $aa' \neq 0$ onto the subvariety where $a_1a_2 \neq 0$, and collapses the subvariety where $aa' = 0$ onto the point $(u,a_1,a_2) = (1,0,0)$. The set of $C$-points in the domain where $aa' = 0$ is contractible, and the set of $C$-points in the target where $a_1a_2 = 0$ admits a deformation retract onto $\{(1,0,0)\}$. Therefore, $\Phi_g$ induces a homotopy equivalence on $C$-points. It is not injective because $\Phi_g(c,a,d,0) = (\frac{1}{c},1,0,0,0) = \Phi_g(c,0,d,a')$ for all $(c,d,a,a')$ such that $aa' \neq -1$, and it is not surjective because the points of the form $(X,1,0,B,D) \in \mathcal{V}_g(C)$ with $B \neq 0$ do not appear in the image. \hfill \Box

4. Configurations of Flags

4.1. In this section, we relate $\mathcal{U}_g$ and $\mathcal{V}_g$ to varieties that were studied in [T]. Henceforth, we fix any field $F$ of good characteristic for $G$, and replace $G$ with its split form over $F$. We also assume that $B_+$ is defined over $F$.

Let $B$ be the flag variety of $G$, i.e., the variety that parametrizes its Borel subgroups. As these subgroups are all self-normalizing and conjugate to one another, there is an isomorphism of varieties:

\[ G/B_+ \xrightarrow{\sim} B \]

\[ xB_+ \mapsto xB_+x^{-1} \]
It transports the $G$-action on $G/B_+$ by left multiplication to the $G$-action on $B$ by left conjugation.

The orbits of the diagonal $G$-action on $B \times B$ can be indexed by the Weyl group $W$. The closure order on the orbits corresponds to the Bruhat order on $W$ induced by the Coxeter presentation. For all $(B_1, B_2) \in B \times B$ and $w \in W$, we write $B_1 \xrightarrow{w} B_2$ to indicate that $(B_1, B_2)$ belongs to the $w$th orbit, in which case we say that it is in relative position $w$. In particular, note that $B_+ \xrightarrow{w_0^{-1}} B_-$ because $B_\pm = \hat{w}_0 B_+ \hat{w}_0^{-1}$.

Under the Bruhat decomposition, (4.1) restricts to an isomorphism

\begin{equation}
U_+ \hat{w} B_+/B_+ \xrightarrow{\sim} \{ B \in B \mid B_+ \xrightarrow{w_0} B \},
\end{equation}

Each side is isomorphic to an affine space of dimension $\ell(w)$, where $\ell : W \to \mathbb{Z}_{\geq 0}$ is the Bruhat length function.

4.2. Fix $g \in G(F)$. Recall that $H_g = B_+ \cap gB_+ g^{-1}$ acts on $\mathcal{V}_g = U_+ U_- \cap gB_+$ according to (1.2). Let

$$
\mathcal{V}_g = \{ B \in B : B_+ \xrightarrow{w_0} B \xrightarrow{w_0} gB_+ g^{-1} \},
$$

and let $H_g$ act on $\mathcal{V}_g$ by left conjugation. We will prove:

**Proposition 11.** There is a $H_g$-equivariant isomorphism of varieties $\mathcal{V}_g \to \mathcal{V}_g$.

We give the proof in two steps. For convenience, we set $\mathcal{Y}_g = gU_+ \hat{w}_0 B_+/B_+ \subseteq G/B_+$. Let $H_g$ act on $\mathcal{Y}_1 \cap \mathcal{Y}_g$ by left multiplication.

**Lemma 12.** The isomorphism (4.2) for $w = w_0$ restricts to an $H_g$-equivariant isomorphism $\mathcal{Y}_1 \cap \mathcal{Y}_g \xrightarrow{\sim} \mathcal{V}_g$.

**Proof.** Recall that (4.2) is $G$-equivariant. Under the action of an element $x$, the $w = w_0$ case is transported to an isomorphism

$$
\mathcal{Y}_g \xrightarrow{\sim} \{ B \in B \mid xB_+ x^{-1} \xrightarrow{w_0} B \}.
$$

On the right-hand side, the direction of the arrow $\xrightarrow{w_0}$ can be reversed because $w_0^{-1} = w_0$. Now take the fiber product of the isomorphisms for $x = 1$ and $x = g$ over the isomorphism (4.1).

In what follows, recall that via the decomposition $B_+ \simeq U_+ \rtimes T$, any element of $B_+$ can be written as $ut$ for some uniquely determined $(u, t) \in U_+ \times T$. As a consequence, we also get a decomposition $\hat{w}_0 B_+ = U_- \hat{w}_0 T = U_- T \hat{w}_0$.

**Lemma 13.** The map

$$
\mathcal{V}_g = U_+ U_- \cap gB_+ \quad \to \quad \mathcal{Y}_1 \cap \mathcal{Y}_g \quad \xrightarrow{x_+ x_- = \text{gut}} \quad x_+ x_- t^{-1} \hat{w}_0 B_+ = g u \hat{w}_0 B_+
$$

is an $H_g$-equivariant isomorphism of varieties.

**Proof.** Let $\mathcal{V}_g' = U_+ \hat{w}_0 B_+ \cap gU_+ \hat{w}_0$. Since the map

$$
\mathcal{V}_g \quad \to \quad \mathcal{V}_g'
$$

$$
x_+ x_- = \text{gut} \quad \to \quad x_+ x_- t^{-1} \hat{w}_0 = g u \hat{w}_0$$
is an isomorphism, it remains to show that the map \( f : \mathcal{V}_g' \to \mathcal{Y}_1 \cap \mathcal{Y}_g \) given by

\[
\mathcal{V}_g' \to \mathcal{V}_g'B_+ \to (\mathcal{V}_g'B_+)/B_+ = \mathcal{Y}_1 \cap \mathcal{Y}_g
\]
is bijective on \( R \)-points for every \( \mathbf{F} \)-algebra \( R \). For convenience, we suppress \( R \) in the notation below.

Let \( yB_+ \in (\mathcal{V}_g'B_+)/B_+ \). Then we can write \( y = uw_0b = gu'u_0b' \) for some \( u, u' \in U_+ \) and \( b, b' \in B_+ \). Therefore, \( yB_+ = f(y(b')^{-1}) \), where \( y(b')^{-1} = gu'u_0 \in \mathcal{V}_g' \). This proves \( f^{-1}(yB_+) \) is nonempty. We claim that \( f^{-1}(yB_+) \) contains only one point. Recall that the map \( U_+ \to U_+\bar{w}_0B_+/B_+ \) that sends \( v \mapsto v\bar{w}_0B_+ \) is an isomorphism. Thus, \( v \neq u \) implies \( v\bar{w}_0B_+ \neq u\bar{w}_0B_+ \). We deduce that

\[
f^{-1}(yB_+) = f^{-1}(uw_0B_+)
\]
\[
\subseteq \bar{u}u_0B_+ \cap gU_+\bar{w}_0
\]
\[
\simeq (\bar{w}_0^{-1}g^{-1}uw_0)B_+ \cap U_+.
\]
But the intersection of \( U_+ \) with any coset of \( B_+ \) contains only one point. \( \square \)

4.3. Let \( \mathcal{O}_w, \mathcal{U}_w, \mathcal{X}_w \) be the varieties defined by

\[
\mathcal{O}_w = \{(B', B'') \in B \times B \mid B' \xrightarrow{w} B''\},
\]
\[
\mathcal{U}_w = \{(u, B') \in \mathcal{U} \times B \mid B' \xrightarrow{w} uB'u^{-1}\},
\]
\[
\mathcal{X}_w = \{(B, B', B'') \in B \times B \times B \mid B' \xrightarrow{w} B \xrightarrow{w_0} B'' \xrightarrow{w} B'\}.
\]
Let \( G \) act on these varieties by \( \text{diagonal} \) left conjugation. We regard \( \mathcal{U}_w \) and \( \mathcal{X}_w \) as varieties over \( \mathcal{O}_w \) via the \( G \)-equivariant maps \( (u, B') \mapsto (B', uB'u^{-1}) \) and \( (B, B', B'') \mapsto (B', B'') \), respectively.

Let \( H_g \) act on \( G \) by right multiplication. For any variety \( X \) with an \( H_g \)-action, let \( H_g \) act diagonally on \( X \times G \), and let \( G \) act on \( (X \times G)/H_g \) by left multiplication on the second factor. Finally, fix a prime \( \ell > 0 \) invertible in \( \mathbf{F} \), so that we can form the equivariant \( \ell \)-adic compactly-supported cohomology groups

\[
H^*_c,\mathcal{H}_g(X) \simeq H^*_c,\mathcal{G}((X \times G)/H_g).
\]

With these conventions, we have:

**Proposition 14.** If \( B_+ \xrightarrow{w} gB_+g^{-1} \), then there are \( G \)-equivariant isomorphisms

\[
(\mathcal{O}_g \times G)/H_g \xrightarrow{\sim} \mathcal{U}_w,
\]
\[
(\mathcal{O}_g \times G)/H_g \xrightarrow{\sim} \mathcal{X}_w.
\]

In particular, they induce isomorphisms on compactly-supported cohomology:

\[
H^*_c,\mathcal{H}_g(\mathcal{O}_g, \mathcal{Q}_\ell) \xrightarrow{\sim} H^*_c,\mathcal{G}(\mathcal{U}_w, \mathcal{Q}_\ell),
\]
\[
H^*_c,\mathcal{H}_g(\mathcal{O}_g, \mathcal{Q}_\ell) \xrightarrow{\sim} H^*_c,\mathcal{G}(\mathcal{X}_w, \mathcal{Q}_\ell).
\]

**Proof.** The maps \( (\mathcal{O}_g \times G)/H_g \to \mathcal{U}_w \) and \( (\mathcal{O}_g \times G)/H_g \to \mathcal{X}_w \) are

\[
[u, x] \mapsto (xux^{-1}, xB_+x^{-1}),
\]
\[
[B, x] \mapsto (xBx^{-1}, xB_+x^{-1}, xgB_+g^{-1}x^{-1}),
\]
respectively. To show that they are isomorphisms: Observe that \( G \) acts transitively on \( G_w \), and the stabilizer of \((B_{+},gB_{+}g^{-1})\) is precisely \( H_g \). The preimage of this point in \( \mathcal{U}_w \), resp. \( \mathcal{F}_w \), is \( \mathcal{U}_g \), resp. \( \mathcal{F}_g \). Therefore, the maps above are the respective pullbacks to \( \mathcal{U}_w \) and \( \mathcal{F}_w \) of the isomorphism \( G/H_g \xrightarrow{\sim} G_w \) that sends \( xH_g \mapsto (xB_{+}x^{-1},gB_{+}x^{-1}) \).

Note that when \( F = C \), the maps on cohomology in Proposition 14 preserve weight filtrations because the maps that induce them are algebraic.

5. Khovanov–Rozansky Homology

5.1. In this section, we prove Theorem 2 by way of more general constructions motivated by knot theory.

Let \( Br^+_W \) be the positive braid monoid of \( W \). It is the monoid freely generated by a set of symbols \( \{\sigma_w\}_{w \in W} \), modulo the relations \( \sigma_{ww'} = \sigma_w \sigma_{w'} \) for all \( w, w' \in W \) such that \( \ell(ww') = \ell(w) + \ell(w') \). The full twist is the element \( \pi = \sigma_{w_0}^2 \in Br^+_W \).

For all \( \beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br^+_W \), we set

\[
\mathcal{U}(\beta) = \{(u, B_1, \ldots, B_k) \in \mathcal{U} \times \mathcal{B}^k \mid u^{-1}B_k u \xrightarrow{w_1} B_1 \xrightarrow{w_2} \cdots \xrightarrow{w_k} B_k\},
\]

\[
\mathcal{F}(\beta) = \{(B_1, \ldots, B_k) \in \mathcal{B}^k \mid B_k \xrightarrow{w_1} B_1 \xrightarrow{w_2} \cdots \xrightarrow{w_k} B_k\}.
\]

Let \( G \) act on these varieties by left conjugation. We regard \( \mathcal{U}(\beta) \) and \( \mathcal{F}(\beta) \) as \( G \)-varieties over \( G_w \), where \( w = w_1 \cdots w_k \in W \), via the equivariant maps \((u, (B_1)_i) \mapsto (B_k, B_1)\) and \((B_1)_i \mapsto (B_k, B_1)\), respectively. Deligne showed that up to isomorphism over \( \mathcal{B} \times \mathcal{B} \), these varieties only depend on \( \beta \), not on the sequence of elements \( w_i \). His full result describes the extent to which the isomorphism can be pinned down uniquely; see [D] for details.

In particular, we have equivariant identifications

\[
\mathcal{U}_w \xrightarrow{\sim} \mathcal{U}(\sigma_w),
\]

\[
\mathcal{F}_w \xrightarrow{\sim} \mathcal{F}(\sigma_w \pi)
\]

via \((u, B_1) = (u, B')\) and \((B_1, B_2, B_3) = (B', B'', B)\).

5.2. If \( W \) is the symmetric group on \( n \) letters, denoted \( S_n \), then the group completion of \( Br_W \) is the group of topological braids on \( n \) strands, denoted \( Br_n \). Any braid can be closed up end-to-end to form a link: that is, an embedding of a disjoint union of circles into 3-dimensional space. Thus there is a close relationship between isotopy invariants of links and functions on the groups \( Br_n \).

In [KR], Khovanov and Rozansky introduced a link invariant valued in triply-graded vector spaces. Its graded dimension can be written as a formal series in variables \( a, q^{\frac{1}{2}}, t \). In [Kh], Khovanov showed how to construct it in terms of class functions on the groups \( Br_n \), and more precisely, in terms of functors on monoidal additive categories attached to the groups \( S_n \). When we set \( t = -1 \), the Khovanov–Rozansky invariant of a link specializes to its so-called HOMFLYPT series, and Khovanov’s functors specialize to class functions originally introduced by Jones and Ocneanu.
The positive braid monoid $Br_W^+$ and its group completion $Br_W$ can actually be defined for any Coxeter group $W$, not just Weyl groups. In [G], Y. Gomi extended the construction of Jones–Ocneanu to finite Coxeter groups. There is a similar extension of Khovanov’s construction, up to a choice of a (faithful) representation on which $W$ acts as a reflection group.

Fix such a representation $V$. For any braid $\beta \in Br_W$, we write $\operatorname{HHH}_V(\beta)$ to denote the Khovanov–Rozansky (KR) homology of $\beta$ with respect to $V$. We will use the grading conventions in [T], so that

$$P_V(\beta) = (at)^{|\beta|}a^{-\dim(V)} \sum_{i,j,k} (a^2q^{\frac{i}{2}}t)^{\dim(V)-i} q^{\frac{i}{2}t-k} \dim \operatorname{HHH}_V^{i+j,k}(\beta)$$

is an isotopy invariant of the link closure of $\beta$. In the case where $W = S_n$, taking $V$ to be the $(n-1)$-dimensional reflection representation yields what is usually called reduced KR homology and denoted $\operatorname{HHH}$, while taking $V$ to be the $n$-dimensional permutation representation yields what is usually called unreduced KR homology and denoted $\operatorname{HHH}$. They are related by

$$\left(\frac{a^{-1} + at}{q^{-2} - q^2}\right) P(\beta) = P(\beta),$$

where $P$ and $P$ denote the series $P_V$ for these respective choices of $V$.

Henceforth, let $r = \dim(V)$ and $N = \dim(\mathcal{R})$. The results below are [T, Cor. 4] and [GHMN, Thm. 1.9].

**Theorem 15.** Suppose that $W$ is the Weyl group of a split reductive group $G$ over $F$ with root lattice $\Phi$, and that $V = \mathbb{Z}\Phi \otimes \mathbb{Z} Q$. Then for any $\beta \in Br_W^+$, we have isomorphisms

$$\begin{align*}
\gr_{\beta}^W H_{c,G}^{j+k+2r}(\mathcal{U}(\beta), \mathcal{Q}_c) &\simeq \operatorname{HHH}_V^{0,j,k}(\beta), \\
\gr_{\beta}^W H_{c,G}^{j+k+2(r+N)}(\mathcal{F}(\beta), \mathcal{Q}_c) &\simeq \operatorname{HHH}_V^{r+j,k}(\beta)
\end{align*}$$

for all $j, k$.

**Theorem 16** (Gorsky–Hogancamp–Mellit–Nakagane). For any integer $n \geq 1$ and $\beta \in Br_n$, we have

$$\operatorname{HHH}^{0,j,k}(\beta) \simeq \operatorname{HHH}^{r+j,k}(\beta \pi)$$

for all $j, k$.

**Proof of Theorem 2.** We must have $B_+ \xrightarrow{w} gB_+g^{-1}$ for some $w \in W$. Combining Proposition 11, Proposition 14, and Theorem 15, we get isomorphisms

$$\begin{align*}
\gr_{\beta}^W H_{c,H}^{j+k+2n}(\mathcal{U}_g, \mathcal{Q}_c) &\simeq \operatorname{HHH}_V^{0,j,k}(\sigma_w), \\
\gr_{\beta}^W H_{c,H}^{j+k+2(n+N)}(\mathcal{F}_g, \mathcal{Q}_c) &\simeq \operatorname{HHH}_V^{r+j,k}(\sigma_w \pi),
\end{align*}$$

where $V = \mathbb{Z}\Phi \otimes \mathbb{Z} Q$ and $\Phi$ is the root lattice of $G$.

If $G = \text{GL}_n$, then $V$ is the permutation representation of $S_n$. So in this case, $\operatorname{HHH}_V = \operatorname{HHH}$, and we are done by Theorem 16. Finally, we bootstrap from $\text{GL}_n$ to any other split reductive group of type $A$ using Lemmas 4 and 5. □
5.3. Theorems 15–16 suggest the following generalization of Conjecture 1.

**Conjecture 17.** For any $\beta \in Br_{W}^{+}$, there is a homotopy equivalence between $\mathcal{U}(\beta)(C)$ and $\mathcal{X}(\beta\pi)(C)$ that matches the weight filtrations on their compactly-supported cohomology.

**Remark 18.** It would be desirable to generalize the map of stacks $[Y_{g}/H_{g}] \rightarrow [\mathcal{U}_{g}/H_{g}]$ that arises from $\Phi_{g}$ to an explicit map $[\mathcal{X}(\beta\pi)/G] \rightarrow [\mathcal{U}(\beta)/G]$ for any positive braid $\beta$. Due to the inexplicit nature of Lemma 13, we have not yet found such a generalization.

6. Point Counts over Finite Fields

6.1. For any braid $\beta \in Br_{n}$, we write $\hat{\beta}$ to denote its link closure. The reduced HOMFLYPT series $P(\hat{\beta})$ is related to the KR homology of $\beta$ by

$$P(\hat{\beta}) = P(\beta)|_{t \rightarrow -1}.$$ 

This is an element of $\mathbb{Z}[q^{\pm 1}]|_{t \rightarrow -1}$. We write $[a^{i}]P(\hat{\beta})$ to denote the coefficient of $a^{i}$ in $P(\hat{\beta})$, viewed as an element of $\mathbb{Z}[q^{\pm 1}]|_{t \rightarrow -1}$.

If $\beta = \sigma_{s_{1}} \cdots \sigma_{s_{t}}$, where the elements $s_{1}, \ldots, s_{t} \in W$ are all simple reflections, then we set $|\beta| = t$. This number only depends on $\beta$. Theorem 16 then specializes to the following result from [K].

**Theorem 19** (Kálmán). For any integer $n \geq 1$ and $\beta \in Br_{n}$, we have

$$[a^{[\beta]+1}]P(\hat{\beta}) = [a^{[\beta]-1}]P(\hat{\beta}\pi).$$

In [T, §8], we generalized Kálmán’s result from $Br_{n}$ to $Br_{W}$. In this section, we review the statement, then explain its relation to the point-counting identity (1.1).

6.2. Let $H_{W}$ be the Iwahori–Hecke algebra of $W$. For our purposes, $H_{W}$ is the quotient of the group algebra $\mathbb{Z}[q^{\pm 1/2}]|Br_{W}|$ by the two-sided ideal

$$\langle (\sigma_{s} - q^{\frac{1}{2}})(\sigma_{s} + q^{-\frac{1}{2}}) \mid \text{simple reflections } s \rangle.$$ 

For any element $\beta \in Br_{W}$, we abuse notation by again writing $\beta$ to denote its image in $H_{W}$.

The sets $\{\sigma_{w}\}_{w \in W}$ and $\{\sigma_{w}^{-1}\}_{w \in W}$ are bases for $H_{W}$ as a free $\mathbb{Z}[q^{\pm 1/2}]$-module. Let $\tau^{\pm} : H_{W} \rightarrow \mathbb{Z}[q^{\pm 1/2}]$ be the $\mathbb{Z}[q^{\pm 1/2}]$-linear functions defined by:

$$\tau^{\pm}(\sigma_{w}) = \begin{cases} 1 & w = e \\ 0 & w \neq e \end{cases}$$

For $W = S_{n}$, comparing $\tau^{\pm}$ with the Jones–Ocneanu trace on $H_{W}$ shows that

$$[a^{[\beta]+1}]P(\hat{\beta}) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{-(n-1)}(-1)^{|[\beta]|}\tau^{\pm}(\beta)$$

for all $\beta \in Br_{n}$. Therefore the following result from [T, §8] generalizes Kálmán’s theorem to arbitrary $W$.

**Theorem 20.** For any finite Coxeter group $W$ and braid $\beta \in Br_{W}$, we have

$$\tau^{-}(\beta) = \tau^{+}(\beta\pi).$$
6.3. We return to the setting of Section 5, so that \( W \) is the Weyl group of \( G \). Under the hypotheses of Theorem 15, the following identities from loc. cit. relate Theorem 20 to point counting:

\[
\frac{|\mathcal{U}(\beta)(F)|}{|G(F)|} = (q - 1)^{-r} (q^{\frac{1}{2}})^{\tau^-(\beta)},
\]
\[
\frac{|\mathcal{X}(\beta)(F)|}{|G(F)|} = (q - 1)^{-r} (q^{\frac{1}{2}})^{\tau^+(\beta)}.
\]

Together they imply:

**Corollary 21.** Keep the hypotheses of Theorem 15. Then for any \( \beta \in Br^+_W \), we have

\[
|U(\beta)(F)| = |X(\beta\pi)(F)|.
\]

We claim that when \( B_+ \) is defined over \( F \), Corollary 21 implies (1.1) from the introduction. The key is that the proof of Proposition 14 also works at the level of \( F \)-points. Thus there are \( G \)-equivariant bijections

\[
(U_g(F) \times G(F))/H_g(F) \to U_w(F),
\]
\[
(X_g(F) \times G(F))/H_g(F) \to X_w(F)
\]

for any \( g \in G(F) \) such that \( B_+ \to gB_+g^{-1} \). Since the quotients are free, we deduce that

\[
|U_g(F)||G(F)| = |U_w(F)||H_g(F)| = |X_w(F)||H_g(F)| = |X_g(F)||G(F)|.
\]

Applying Proposition 11, we arrive at

\[
|U_g(F)| = |X_g(F)| = |Y_g(F)|, \text{ which is (1.1).}
\]

**Remark 22.** The original identity proved by Kawanaka was

\[
|(\mathcal{U} \cap U_+ \mathcal{w} B_+)(F)| = |(U_- \cap U_+ \mathcal{w} B_+)(F)|
\]

for all \( w \in W \) [Ka, Cor. 4.2]. This is equivalent to (1.1) as long as \( B_+ \) is defined over \( F \). For when the latter holds, an argument similar to the proof of Lemma 6 shows that the \( F \)-point counts of \( \mathcal{U}_g \) and \( \mathcal{X}_g \) remain constant as \( g \) runs over elements of \( U_+ \mathcal{w} B_+(F) \).

**References**


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