UNIPOTENT ELEMENTS AND TWISTING IN LINK HOMOLOGY

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Abstract. Let $\mathcal{U}$ be the unipotent variety of a complex reductive group G. Fix opposed Borel subgroups $B_\pm \subseteq G$ with unipotent radicals $U_\pm$. The map that sends $x_+x_- \mapsto x_+x_-x_+^{-1}$ for all $x_\pm \in U_\pm$ restricts to a map from $U_+U_- \cap gB_+$ into $\mathcal{U} \cap gB_+$, for any g. We conjecture that the restricted map forms half of a homotopy equivalence between these varieties, and thus, induces a weight-preserving isomorphism between their compactly-supported cohomologies. Noting that the map is equivariant with respect to certain actions of $B_+ \cap gB_+g^{-1}$, we prove for type A that an equivariant analogue of this isomorphism exists. Curiously, this follows from a certain duality in Khovanov–Rozansky homology, a tool from knot theory.

1. Introduction

1.1. Let G be a complex, connected, reductive algebraic group. Let B_+ and B_- be opposed Borel subgroups of G, and let $U_\pm$ be the unipotent radical of $B_\pm$. For instance, if G is the group of invertible $n \times n$ matrices GL_n, then we can choose U_+, *resp.* U_-, to be the subgroup of upper-triangular, *resp.* lower-triangular, matrices with 1's along the diagonal. Every element of U_+U_- can be written uniquely in the form x_+x_-, where $x_\pm \in U_\pm$.

Let $\mathcal{U} \subseteq G$ be the closed subvariety of unipotent elements. With the notation above, there is a map $\Phi : U_+U_- \to \mathcal{U}$ defined by

$$\Phi(x_+x_-) = x_+x_-x_+^{-1}.$$

Note that U_+U_- is an affine space, whereas $\mathcal{U}$ is usually singular. Nonetheless, in the analytic topology, the sets $U_+U_-(\mathbf{C})$ and $\mathcal{U}(\mathbf{C})$ are homotopy equivalent, as they are both contractible.

For any $g \in G(\mathbf{C})$, the map Φ restricts to a map $\Phi_g : \mathcal{V}_g \to \mathcal{U}_g$, where

$$\mathcal{U}_g = \mathcal{U} \cap gB_+,$$
$$\mathcal{V}_g = U_+U_- \cap gB_+.$$

Note that $\mathcal{U}_g, \mathcal{V}_g, \Phi_g$ only depend on the coset gB_+. We endow $\mathcal{U}_g(\mathbf{C})$ and $\mathcal{V}_g(\mathbf{C})$ with the analytic topology. The new idea proposed in this note is:

Conjecture 1. *The map Φ_g defines half of a homotopy equivalence between $\mathcal{U}_g(\mathbf{C})$ and $\mathcal{V}_g(\mathbf{C})$.*

The simplest case is $g \in B_+(\mathbf{C})$. Here, $\mathcal{U}_g(\mathbf{C}) = U_+(\mathbf{C}) = \mathcal{V}_g(\mathbf{C})$ and Φ_g is a retract from the affine space $U_+(\mathbf{C})$ onto the point corresponding to the identity of $G(\mathbf{C})$. In general, neither $\mathcal{U}_g(\mathbf{C})$ nor $\mathcal{V}_g(\mathbf{C})$ will be contractible, as we will show in Section 3 through examples.

We emphasize that for an arbitrary map of varieties $\Phi : Y \to X$ that induces a homotopy equivalence on $\mathbf{C}$-points, there may not exist a nonconstant map of varieties $X \to S$ such that Φ induces a homotopy equivalence of fibers $Y_s(\mathbf{C}) \to X_s(\mathbf{C})$ for every $s \in S(\mathbf{C})$. For instance, take Φ to be the projection from a quadric cone onto its axis of symmetry. Both total spaces are contractible, so Φ is automatically a homotopy equivalence. For the map of fibers over s to be a homotopy equivalence as well, $X_s(\mathbf{C})$ must contract onto the origin of $X(\mathbf{C}) = \mathbf{C}$. Since $X \to S$ is nonconstant, some s must violate this condition.

1.2. Some motivation for Conjecture 1 comes from classical results about finite groups of Lie type. To state them, let us implicitly replace G with its split form over a finite field $\mathbf{F}$ of good characteristic.

In [S, Thm. 15.1], Steinberg showed the identity $|\mathcal{U}(\mathbf{F})| = |U_+U_-(\mathbf{F})|$. In [Ka, §4], Kawanaka showed an identity equivalent to

$$|\mathcal{U}_g(\mathbf{F})| = |\mathcal{V}_g(\mathbf{F})|; \tag{1.1}$$

see Remark 22. More recently, Lusztig has given a new proof of (1.1) in [L].

Conjecture 1 essentially implies (1.1). Indeed, once one checks that $\mathcal{U}_g$ and $\mathcal{V}_g$ have the same dimension, Conjecture 1 implies that Φ_g induces an isomorphism between the rational, compactly-supported cohomologies of $\mathcal{U}_g$ and $\mathcal{V}_g$. Since Φ_g is algebraic, this isomorphism must match their weight filtrations in the sense of mixed Hodge theory. One can further check that both sides of (1.1) are polynomial functions of $|\mathbf{F}|$, so by the results explained in [Kat], the virtual weight polynomials of $\mathcal{U}_g$ and $\mathcal{V}_g$ specialize to their $\mathbf{F}$-point counts.

Our main result is evidence for an equivariant analogue of Conjecture 1. Let $T = B_+ \cap B_-$, so that $B_+ = TU_+ \simeq T \ltimes U_+$. The map Φ transports the B_+-action on U_+U_- defined by

$$tu \cdot x_+x_- = (tux_+t^{-1})(tx_-t^{-1}) \tag{1.2}$$

for all $(t, u) \in T \times U_+$ onto the B_+-action on $\mathcal{U}$ by left conjugation. Setting

$$H_g = B_+ \cap gB_+g^{-1},$$

we find that these B_+-actions restrict to H_g-actions on $\mathcal{V}_g$ and $\mathcal{U}_g$, respectively. The B_+-equivariance of Φ thus restricts to H_g-equivariance of Φ_g. Though we do not have a general theorem about Φ_g itself, we prove:

Theorem 2. *If G is a split reductive group of type A over $\mathbf{F}$, then for all $g \in G(\mathbf{F})$, there is an isomorphism of bigraded vector spaces*

$$\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,H_g}(\mathcal{U}_g, \bar{\mathbf{Q}}_\ell) \simeq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,H_g}(\mathcal{V}_g, \bar{\mathbf{Q}}_\ell),$$

where $\mathrm{H}^*_{c,H_g}(-, \bar{\mathbf{Q}}_\ell)$ denotes H_g-equivariant, compactly-supported ℓ-adic cohomology and W denotes its weight filtration.

The proof in Sections 4–5 uses ideas from the rather different world of low-dimensional topology, as we now explain.

Let $\mathcal{X}_g$ be the variety of Borel subgroups of G in generic position with respect to both B_+ and gB_+g^{-1}. In Section 4, we will construct an isomorphism $\mathcal{X}_g \to \mathcal{V}_g$ that transports the H_g-action on $\mathcal{X}_g$ by left conjugation to the H_g-action on $\mathcal{V}_g$ described above.

The variety $\mathcal{X}_g$ is closely related to the so-called braid varieties that have been studied recently by several authors, including [M, SW, CGGS, GL]. To be more precise: Let W be the Weyl group of G, and let Br_W^+ be the positive braid monoid of W. By definition, Br_W^+ is generated by elements σ_w for each $w \in W$ modulo

$$\sigma_{ww'} = \sigma_w\sigma_{w'} \qquad\qquad \text{whenever } \ell(ww') = \ell(w) + \ell(w'),$$

where ℓ is the Bruhat length function on W. A braid variety is a configuration space of tuples of Borels, where the relative positions of cyclically consecutive Borels have constraints determined by a fixed element $\beta \in Br_W^+$. Note that there is a central element $\pi \in Br_W^+$ known as the full twist and given by $\pi = \sigma_{w_0}^2$, where w_0 is the longest element of W. In the notation of [T, Appendix B], the variety $\mathcal{X}_g$ is isomorphic to the braid variety attached to $\sigma_w\pi$, where w is the relative position of the pair (B_+, gB_+g^{-1}).

In [T], it was shown that the weight filtration on the equivariant, compactly-supported cohomology of the braid variety of β encodes a certain summand of a certain triply-graded vector space attached to β, known as its *HOMFLYPT* or *Khovanov–Rozansky (KR) homology*. When W is the symmetric group S_n, the braid β represents the isotopy class of a topological braid on n strands, and the KR homology of β is an isotopy invariant of the link closure of β, up to grading shifts [KR, Kh].

At the same time, when w is the relative position of (B_+, gB_+g^{-1}), it turns out that $\mathcal{U}_g$ is closely related to another variety attached to σ_w in [T]. Just as $\mathcal{X}_g$ encodes the "highest a-degree" of the KR homology of $\sigma_w\pi$, so $\mathcal{U}_g$ encodes the "lowest a-degree" of the KR homology of σ_w. For $W = S_n$, Gorsky–Hogancamp–Mellit–Nagakane have established an isomorphism between these bigraded vector spaces for general braids β, not just σ_w, which they deduce from an analogue of Serre duality for the homotopy category of Soergel bimodules for S_n [GHMN]. Our Theorem 2 follows from this isomorphism.

This proof suggests that we regard Conjecture 1 as a *geometric* realization of the (purely algebraic) Serre duality of [GHMN]. Moreover, it suggests an extension of Conjecture 1 with positive braids in place of elements of W. We state the extended conjecture in Section 5.

The isomorphism of *ibid.* categorifies an earlier identity of Kálmán, relating the bivariate HOMFLY series of the link closures of β and $\beta\pi$. In [T], we generalized Kálmán's theorem to the braids associated with arbitrary finite Coxeter groups. (We expect, but do not show, that [GHMN] admits a similar generalization.) For Weyl groups, we also interpreted our result as an identity of $\mathbf{F}$-point counts. In Section 6, we review these results, and explain how they recover (1.1).

1.3. **Acknowledgments.** I thank George Lusztig and Zhiwei Yun for helpful discussions. During this work, I was supported by a Mathematical Sciences Postdoctoral Fellowship (grant #2002238) from the National Science Foundation.

2. Reductions

In this section, we collect lemmas about easy reductions and special cases of the conjecture.

Lemma 3. *If $G = G_1 \times G_2$, where G_1, G_2 are again reductive, then Conjecture 1 holds if and only if it holds with G_1 in place of G and with G_2 in place of G.*

Proof. We can factor $U_\pm = U_{\pm,1} \times U_{\pm,2}$ and $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ and $\Phi = \Phi^{(1)} \times \Phi^{(2)}$, where $U_{\pm,i}, \mathcal{U}_i, \Phi^{(i)}$ are the analogues of $U_\pm, \mathcal{U}, \Phi$ with G_i in place of G. $\square$

Lemma 4. *Both $\mathcal{U}_g$ and $\mathcal{V}_g$ are contained within the derived subgroup $G^{\mathrm{der}} \subseteq G$. In particular, we can assume G is semisimple in Conjecture 1.*

Proof. It is enough to show $\mathcal{U} \subseteq G^{\mathrm{der}}$, because in that case, we also have $U_+U_- \subseteq G^{\mathrm{der}}G^{\mathrm{der}} = G^{\mathrm{der}}$. Suppose instead that $\mathcal{U} \not\subseteq G^{\mathrm{der}}$. Then G/G^{der} contains nontrivial unipotent elements, because homomorphisms of algebraic groups preserve Jordan decompositions [Mi, §9.21]. But G/G^{der} is isogeneous to the center of G by [Mi, Ex. 19.25], so it is a torus, in which the only unipotent element is the identity. $\square$

Lemma 5. *The map Φ_g is unchanged, up to composition with maps that induce homeomorphisms on $\mathbf{C}$-points, when we replace G with the adjoint group $G^{\mathrm{ad}} = G/Z(G)$ and g with its image in G^{ad}. In particular, Conjecture 1 is equivalent to its analogue where we replace G with any quotient by a central isogeny, and g with its image in that quotient.*

Proof. Let $\bar{U}_\pm$ and $\bar{\mathcal{U}}$ be the respective analogues of $U_\pm$ and $\mathcal{U}$ with G^{ad} in place of G. Again using the preservation of Jordan decomposition, and the fact that central elements of G are semisimple, we can check that the quotient map $G \to G^{\mathrm{ad}}$ restricts to maps $U^\pm \xrightarrow{\sim} \bar{U}^\pm$ and $\mathcal{U} \to \bar{\mathcal{U}}$ that give rise to bijections on field-valued points. Writing $\bar{\Phi} : \bar{U}_+\bar{U}_- \to \bar{\mathcal{U}}$ for the analogue of Φ, we see that the diagram

$$\begin{tikzcd} U_+U_- \arrow[r, "\Phi"] \arrow[d, "\wr"'] & \mathcal{U} \arrow[d, "\wr"] \\ \bar{U}_+\bar{U}_- \arrow[r, "\bar{\Phi}"] & \bar{\mathcal{U}} \end{tikzcd}$$

commutes. Now the result follows. $\square$

The following lemma is motivated by the Bruhat decomposition

$$G = \coprod_{w \in W} U_+ \dot{w} B_+,$$

where $w \mapsto \dot{w}$ is any choice of set-theoretic lift from $W \simeq N_G(T)/T$ into $N_G(T)$. Note that $\dot{w}B_+$ only depends on w because $T \subseteq B_+$.

Lemma 6. *If Conjecture 1 holds for some $g \in G(\mathbf{C})$, then it holds with ug in place of g, for all $u \in U_+(\mathbf{C})$. In particular, if Conjecture 1 holds for all $g \in N_G(T)(\mathbf{C})$, then it holds in general.*

Proof. We can factor Φ_{ug} as a composition

$$\mathcal{V}_{ug} \xrightarrow{\sim} \mathcal{V}_g \xrightarrow{\Phi_g} \mathcal{U}_g \xrightarrow{\sim} \mathcal{U}_{ug},$$

where the first arrow is left multiplication by u^{-1}, and the last arrow is left conjugation by u. Since these are both isomorphisms of varieties, we get the first assertion of the lemma. The second follows from the first via Bruhat. $\square$

If P_+ and P_- are opposed parabolic subgroups of G containing B_+ and B_-, respectively, and $L = P_+ \cap P_-$ is their common Levi subgroup, then we write $U_{\pm,P}$ for the unipotent radical of $P_\pm$, and write $B_{\pm,L}, U_{\pm,L}, \mathcal{U}_L$ for the analogues of $B_\pm, U_\pm, \mathcal{U}$ with L in place of G. Thus we have

$$\begin{aligned}
P_\pm &= LU_{\pm,P} \simeq L \ltimes U_{\pm,P},\\
B_\pm &= B_{\pm,L}U_{\pm,P} \simeq B_{\pm,L} \ltimes U_{\pm,P},\\
U_\pm &= U_{\pm,L}U_{\pm,P} \simeq U_{\pm,L} \ltimes U_{\pm,P}.
\end{aligned}$$

If $g \in L$, then we write $\mathcal{U}_{L,g}, \mathcal{V}_{L,g}, \Phi_{L,g}$ for the analogues of $\mathcal{U}_g, \mathcal{V}_g, \Phi_g$ with L in place of G.

Lemma 7. *Let $P_\pm \supseteq B_\pm$ be opposed parabolic subgroups of G, and let $L = P_+ \cap P_-$. Then for all $g \in L(\mathbf{C})$, we have isomorphisms of algebraic varieties*

$$\begin{aligned}
\mathcal{U}_g &\simeq \mathcal{U}_{L,g}U_{+,P} \simeq \mathcal{U}_{L,g} \times U_{+,P},\\
\mathcal{V}_g &\simeq \mathcal{V}_{L,g}U_{+,P} \simeq \mathcal{V}_{L,g} \times U_{+,P}.
\end{aligned}$$

In particular, if $g \in G(\mathbf{C})$ belongs to a Levi subgroup of G, then in Conjecture 1, we can replace G with that Levi subgroup.

Proof. Since the decomposition $P_+ \simeq L \ltimes U_{+,P}$ preserves Jordan decompositions, we have $\mathcal{U} \cap P_+ = \mathcal{U}_L U_{+,P} \simeq \mathcal{U}_L \times U_{+,P}$. Intersecting with gB_+, we get

$$\mathcal{U} \cap gB_+ = (\mathcal{U}_L \cap gB_{+,L})U_{+,P} \simeq (\mathcal{U}_L \cap gB_{+,L}) \times U_{+,P}.$$

Next, $U_+U_- \cap gB_+ \subseteq gB_+ \subseteq P_+$ and $U_- \cap P_+ = U_{-,L} \cap P_+$ together imply $U_+U_- \cap gB_+ = U_+U_{-,L} \cap gB_+$, from which

$$\begin{aligned}
U_+U_- \cap gB_+ &= U_{+,L}U_{+,P}U_{-,L} \cap gB_{+,L}U_{+,P}\\
&= (U_{+,L}U_{-,L} \cap gB_{+,L})U_{+,P}\\
&\simeq (U_{+,L}U_{-,L} \cap gB_{+,L}) \times U_{+,P}.
\end{aligned}$$

So it remains to prove the last assertion in the lemma. For this, it is more convenient to use the decomposition $\mathcal{V}_g = U_{+,P}\mathcal{V}_{L,g} \simeq U_{+,P} \times \mathcal{V}_{L,g}$. For all $(x, y_+, y_-) \in U_{+,P} \times U_{+,L} \times U_{-,L}$, observe that

$$\Phi_g(xy_+y_-) = xy_+y_-y_+^{-1}x^{-1} = \mathrm{Ad}_x(\Phi_{L,g}(y_+y_-)),$$

where $\mathrm{Ad}_x(u) = xux^{-1}$. A choice of deformation retract from $U_{+,P}(\mathbf{C})$ onto $\{1\}$ induces a homotopy from $\mathrm{Ad}_x : \mathcal{U}_g \to \mathcal{U}_g$ onto $\mathrm{id} : \mathcal{U}_g \to \mathcal{U}_g$, which in turn induces a homotopy from $\Phi_g : \mathcal{V}_g \to \mathcal{U}_g$ onto the composition

$$\mathcal{V}_g \xrightarrow{p_{L,g}} \mathcal{V}_{L,g} \xrightarrow{\Phi_{L,g}} \mathcal{U}_{L,g} \xrightarrow{i_{L,g}} \mathcal{U}_g,$$

where $p_{L,g} : \mathcal{V}_g \to \mathcal{V}_{L,g}$ is the retract induced by the projection $U_{+,P}(\mathbf{C}) \to \{1\}$ and $i_{L,g} : \mathcal{U}_{L,g} \to \mathcal{U}_g$ is the section induced by the inclusion $\{1\} \to U_{+,P}(\mathbf{C})$. Therefore, $\Phi_{L,g}$ being half of a homotopy equivalence is equivalent to Φ_g being half of a homotopy equivalence. □

Lemma 8. *Conjecture 1 holds for $g \in B_+(\mathbf{C})$.*

As observed in the introduction, this can be proved by computing $\mathcal{U}_g(\mathbf{C}), \mathcal{V}_g(\mathbf{C})$, and Φ_g directly. Alternatively:

Proof. Since $\mathcal{U}_g, \mathcal{V}_g, \Phi_g$ only depend on gB_+, we can assume $g = 1$. Then Lemma 7 reduces us to the case $G = T$, where $\mathcal{U}_g = \{1\} = \mathcal{V}_g$. □

3. Examples

3.1. By Lemmas 3–5, it suffices to check Conjecture 1 for one representative G from each central isogeny class of semisimple algebraic group with connected Dynkin diagram. Moreover, by Lemma 6, it suffices to fix a lift $w \mapsto \dot{w}$ from W into $N_G(T)$ and check Conjecture 1 for cosets of the form $gB_+ = \dot{w}B_+$. In this section, we settle the conjecture completely for $G \in \{\mathrm{SL}_2, \mathrm{SL}_3\}$, and check one further case in which $G = \mathrm{Sp}_4$.

Without loss of generality, we can always take U_+, *resp.* U_-, to be the group of unipotent upper-triangular, *resp.* unipotent lower-triangular, matrices in G. To produce defining equations for $\mathcal{U}$, we use the coefficients of the map that sends $g \in G$ to its characteristic polynomial. We write e for the identity element of W.

3.2. The Group SL_2.

We have

$$\mathcal{U} = \{g \in \mathrm{SL}_2 \mid \mathrm{tr}(g) = 2\}.$$

The map $\Phi : U_+U_- \to \mathcal{U}$ is

$$\Phi\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ b' & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 + bb' & -b^2b' \\ b' & 1 - bb' \end{pmatrix}.$$

We can write $W = \{e, w_0\}$. By Lemmas 6 and 8, it suffices to check $gB_+ = \dot{w}_0B_+$. The varieties $\mathcal{U}_g$ and $\mathcal{V}_g$ are

$$\mathcal{U}_g = \left\{\begin{pmatrix} & -\frac{1}{X} \\ X & 2 \end{pmatrix} \;\middle|\; X \neq 0\right\},$$

$$\mathcal{V}_g = \left\{\begin{pmatrix} & -\frac{1}{b'} \\ b' & 1 \end{pmatrix} \;\middle|\; b' \neq 0\right\}.$$

The coordinates define isomorphisms $b' : \mathcal{U}_g \xrightarrow{\sim} \mathbf{G}_m$ and $X : \mathcal{V}_g \xrightarrow{\sim} \mathbf{G}_m$. The map Φ_g is an isomorphism of varieties, corresponding to setting $X = b'$.

3.3. **The Group** SL_3**.** Let $\Lambda^2(g)$ denote the exterior square of a matrix g. From the identity $2\,\mathrm{tr}(\Lambda^2(g)) = \mathrm{tr}(g)^2 - \mathrm{tr}(g^2)$, we have

$$\mathcal{U} = \left\{ g \in \mathrm{SL}_3 \;\middle|\; \begin{matrix} \mathrm{tr}(g) = 3, \\ \mathrm{tr}(\Lambda^2(g)) = 3 \end{matrix} \right\} = \left\{ g \in \mathrm{SL}_3 \;\middle|\; \begin{matrix} \mathrm{tr}(g) = 3, \\ \mathrm{tr}(g^2) = 3 \end{matrix} \right\}.$$

The map $\Phi : U_+U_- \to \mathcal{U}$ is

$$\Phi\left(\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ a' & 1 & \\ b' & c' & 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 + aa' + bb' & bc' - a^2a' - abb' & -aba' - b^2b' - bcc' + a^2ca' + abcb' \\ a' + cb' & 1 - aa' + cc' - acb' & -ba' + aca' - bcb' - c^2c' + ac^2b' \\ b' & c' - ab' & 1 - bb' - cc' + acb' \end{pmatrix}.$$

We can write $W = \{e, s, t, ts, st, w_0\}$, where s and t are the simple reflections. The simple reflections lift to elements of $N_G(T)$ contained in proper Levi subgroups of G, so by the SL_2 case and Lemmas 4, 6, and 7, it remains to consider $gB_+ = \dot{w}B_+$ for $w \in \{ts, st, w_0\}$. In what follows, we choose $\dot{s}, \dot{t}$ so that

$$\dot{s}B_+ \subseteq \left\{\begin{pmatrix} & * & * \\ * & * & * \\ & & * \end{pmatrix}\right\} \quad \text{and} \quad \dot{t}B_+ \subseteq \left\{\begin{pmatrix} * & * & * \\ & & * \\ & * & * \end{pmatrix}\right\}.$$

3.3.1. If $w = ts$, then the varieties $\mathcal{U}_g$ and $\mathcal{V}_g$ are

$$\mathcal{U}_g = \left\{\begin{pmatrix} & Y & C \\ & & Z \\ \frac{1}{YZ} & -\frac{1}{Z}(3 + \frac{C}{YZ}) & 3 \end{pmatrix} \;\middle|\; Y, Z \neq 0 \right\},$$

$$\mathcal{V}_g = \left\{\begin{pmatrix} & -\frac{1}{a'} & b \\ & & c \\ -\frac{a'}{c} & -\frac{1}{c} & 1 \end{pmatrix} \;\middle|\; a', c \neq 0 \right\}.$$

The coordinates define isomorphisms $(Y, Z, C) : \mathcal{U}_g \xrightarrow{\sim} \mathbf{G}_m^2 \times \mathbf{A}^1$ and $(c, a', b) : \mathcal{V}_g \xrightarrow{\sim} \mathbf{G}_m^2 \times \mathbf{A}^1$. The map

$$\Phi_g(x_+x_-) = \begin{pmatrix} & -\frac{1}{a'} & b + \frac{c}{a'} \\ & & c \\ -\frac{a'}{c} & -\frac{1}{c}(2 - \frac{ba'}{c}) & 3 \end{pmatrix}$$

is an isomorphism of varieties, corresponding to $(Y, Z, C) = (-\frac{1}{a'}, c, b + \frac{c}{a'})$.

3.3.2. If $w = st$, then the varieties are

$$\mathcal{U}_g = \left\{\begin{pmatrix} & & \frac{1}{XY} \\ X & A & -\frac{1}{Y}(3 - 3A + A^2) \\ & Y & 3 - A \end{pmatrix} \;\middle|\; X, Y \neq 0 \right\},$$

$$\mathcal{V}_g = \left\{\begin{pmatrix} & & b \\ a' & 1 + \frac{c}{ba'} & c \\ & \frac{1}{ba'} & 1 \end{pmatrix} \;\middle|\; b, a' \neq 0 \right\}.$$

The coordinates define isomorphisms $(X, Y, A) : \mathcal{U}_g \xrightarrow{\sim} \mathbf{G}_m^2 \times \mathbf{A}^1$ and $(b, a', c) : \mathcal{V}_g \xrightarrow{\sim} \mathbf{G}_m^2 \times \mathbf{A}^1$. The map

$$\Phi_g(x_+x_-) = \begin{pmatrix} & & b \\ a' & 2 + \frac{c}{ba'} & -(ba' + \frac{c^2}{ba'} + c) \\ & \frac{1}{ba'} & 1 - \frac{c}{ba'} \end{pmatrix}$$

is an isomorphism of varieties, corresponding to $(X, Y, A) = (a', \frac{1}{ba'}, 1 + \frac{c}{ba'})$.

3.3.3. If $w = w_0$, then the varieties are

$$\mathcal{U}_g = \left\{ \begin{pmatrix} & & Z \\ & -\frac{1}{XZ} & C \\ X & A & 3 + \frac{1}{XZ} \end{pmatrix} \;\middle|\; \begin{array}{l} X, Z \neq 0, \\ (1 + \frac{1}{XZ})^3 + \frac{AC}{XZ} = 0 \end{array} \right\},$$

$$\mathcal{V}_g = \left\{ \begin{pmatrix} & & b \\ & 1 + c'c & c \\ b' & c' & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} b, b' \neq 0, \\ 1 + bb' + (bb')(cc') = 0 \end{array} \right\}.$$

The coordinates define isomorphisms

$$\mathcal{U}_g \xrightarrow{\sim} \{(X, Z, A, C) \in \mathbf{G}_m^2 \times \mathbf{A}^2 \mid (1 + \tfrac{1}{XZ})^3 + \tfrac{AC}{XZ} = 0\},$$
$$\mathcal{V}_g \xrightarrow{\sim} \{(b, b', c, c') \in \mathbf{G}_m^2 \times \mathbf{A}^2 \mid 1 + bb' + (bb')(cc') = 0\}.$$

The map

$$\Phi_g(x_+x_-) = \begin{pmatrix} & & b \\ & 1 + cc' & (1 + \frac{1}{bb'})c \\ b' & (1 + bb')c' & 2 - cc' \end{pmatrix}$$

corresponds to setting $(X, Z, A, C) = (b', b, (1 + bb')c', (1 + \frac{1}{bb'})c)$. Note that $\mathcal{U}_g$ and $\mathcal{V}_g$ are *not* isomorphic as varieties.

Proposition 9. *For $G = \mathrm{SL}_3$ and $gB_+ = \dot{w}_0 B_+$, the map Φ_g is neither injective nor surjective on $\mathbf{C}$-points, but does define half of a homotopy equivalence.*

Proof. Let

$$\mathcal{U}_g^\dagger = \{(u, A, C) \in \mathbf{G}_m \times \mathbf{A}^2 \mid AC = -(1 + u)(1 + \tfrac{1}{u})^2\},$$
$$\mathcal{V}_g^\dagger = \{(u, c, c') \in \mathbf{G}_m \times \mathbf{A}^2 \mid cc' = -(1 + \tfrac{1}{u})\}.$$

Let $\Phi_g^\dagger : \mathcal{V}_g^\dagger \to \mathcal{U}_g^\dagger$ be the map $\Phi_g^\dagger(u, c, c') = (u, (1 + u)c', (1 + \frac{1}{u})c)$. Then Φ_g is a pullback of $\Phi_g^\dagger$, so it suffices to show the claim of the proposition with $\mathcal{U}_g^\dagger, \mathcal{V}_g^\dagger, \Phi_g^\dagger$ in place of $\mathcal{U}_g, \mathcal{V}_g, \Phi_g$.

Observe that $\Phi_g^\dagger$ preserves $u \in \mathbf{G}_m$. Over the subvariety of $\mathbf{G}_m$ where $u \neq -1$, the fibers of $\mathcal{U}_g^\dagger$ and $\mathcal{V}_g^\dagger$ are copies of $\mathbf{G}_m$: say, via the coordinates A and c'. In these coordinates, $\Phi_g^\dagger$ amounts to rotating $\mathbf{G}_m$ by $1 + u$. Over the point $u = -1$, the fibers are copies of the transverse intersection of two lines. Altogether, $\mathcal{U}_g^\dagger(\mathbf{C})$ and $\mathcal{V}_g^\dagger(\mathbf{C})$ are both homotopic to pinched tori, and $\Phi_g^\dagger$ induces a self-map of the pinched torus that preserves its longitude and top homology. Thus $\Phi_g^\dagger$ fits into a homotopy equivalence. It is neither injective nor surjective because $\Phi_g^\dagger(-1, c, c') = (-1, 0, 0)$. □

3.4. **The Group** Sp_4**.** We set $\mathrm{Sp}_4 = \{g \in \mathrm{GL}_4 \mid g^t J g = J\}$, where

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

For Sp_4, the only nontrivial coefficients of the characteristic polynomial are $\mathrm{tr}(g) = \mathrm{tr}(\Lambda^3(g))$ and $\mathrm{tr}(\Lambda^2(g))$, so similarly to SL_3, we have:

$$\mathscr{U} = \left\{ g \in \mathrm{Sp}_4 \,\middle|\, \begin{array}{l} \mathrm{tr}(g) = 4, \\ \mathrm{tr}(\Lambda^2(g)) = 4 \end{array} \right\} = \left\{ g \in \mathrm{Sp}_4 \,\middle|\, \begin{array}{l} \mathrm{tr}(g) = 4, \\ \mathrm{tr}(g^2) = 4 \end{array} \right\}.$$

The map $\Phi : U_+ U_- \to \mathscr{U}$ is

$$\Phi\left(\begin{pmatrix} 1 & a & b+ad & c \\ & 1 & 2d & b-ad \\ & & 1 & -a \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ a' & 1 & & \\ b'+a'd' & 2d' & 1 & \\ c' & b'-a'd' & -a' & 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1+f_1+g_1 & f_{1,2}+g_{1,2} & f_{1,3}+g_{1,3} & h_{1,4} \\ f_{2,1}+g_{2,1} & 1-f_1+g_2 & h_{2,3} & f_{1,3}-g_{1,3} \\ f_{3,1}+g_{3,1} & h_{3,2} & 1-f_1-g_2 & f_{1,2}-g_{1,2} \\ h_{4,1} & f_{3,1}-g_{3,1} & f_{2,1}-g_{2,1} & 1+f_1-g_1 \end{pmatrix},$$

where we set

$$\begin{aligned} f_1 &= ba'd' + ada'd', \\ g_1 &= aa' + bb' + cc' + adb', \\ g_2 &= -aa' + bb' + 4dd' - abc' - 3adb' + a^2dc' \end{aligned}$$

and

$$\begin{aligned} f_{1,2} &= -(c + ab + a^2d)a'd', \\ g_{1,2} &= 2bd' + cb' - a(aa' + bb' + cc' - 2dd' + adb'), \\ f_{1,3} &= -ca' - b(aa' + bb' + cc' + 4dd') - 2cdb' + ad(aa' + cc' - 4dd' + adb'), \\ g_{1,3} &= -(b^2 - 2cd - a^2d^2)a'd', \\ f_{2,1} &= 2da'd', \\ g_{2,1} &= a' + bc' + 2db' - adc', \\ f_{3,1} &= b' - ac', \\ g_{3,1} &= a'd' \end{aligned}$$

and

$$\begin{aligned} h_{1,4} &= -c(2aa' + 2bb' + cc') - 2b^2d' - 2ad(cb' + 2bd' + add'), \\ h_{2,3} &= -2ba' + 2d(aa' - 2bb' - 4dd') - b^2c' + ad(2bc' + 4db' - adc'), \\ h_{3,2} &= 2d' - 2ab' + a^2c', \\ h_{4,1} &= c'. \end{aligned}$$

We can write $W = \{e, s, t, ts, st, sts, tst, w_0\}$, where s and t are the simple reflections. By the SL_2 case and Lemmas 4, 6, and 7, it remains to consider $gB_+ = \dot{w}B_+$ for $w \in \{ts, st, sts, tst, w_0\}$.

Below, we will only check $w = sts$. Without loss of generality, we can assume

$$s\dot{t}sB_+ = \left\{ \begin{pmatrix} & & & \frac{1}{X} \\ & Y & 2YD & Y(B-AD) \\ & & \frac{1}{Y} & -\frac{A}{Y} \\ -X & -XA & -X(B+AD) & C \end{pmatrix} \;\middle|\; X, Y \neq 0 \right\}.$$

The varieties $\mathcal{U}_g$ and $\mathcal{V}_g$ are

$$\mathcal{U}_g = \left\{ \begin{pmatrix} & & & \frac{1}{X} \\ & Y & 2YD & Y(B-AD) \\ & & \frac{1}{Y} & -\frac{A}{Y} \\ -X & -XA & -X(B+AD) & 4 - Y - \frac{1}{Y} \end{pmatrix} \;\middle|\; \begin{array}{l} X, Y \neq 0, \\ XA(Y(B-AD) - \frac{1}{Y}(B+AD)) \\ = \frac{1}{Y^2}(1-Y)^4 \end{array} \right\},$$

$$\mathcal{V}_g = \left\{ \begin{pmatrix} & & & c \\ & \frac{1}{1+aa'} & 2(1+aa')d - c(a')^2 & ca' - 2ad \\ & & 1+aa' & -a \\ -\frac{1}{c} & -\frac{a}{c(1+aa')} & -a' & 1 \end{pmatrix} \;\middle|\; c, 1+aa' \neq 0 \right\}.$$

The coordinates define isomorphisms

$$\mathcal{U}_g \xrightarrow{\sim} \left\{ (X, Y, A, B, D) \in \mathbf{G}_m^2 \times \mathbf{A}^3 \;\middle|\; \begin{array}{l} XA(Y(B-AD) - \frac{1}{Y}(B+AD)) \\ = \frac{1}{Y^2}(1-Y)^4 \end{array} \right\},$$

$$\mathcal{V}_g \xrightarrow{\sim} \{(c, a, d, a') \in \mathbf{G}_m \times \mathbf{A}^3 \mid 1 + aa' \neq 0\}.$$

The map

$$\Phi_g(x_+x_-) = \begin{pmatrix} & & & c \\ & \frac{1}{1+aa'} & 2ada'(\frac{2+aa'}{1+aa'}) - c(a')^2 & 2a^2da' - \frac{a^2c(a')^3}{1+aa'} \\ & & 1+aa' & a^2a' \\ -\frac{1}{c} & \frac{a^2a'}{c(1+aa')} & -\frac{2a^2da'}{c(1+aa')} & 3 - aa' - \frac{1}{1+aa'} \end{pmatrix}$$

corresponds to setting

$$\begin{pmatrix} X \\ Y \\ A \\ B \\ D \end{pmatrix} = \begin{pmatrix} \frac{1}{c} \\ \frac{1}{1+aa'} \\ -\frac{a^2a'}{1+aa'} \\ a^2da'(1 + aa' + \frac{1}{1+aa'}) - \frac{1}{2}a^2c(a')^3 \\ ada'(2+aa') - \frac{1}{2}c(a')^2(1+aa') \end{pmatrix}.$$

Proposition 10. *For $G = \mathrm{Sp}_4$ and $gB_+ = s\dot{t}sB_+$, the map Φ_g is neither injective nor surjective on $\mathbf{C}$-points, but does define half of a homotopy equivalence.*

Proof. The map Φ_g fits into a commutative diagram

$$\begin{tikzcd} \mathscr{V}_g \arrow[r, "\Phi_g"] \arrow[d, "\Psi_{\mathscr{V}}"'] & \mathscr{U}_g \arrow[d, "\Psi_{\mathscr{U}}"] \\ \mathscr{V}_g^\dagger \arrow[r, "\Phi_g^\dagger"] & \mathscr{U}_g^\dagger \end{tikzcd}$$

where the new varieties are

$$\begin{aligned} \mathscr{U}_g^\dagger &= \{(X, Y, A, B_-, B_+) \in \mathbf{G}_m^2 \times \mathbf{A}^3 \mid XA(YB_- - \tfrac{B_+}{Y}) = \tfrac{1}{Y^2}(1-Y)^4\}, \\ \mathscr{V}_g^\dagger &= \{(c, u, a_1, a_2, d') \in \mathbf{G}_m^2 \times \mathbf{A}^3 \mid (u-1)^4 = a_1a_2\}, \end{aligned}$$

and the new maps are

$$\begin{aligned} \Phi_g^\dagger(c, u, a_1, a_2, d') &= (\tfrac{1}{c}, \quad \tfrac{1}{u}, \quad -\tfrac{a_1}{u}, \quad 2ud' - ca_2, \quad \tfrac{2d'}{u}), \\ \Psi_{\mathscr{U}}(X, Y, A, B, D) &= (X, \quad Y, \quad A, \quad B - AD, \quad B + AD), \\ \Psi_{\mathscr{V}}(c, a, d, a') &= (c, \quad 1 + aa', \quad a^2a', \quad a^2(a')^3, \quad a^2da'). \end{aligned}$$

The map $\Phi_g^\dagger$ is algebraically invertible, so to show that Φ_g induces a homotopy equivalence, it remains to study the topology of the maps $\Psi_{\mathscr{U}}$ and $\Psi_{\mathscr{V}}$.

To show that $\Psi_{\mathscr{U}}$ induces a homotopy equivalence, we first note that it preserves $(X, A) \in \mathbf{A}^2$. Over the subvariety of $\mathbf{A}^2$ where $A \neq 0$, it is invertible. Over the line $A = 0$, the defining equations of $\mathscr{U}_g$ and $\mathscr{U}_g^\dagger$ both simplify to $(1-Y)^4 = 0$, which has the unique solution $Y = 1$ over $\mathbf{C}$, so over this line, the fibers of $\mathscr{U}_g(\mathbf{C})$ and $\mathscr{U}_g^\dagger(\mathbf{C})$ are contractible, being copies of $\mathbf{C}^2$.

To show that $\Psi_{\mathscr{V}}$ induces a homotopy equivalence, it suffices to show the same for the map from $\{(a, a') \in \mathbf{A}^2 \mid aa' \neq -1\}$ into $\{(u, a_1, a_2) \in \mathbf{G}_m \times \mathbf{A}^2 \mid (u-1)^4 = a_1a_2\}$ that sends $(a, a') \mapsto (1 + aa', a^2a', a^2(a')^3)$. Indeed, this map restricts to an isomorphism from the subvariety where $aa' \neq 0$ onto the subvariety where $a_1a_2 \neq 0$, and collapses the subvariety where $aa' = 0$ onto the point $(u, a_1, a_2) = (1, 0, 0)$. The set of $\mathbf{C}$-points in the domain where $aa' = 0$ is contractible, and the set of $\mathbf{C}$-points in the target where $a_1a_2 = 0$ admits a deformation retract onto $\{(1, 0, 0)\}$.

Therefore, Φ_g induces a a homotopy equivalence on $\mathbf{C}$-points. It is not injective because $\Phi_g(c, a, d, 0) = (\frac{1}{c}, 1, 0, 0, 0) = \Phi_g(c, 0, d, a')$ for all (c, d, a, a') such that $aa' \neq -1$, and it is not surjective because the points of the form $(X, 1, 0, B, D) \in \mathscr{V}_g(\mathbf{C})$ with $B \neq 0$ do not appear in the image. □

4. Configurations of Flags

4.1. In this section, we relate $\mathscr{U}_g$ and $\mathscr{V}_g$ to varieties that were studied in [T]. Henceforth, we fix any field $\mathbf{F}$ of good characteristic for G, and replace G with its split form over $\mathbf{F}$. We also assume that B_+ is defined over $\mathbf{F}$.

Let $\mathscr{B}$ be the flag variety of G, *i.e.*, the variety that parametrizes its Borel subgroups. As these subgroups are all self-normalizing and conjugate to one another, there is an isomorphism of varieties:

$$\begin{array}{ccc} G/B_+ & \xrightarrow{\sim} & \mathscr{B} \\ xB_+ & \mapsto & xB_+x^{-1} \end{array} \tag{4.1}$$

It transports the G-action on G/B_+ by left multiplication to the G-action on $\mathcal{B}$ by left conjugation.

The orbits of the diagonal G-action on $\mathcal{B} \times \mathcal{B}$ can be indexed by the Weyl group W. The closure order on the orbits corresponds to the Bruhat order on W induced by the Coxeter presentation. For all $(B_1, B_2) \in \mathcal{B} \times \mathcal{B}$ and $w \in W$, we write $B_1 \xrightarrow{w} B_2$ to indicate that (B_1, B_2) belongs to the wth orbit, in which case we say that it is in *relative position* w. In particular, note that $B_+ \xrightarrow{w_0} B_-$ because $B_- = \dot{w}_0 B_+ \dot{w}_0^{-1}$.

Under the Bruhat decomposition, (4.1) restricts to an isomorphism

$$U_+ \dot{w} B_+/B_+ \xrightarrow{\sim} \{B \in \mathcal{B} \mid B_+ \xrightarrow{w} B\}. \tag{4.2}$$

Each side is isomorphic to an affine space of dimension $\ell(w)$, where $\ell : W \to \mathbf{Z}_{\geq 0}$ is the Bruhat length function.

4.2. Fix $g \in G(\mathbf{F})$. Recall that $H_g = B_+ \cap gB_+g^{-1}$ acts on $\mathcal{V}_g = U_+U_- \cap gB_+$ according to (1.2). Let

$$\mathcal{X}_g = \{B \in \mathcal{B} : B_+ \xrightarrow{w_0} B \xrightarrow{w_0} gB_+g^{-1}\},$$

and let H_g act on $\mathcal{X}_g$ by left conjugation. We will prove:

Proposition 11. *There is a H_g-equivariant isomorphism of varieties $\mathcal{X}_g \to \mathcal{V}_g$.*

We give the proof in two steps. For convenience, we set $\mathcal{Y}_g = gU_+\dot{w}_0B_+/B_+ \subseteq G/B_+$. Let H_g act on $\mathcal{Y}_1 \cap \mathcal{Y}_g$ by left multiplication.

Lemma 12. *The isomorphism (4.2) for $w = w_0$ restricts to an H_g-equivariant isomorphism $\mathcal{Y}_1 \cap \mathcal{Y}_g \xrightarrow{\sim} \mathcal{X}_g$.*

Proof. Recall that (4.2) is G-equivariant. Under the action of an element x, the $w = w_0$ case is transported to an isomorphism

$$\mathcal{Y}_x \xrightarrow{\sim} \{B \in \mathcal{B} \mid xB_+x^{-1} \xrightarrow{w_0} B\}.$$

On the right-hand side, the direction of the arrow $\xrightarrow{w_0}$ can be reversed because $w_0^{-1} = w_0$. Now take the fiber product of the isomorphisms for $x = 1$ and $x = g$ over the isomorphism (4.1). $\square$

In what follows, recall that via the decomposition $B_+ \simeq U_+ \rtimes T$, any element of B_+ can be written as ut for some uniquely determined $(u, t) \in U_+ \times T$. As a consequence, we also get a decomposition $\dot{w}_0B_+ = U_-\dot{w}_0T = U_-T\dot{w}_0$.

Lemma 13. *The map*

$$\begin{array}{rcl} \mathcal{V}_g = U_+U_- \cap gB_+ & \to & \mathcal{Y}_1 \cap \mathcal{Y}_g \\ x_+x_- = gut & \mapsto & x_+x_-t^{-1}\dot{w}_0B_+ = gu\dot{w}_0B_+ \end{array}$$

is an H_g-equivariant isomorphism of varieties.

Proof. Let $\mathcal{V}_g' = U_+\dot{w}_0B_+ \cap gU_+\dot{w}_0$. Since the map

$$\begin{array}{rcl} \mathcal{V}_g & \to & \mathcal{V}_g' \\ x_+x_- = gut & \mapsto & x_+x_-t^{-1}\dot{w}_0 = gu\dot{w}_0 \end{array}$$

is an isomorphism, it remains to show that the map $f : \mathcal{V}'_g \to \mathcal{Y}_1 \cap \mathcal{Y}_g$ given by

$$\mathcal{V}'_g \to \mathcal{V}'_g B_+ \to (\mathcal{V}'_g B_+)/B_+ = \mathcal{Y}_1 \cap \mathcal{Y}_g$$

is bijective on R-points for every $\mathbf{F}$-algebra R. For convenience, we suppress R in the notation below.

Let $yB_+ \in (\mathcal{V}'_g B_+)/B_+$. Then we can write $y = u\dot{w}_0 b = gu'\dot{w}_0 b'$ for some $u, u' \in U_+$ and $b, b' \in B_+$. Therefore, $yB_+ = f(y(b')^{-1})$, where $y(b')^{-1} = gu'\dot{w}_0 \in \mathcal{V}'_g$. This proves $f^{-1}(yB_+)$ is nonempty. We claim that $f^{-1}(yB_+)$ contains only one point. Recall that the map $U_+ \to U_+\dot{w}_0 B_+/B_+$ that sends $v \mapsto v\dot{w}_0 B_+$ is an isomorphism. Thus, $v \neq u$ implies $v\dot{w}_0 B_+ \neq u\dot{w}_0 B_+$. We deduce that

$$\begin{aligned} f^{-1}(yB_+) &= f^{-1}(u\dot{w}_0 B_+) \\ &\subseteq u\dot{w}_0 B_+ \cap gU_+\dot{w}_0 \\ &\simeq (\dot{w}_0^{-1} g^{-1} u\dot{w}_0)B_+ \cap U_-. \end{aligned}$$

But the intersection of U_- with any coset of B_+ contains only one point. $\square$

4.3. Let $\mathcal{O}_w, \mathcal{U}_w, \mathcal{X}_w$ be the varieties defined by

$$\begin{aligned} \mathcal{O}_w &= \{(B', B'') \in \mathcal{B} \times \mathcal{B} \mid B' \xrightarrow{w} B''\}, \\ \mathcal{U}_w &= \{(u, B') \in \mathcal{U} \times \mathcal{B} \mid B' \xrightarrow{w} uB'u^{-1}\}, \\ \mathcal{X}_w &= \{(B, B', B'') \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \mid B' \xrightarrow{w_0} B \xrightarrow{w_0} B'' \xleftarrow{w} B'\}. \end{aligned}$$

Let G act on these varieties by (diagonal) left conjugation. We regard $\mathcal{U}_w$ and $\mathcal{X}_w$ as varieties over $\mathcal{O}_w$ via the G-equivariant maps $(u, B') \mapsto (B', uB'u^{-1})$ and $(B, B', B'') \mapsto (B', B'')$, respectively.

Let H_g act on G by right multiplication. For any variety X with an H_g-action, let H_g act diagonally on $X \times G$, and let G act on $(X \times G)/H_g$ by left multiplication on the second factor. Finally, fix a prime $\ell > 0$ invertible in $\mathbf{F}$, so that we can form the equivariant ℓ-adic compactly-supported cohomology groups

$$\mathrm{H}^*_{c,H_g}(X) \simeq \mathrm{H}^*_{c,G}((X \times G)/H_g).$$

With these conventions, we have:

Proposition 14. *If $B_+ \xrightarrow{w} gB_+g^{-1}$, then there are G-equivariant isomorphisms*

$$\begin{aligned} (\mathcal{U}_g \times G)/H_g &\xrightarrow{\sim} \mathcal{U}_w, \\ (\mathcal{X}_g \times G)/H_g &\xrightarrow{\sim} \mathcal{X}_w. \end{aligned}$$

In particular, they induce isomorphisms on compactly-supported cohomology:

$$\begin{aligned} \mathrm{H}^*_{c,H_g}(\mathcal{U}_g, \bar{\mathbf{Q}}_\ell) &\xrightarrow{\sim} \mathrm{H}^*_{c,G}(\mathcal{U}_w, \bar{\mathbf{Q}}_\ell), \\ \mathrm{H}^*_{c,H_g}(\mathcal{X}_g, \bar{\mathbf{Q}}_\ell) &\xrightarrow{\sim} \mathrm{H}^*_{c,G}(\mathcal{X}_w, \bar{\mathbf{Q}}_\ell). \end{aligned}$$

Proof. The maps $(\mathcal{U}_g \times G)/H_g \to \mathcal{U}_w$ and $(\mathcal{X}_g \times G)/H_g \to \mathcal{X}_w$ are

$$\begin{aligned} [u, x] &\mapsto (xux^{-1}, xB_+x^{-1}), \\ [B, x] &\mapsto (xBx^{-1}, xB_+x^{-1}, xgB_+g^{-1}x^{-1}), \end{aligned}$$

respectively. To show that they are isomorphisms: Observe that G acts transitively on $\mathcal{O}_w$, and the stabilizer of (B_+, gB_+g^{-1}) is precisely H_g. The preimage of this point in $\mathcal{U}_w$, *resp.* $\mathcal{X}_w$, is $\mathcal{U}_g$, *resp.* $\mathcal{X}_g$. Therefore, the maps above are the respective pullbacks to $\mathcal{U}_w$ and $\mathcal{X}_w$ of the isomorphism $G/H_g \xrightarrow{\sim} \mathcal{O}_w$ that sends $xH_g \mapsto (xB_+x^{-1}, xgB_+g^{-1}x^{-1})$. $\square$

Note that when $\mathbf{F} = \mathbf{C}$, the maps on cohomology in Proposition 14 preserve weight filtrations because the maps that induce them are algebraic.

5. Khovanov–Rozansky Homology

5.1. In this section, we prove Theorem 2 by way of more general constructions motivated by knot theory.

Let Br_W^+ be the *positive braid monoid* of W. It is the monoid freely generated by a set of symbols $\{\sigma_w\}_{w \in W}$, modulo the relations $\sigma_{ww'} = \sigma_w \sigma_{w'}$ for all $w, w' \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$. The *full twist* is the element $\pi = \sigma_{w_0}^2 \in Br_W^+$.

For all $\beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br_W^+$, we set

$$\mathcal{U}(\beta) = \{(u, B_1, \ldots, B_k) \in \mathcal{U} \times \mathcal{B}^k \mid u^{-1}B_k u \xrightarrow{w_1} B_1 \xrightarrow{w_2} \cdots \xrightarrow{w_k} B_k\},$$
$$\mathcal{X}(\beta) = \{(B_1, \ldots, B_k) \in \mathcal{B}^k \mid B_k \xrightarrow{w_1} B_1 \xrightarrow{w_2} \cdots \xrightarrow{w_k} B_k\}.$$

Let G act on these varieties by left conjugation. We regard $\mathcal{U}(\beta)$ and $\mathcal{X}(\beta)$ as varieties over $\mathcal{O}_w$, where $w = w_1 \cdots w_k \in W$, via the equivariant maps $(u, (B_i)_i) \mapsto (B_k, B_1)$ and $(B_i)_i \mapsto (B_k, B_1)$, respectively. Deligne showed that up to isomorphism over $\mathcal{B} \times \mathcal{B}$, these varieties only depend on β, not on the sequence of elements w_i. His full result describes the extent to which the isomorphism can be pinned down uniquely; see [D] for details.

In particular, we have equivariant identifications

$$\mathcal{U}_w \xrightarrow{\sim} \mathcal{U}(\sigma_w),$$
$$\mathcal{X}_w \xrightarrow{\sim} \mathcal{X}(\sigma_w \pi)$$

via $(u, B_1) = (u, B')$ and $(B_1, B_2, B_3) = (B', B'', B)$.

5.2. If W is the symmetric group on n letters, denoted S_n, then the group completion of Br_W is the group of topological braids on n strands, denoted Br_n. Any braid can be closed up end-to-end to form a link: that is, an embedding of a disjoint union of circles into 3-dimensional space. Thus there is a close relation between isotopy invariants of links and functions on the groups Br_n.

In [KR], Khovanov and Rozansky introduced a link invariant valued in triply-graded vector spaces. Its graded dimension can be written as a formal series in variables $a, q^{\frac{1}{2}}, t$. In [Kh], Khovanov showed how to construct it in terms of class functions on the groups Br_n, and more precisely, in terms of functors on monoidal additive categories attached to the groups S_n. When we set $t = -1$, the Khovanov–Rozansky invariant of a link specializes to its so-called HOMFLYPT series, and Khovanov's functors specialize to class functions originally introduced by Jones and Ocneanu.

The positive braid monoid Br_W^+ and its group completion Br_W can actually be defined for any Coxeter group W, not just Weyl groups. In [G], Y. Gomi extended the construction of Jones–Ocneanu to *finite* Coxeter groups. There is a similar extension of Khovanov's construction, up to a choice of a (faithful) representation on which W acts as a reflection group.

Fix such a representation V. For any braid $\beta \in Br_W$, we write $\mathsf{HHH}_V(\beta)$ to denote the *Khovanov–Rozansky (KR) homology* of β with respect to V. We will use the grading conventions in [T], so that

$$\mathsf{P}_V(\beta) = (at)^{|\beta|} a^{-\dim(V)} \sum_{i,j,k} (a^2 q^{\frac{1}{2}} t)^{\dim(V)-i} q^{\frac{j}{2}} t^{-k} \dim \mathsf{HHH}_V^{i,i+j,k}(\beta)$$

is an isotopy invariant of the link closure of β. In the case where $W = S_n$, taking V to be the $(n-1)$-dimensional reflection representation yields what is usually called *reduced KR homology* and denoted HHH, while taking V to be the n-dimensional permutation representation yields what is usually called *unreduced KR homology* and denoted $\overline{\mathsf{HHH}}$. They are related by

$$\left(\frac{a^{-1} + at}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}\right) \mathsf{P}(\beta) = \overline{\mathsf{P}}(\beta),$$

where P and $\overline{\mathsf{P}}$ denote the series P_V for these respective choices of V.

Henceforth, let $r = \dim(V)$ and $N = \dim(\mathcal{B})$. The results below are [T, Cor. 4] and [GHMN, Thm. 1.9].

Theorem 15. *Suppose that W is the Weyl group of a split reductive group G over $\mathbf{F}$ with root lattice Φ, and that $V = \mathbf{Z}\Phi \otimes_{\mathbf{Z}} \mathbf{Q}$. Then for any $\beta \in Br_W^+$, we have isomorphisms*

$$\mathrm{gr}^{\mathsf{W}}_{j+2r} \mathrm{H}^{j+k+2r}_{c,G}(\mathcal{U}(\beta), \bar{\mathbf{Q}}_\ell) \simeq \mathsf{HHH}_V^{0,j,k}(\beta),$$
$$\mathrm{gr}^{\mathsf{W}}_{j+2(r+N)} \mathrm{H}^{j+k+2(r+N)}_{c,G}(\mathcal{X}(\beta), \bar{\mathbf{Q}}_\ell) \simeq \mathsf{HHH}_V^{r,r+j,k}(\beta)$$

for all j, k.

Theorem 16 (Gorsky–Hogancamp–Mellit–Nakagane)**.** *For any integer $n \geq 1$ and $\beta \in Br_n$, we have*

$$\overline{\mathsf{HHH}}^{0,j,k}(\beta) \simeq \overline{\mathsf{HHH}}^{r,r+j,k}(\beta\pi)$$

for all j, k.

Proof of Theorem 2. We must have $B_+ \xrightarrow{w} gB_+g^{-1}$ for some $w \in W$. Combining Proposition 11, Proposition 14, and Theorem 15, we get isomorphisms

$$\mathrm{gr}^{\mathsf{W}}_{j+2n} \mathrm{H}^{j+k+2n}_{c,H_g}(\mathcal{U}_g, \bar{\mathbf{Q}}_\ell) \simeq \mathsf{HHH}_V^{0,j,k}(\sigma_w),$$
$$\mathrm{gr}^{\mathsf{W}}_{j+2(n+N)} \mathrm{H}^{j+k+2(n+N)}_{c,H_g}(\mathcal{X}_g, \bar{\mathbf{Q}}_\ell) \simeq \mathsf{HHH}_V^{r,r+j,k}(\sigma_w\pi),$$

where $V = \mathbf{Z}\Phi \otimes_{\mathbf{Z}} \mathbf{Q}$ and Φ is the root lattice of G.

If $G = \mathrm{GL}_n$, then V is the permutation representation of S_n. So in this case, $\mathsf{HHH}_V = \overline{\mathsf{HHH}}$, and we are done by Theorem 16. Finally, we bootstrap from GL_n to any other split reductive group of type A using Lemmas 4 and 5. $\square$

5.3. Theorems 15–16 suggest the following generalization of Conjecture 1.

Conjecture 17. *For any $\beta \in Br_W^+$, there is a homotopy equivalence between $\mathcal{U}(\beta)(\mathbf{C})$ and $\mathcal{X}(\beta\pi)(\mathbf{C})$ that matches the weight filtrations on their compactly-supported cohomology.*

Remark 18. It would be desirable to generalize the map of stacks $[\mathcal{V}_g/H_g] \to [\mathcal{U}_g/H_g]$ that arises from Φ_g to an explicit map $[\mathcal{X}(\beta\pi)/G] \to [\mathcal{U}(\beta)/G]$ for any positive braid β. Due to the inexplicit nature of Lemma 13, we have not yet found such a generalization.

6. Point Counts over Finite Fields

6.1. For any braid $\beta \in Br_n$, we write $\hat{\beta}$ to denote its link closure. The *reduced HOMFLYPT series* $\mathbf{P}(\hat{\beta})$ is related to the KR homology of β by

$$\mathbf{P}(\hat{\beta}) = \mathsf{P}(\beta)|_{t \to -1}.$$

This is an element of $\mathbf{Z}[\![q^{\frac{1}{2}}]\!][q^{-\frac{1}{2}}][a^{\pm 1}]$. We write $[a^i]\mathbf{P}(\hat{\beta})$ to denote the coefficient of a^i in $\mathbf{P}(\hat{\beta})$, viewed as an element of $\mathbf{Z}[\![q^{\frac{1}{2}}]\!][q^{-\frac{1}{2}}]$.

If $\beta = \sigma_{s_1} \cdots \sigma_{s_\ell}$, where the elements $s_1, \ldots, s_\ell \in W$ are all simple reflections, then we set $|\beta| = \ell$. This number only depends on β. Theorem 16 then specializes to the following result from [K].

Theorem 19 (Kálmán)**.** *For any integer $n \geq 1$ and $\beta \in Br_n$, we have*

$$[a^{|\beta|-n+1}]\mathbf{P}(\hat{\beta}) = [a^{|\beta|+n-1}]\mathbf{P}(\widehat{\beta\pi}).$$

In [T, §8], we generalized Kálmán's result from Br_n to Br_W. In this section, we review the statement, then explain its relation to the point-counting identity (1.1).

6.2. Let H_W be the *Iwahori–Hecke algebra* of W. For our purposes, H_W is the quotient of the group algebra $\mathbf{Z}[q^{\pm\frac{1}{2}}][Br_W]$ by the two-sided ideal

$$\langle (\sigma_s - q^{\frac{1}{2}})(\sigma_s + q^{-\frac{1}{2}}) \mid \text{simple reflections } s \rangle.$$

For any element $\beta \in Br_W$, we abuse notation by again writing β to denote its image in H_W.

The sets $\{\sigma_w\}_{w \in W}$ and $\{\sigma_w^{-1}\}_{w \in W}$ are bases for H_W as a free $\mathbf{Z}[q^{\pm\frac{1}{2}}]$-module. Let $\tau^\pm : H_W \to \mathbf{Z}[q^{\pm\frac{1}{2}}]$ be the $\mathbf{Z}[q^{\pm\frac{1}{2}}]$-linear functions defined by:

$$\tau^\pm(\sigma_w^{\pm 1}) = \begin{cases} 1 & w = e \\ 0 & w \neq e \end{cases}$$

For $W = S_n$, comparing $\tau^\pm$ with the Jones–Ocneanu trace on H_W shows that

$$[a^{|\beta| \pm (n-1)}]\mathbf{P}(\hat{\beta}) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{-(n-1)}(-1)^{|\beta|}\tau^\pm(\beta)$$

for all $\beta \in Br_n$. Therefore the following result from [T, §8] generalizes Kálmán's theorem to arbitrary W.

Theorem 20. *For any finite Coxeter group W and braid $\beta \in Br_W$, we have*

$$\tau^-(\beta) = \tau^+(\beta\pi).$$

6.3. We return to the setting of Section 5, so that W is the Weyl group of G. Under the hypotheses of Theorem 15, the following identities from *loc. cit.* relate Theorem 20 to point counting:

$$\frac{|\mathcal{U}(\beta)(\mathbf{F})|}{|G(\mathbf{F})|} = (q-1)^{-r}(q^{\frac{1}{2}})^{|\beta|}\tau^{-}(\beta),$$
$$\frac{|\mathcal{X}(\beta)(\mathbf{F})|}{|G(\mathbf{F})|} = (q-1)^{-r}(q^{\frac{1}{2}})^{|\beta|}\tau^{+}(\beta).$$

Together they imply:

Corollary 21. *Keep the hypotheses of Theorem 15. Then for any $\beta \in Br_W^+$, we have $|\mathcal{U}(\beta)(\mathbf{F})| = |\mathcal{X}(\beta\pi)(\mathbf{F})|$.*

We claim that when B_+ is defined over $\mathbf{F}$, Corollary 21 implies (1.1) from the introduction. The key is that the proof of Proposition 14 also works at the level of $\mathbf{F}$-points. Thus there are G-equivariant bijections

$$(\mathcal{U}_g(\mathbf{F}) \times G(\mathbf{F}))/H_g(\mathbf{F}) \xrightarrow{\sim} \mathcal{U}_w(\mathbf{F}),$$
$$(\mathcal{X}_g(\mathbf{F}) \times G(\mathbf{F}))/H_g(\mathbf{F}) \xrightarrow{\sim} \mathcal{X}_w(\mathbf{F})$$

for any $g \in G(\mathbf{F})$ such that that $B_+ \xrightarrow{w} gB_+g^{-1}$. Since the quotients are free, we deduce that

$$|\mathcal{U}_g(\mathbf{F})||G(\mathbf{F})| = |\mathcal{U}_w(\mathbf{F})||H_g(\mathbf{F})| = |\mathcal{X}_w(\mathbf{F})||H_g(\mathbf{F})| = |\mathcal{X}_g(\mathbf{F})||G(\mathbf{F})|.$$

Applying Proposition 11, we arrive at $|\mathcal{U}_g(\mathbf{F})| = |\mathcal{X}_g(\mathbf{F})| = |\mathcal{V}_g(\mathbf{F})|$, which is (1.1).

Remark 22. The original identity proved by Kawanaka was

$$|(\mathcal{U} \cap U_+\dot{w}B_+)(\mathbf{F})| = |(U_+U_- \cap U_+\dot{w}B_+)(\mathbf{F})|$$

for all $w \in W$ [Ka, Cor. 4.2]. This is equivalent to (1.1) as long as B_+ is defined over $\mathbf{F}$. For when the latter holds, an argument similar to the proof of Lemma 6 shows that the $\mathbf{F}$-point counts of $\mathcal{U}_g$ and $\mathcal{V}_g$ remain constant as g runs over elements of $U_+\dot{w}B_+(\mathbf{F})$.

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