Artin Braids from Infinitesimal Loops

Minh-Tâm Quang Trinh

Massachusetts Institute of Technology
\[ \Delta = \text{Spec } \mathbb{C}[x] = \text{infinitesimal disk} \]
\[ \eta = \text{Spec } \mathbb{C}((x)) = \text{infinitesimal loop} \]
\[ \pi_1^\et (\eta) \simeq \hat{\mathbb{Z}} = \lim_{\leftarrow n} \mathbb{Z}/n\mathbb{Z} \]

Suppose \( W \curvearrowright V \) by real reflections.
\[ Br_W := \pi_1^{\text{top}}(V^\circ/W) \]
where \( V^\circ \subseteq V \) is the free locus.

\textbf{Ex} \quad \text{For } S_n \curvearrowright \mathbb{C}^n, \text{ recover that links of plane curves are \textit{iterated torus} type.}

\textbf{Prob} \quad \text{Classify algebraic braids.}

\[ \pi_W \in Z(Br_W) \text{ full twist} \]

\textbf{Thm} \quad \text{If } [\beta] = [\beta_f] \text{ is algebraic, then}
\[ \beta^n \sim \pi_{W_1}^{e_1} \cdots \pi_{W_k}^{e_k} \]
for some \( n \) and \( W = W_1 \supseteq \cdots \supseteq W_k \).

If \( f \) extends to \( \Delta \rightarrow V / W \), then can take \( n, e_1, \ldots, e_k \geq 0 \).

\[ \text{Maps}(\eta, V^\circ/W) \xrightarrow{\pi_1^\et} \hat{Br}_W \] \text{ conjugacy}
\[ f \mapsto [\beta_f] \]

\textbf{Df} \quad [\beta] \text{ algebraic iff } \exists f \text{ st } [\beta] = [\beta_f].

\textbf{Cor} \quad \text{If } \Sigma(\beta) \text{ is the set of Burau eigenvalues of } \beta, \text{ then}
\[ \max_{\lambda_q \in \Sigma(\beta)} \max_{|q|=1} \lambda_q = 1. \]

For \( S_n \curvearrowright \mathbb{C}^n \), stronger: reducible with periodic components.

\textbf{Proof of Thm} \quad \text{Reduce to } f \text{ lifting to } \tilde{f} : \Delta \rightarrow V. \text{ Then lift to \textit{wonderful compactification} } \tilde{V} \supseteq V^\circ:

\[ D = \tilde{V} - V^\circ \text{ has normal crossings.} \]
\[ \text{Winding numbers } e_i \text{ are intersection numbers with components of } D. \]

\textit{Image: Feichtner, MSRI 52 (2005), 333-360}