THE HILB-VS-QUOT CONJECTURE

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Abstract. Let R be the complete local ring of a complex plane curve germ and S its normalization. We propose a "Hilb-vs-Quot" conjecture relating the virtual weight polynomials of the Hilbert schemes of R to those of the Quot schemes that parametrize R-submodules of S. By relating the Quot side to a type of compactified Picard scheme, we show that our conjecture generalizes a conjecture of Cherednik's, and that it would relate the perverse filtration on the cohomology of the Picard side to a more elementary filtration. Next, we propose a Quot version of the Oblomkov–Rasmussen–Shende conjecture, relating parabolic refinements of our Quot schemes to Khovanov–Rozansky link homology. It becomes equivalent to the original version under (refined) Hilb-vs-Quot, but can also be strengthened to incorporate polynomial actions and y-ification. For germs $y^n = x^d$, where n is either coprime to or divides d, we prove the Quot version of ORS through combinatorics. When $n = 3$ and $3 \nmid d$, we deduce Hilb-vs-Quot by an asymptotic argument, and hence, establish the original ORS conjecture for these germs.

Contents

1. Introduction

1.1. Let R be the complete local ring of a complex algebraic plane curve germ: a reduced, complete, local $\mathbf{C}$-algebra of Krull dimension 1, embedding dimension at most 2, and residue field $\mathbf{C}$. Let $R \hookrightarrow S$ be the normalization of R. For any finitely-generated R-module E, let $\mathcal{Q}uot^\ell(E)$ denote the Quot scheme whose $\mathbf{C}$-points parametrize submodules of E of $\mathbf{C}$-codimension ℓ. It is a scheme of finite type. When $E = R$, it is the Hilbert scheme of ℓ points on $\mathrm{Spec}(R)$. We write

$$\mathcal{H}^\ell = \mathcal{Q}uot^\ell(R) \quad \text{and} \quad \mathcal{Q}^\ell = \mathcal{Q}uot^\ell(S).$$

For any $\mathbf{C}$-scheme of finite type X, let $\chi(X, \mathsf{t}) \in \mathbf{Z}[\mathsf{t}]$ denote the virtual weight polynomial of X in the sense of mixed Hodge theory. It satisfies the cut-and-paste relation $\chi(X, \mathsf{t}) = \chi(Z, \mathsf{t}) + \chi(X - Z, \mathsf{t})$ for any closed subscheme $Z \subseteq X$, and specializes to the Euler characteristic of X at $\mathsf{t} = -1$. Let

$$\mathsf{Hilb}(\mathsf{q}, \mathsf{t}) = \sum_{\ell \geq 0} \mathsf{q}^\ell \chi(\mathcal{H}^\ell, \mathsf{t}) \quad \text{and} \quad \mathsf{Quot}(\mathsf{q}, \mathsf{t}) = \sum_{\ell \geq 0} \mathsf{q}^\ell \chi(\mathcal{Q}^\ell, \mathsf{t}),$$

viewed as formal power series in q and t. We start by proposing the following conjecture relating the Hilbert schemes $\mathcal{H}^\ell$ to the Quot schemes $\mathcal{Q}^\ell$.

Conjecture 1 (Hilb-vs-Quot). *For any plane curve germ,*

$$\mathsf{Hilb}(\mathsf{q}, \mathsf{t}) = \mathsf{Quot}(\mathsf{q}, \mathsf{q}^{\frac{1}{2}}\mathsf{t}).$$

Our first goal in this paper is to show that Conjecture 1 extends a conjecture of Cherednik's to plane curve germs with multiple branches. Our second goal is to clarify a conjecture of Oblomkov–Rasmussen–Shende (ORS) relating the Hilbert schemes of a plane curve germ to the Khovanov–Rozansky (KhR) homology of its link. Under a parabolic refinement of Conjecture 1, the ORS Conjecture becomes equivalent to a version for the Quot schemes $\mathcal{Q}^\ell$ that is more tractable.

For germs of the form $y^n = x^d$, where n is either coprime to or divides d, we will prove the Quot version of ORS. For germs $y^3 = x^d$ with d coprime to 3, we will prove all of the conjectures above. In addition, we will propose a refinement to the Hilb-vs-Quot conjecture that incorporates known polynomial actions on the link homology and on its y-ification.

1.2. We first review Cherednik's conjecture. Let K be the ring of fractions of S. The compactified Picard scheme of R is a reduced ind-scheme $\overline{\mathcal{P}}$ over $\mathbf{C}$ whose points parametrize finitely-generated R-submodules $M \subseteq K$ such that $KM = K$. Let $c : \overline{\mathcal{P}}(\mathbf{C}) \to \mathbf{Z}_{\geq 0}$ be the constructible *gap function* given by

$$c(M) = \dim_{\mathbf{C}}((SM)/M).$$

Before Cherednik, versions of this function appeared in works of Greuel–Pfister [GP] and Gorsky–Mazin [GM13]. It takes values between $0 = c(S)$ and the delta invariant $\delta := c(R)$.

There is a sub-ind-scheme $\overline{\mathcal{J}} \subseteq \overline{\mathcal{P}}$ parametrizing those $M \subseteq K$ for which $\dim_{\mathbf{C}}(M/(M \cap R)) = \dim_{\mathbf{C}}(R/(M \cap R))$. It is a connected component of $\overline{\mathcal{P}}$, and for R irreducible, *a.k.a.* unibranch, it is a projective variety. In the unibranch case, $\overline{\mathcal{J}}$ has appeared under several different names, including the "Jacobi factor" or "(local) compactified Jacobian" of R.

For $0 \leq c \leq \delta$, let $\overline{\mathcal{J}}(c) \subseteq \overline{\mathcal{J}}$ be the constructible subvariety parametrizing the modules M where $c(M) = c$. In [C, Conj. 4.5], Cherednik essentially conjectured that when R is unibranch,

$$\mathsf{Hilb}(\mathsf{q}, \mathsf{t}) \stackrel{?}{=} \frac{1}{1 - \mathsf{q}} \sum_{0 \leq c \leq \delta} \mathsf{q}^c \chi(\overline{\mathcal{J}}(c), \mathsf{q}^{\frac{1}{2}}\mathsf{t}). \tag{1.1}$$

More precisely, the conjecture of *op. cit.* is stated in terms of point counts over finite fields, rather than virtual weight polynomials. For further discussion of how

Cherednik's conjecture relates to (1.1), see §2.7. In certain "torus knot" cases that we will discuss later, (1.1) was anticipated by Gorsky [ORS, Conj. A.12].

We will show that Conjecture 1 generalizes (1.1) beyond the unibranch case. For this, fix a uniformization $S \simeq \prod_{i=1}^{b} \mathbf{C}[\![\varpi_i]\!]$. The scaling action of $K^\times$ on $\overline{\mathcal{P}}$ restricts to a free action of the lattice $\Gamma \subseteq K^\times$ defined by

$$\Gamma = \{\vec{\varpi}^{\vec{x}} := \varpi_1^{x_1} \cdots \varpi_b^{x_b} \mid x_1, \ldots, x_b \in \mathbf{Z}\}.$$

The quotient $\overline{\mathcal{P}}/\Gamma$ is a projective variety, essentially by work of Kazhdan–Lusztig [KL]. In the unibranch case, where $b = 1$ and $\Gamma \simeq \mathbf{Z}$, we have $\overline{\mathcal{P}}/\Gamma \simeq \overline{\mathcal{J}}$.

Like before, let $\overline{\mathcal{P}}(c) \subseteq \overline{\mathcal{P}}$ parametrize the modules M where $c(M) = c$. As the function c is invariant under the $K^\times$-action on $\overline{\mathcal{P}}$, the Γ-action on $\overline{\mathcal{P}}$ descends to a Γ-action on $\overline{\mathcal{P}}(c)$. When $b = 1$, the identity

$$\mathsf{Hilb}(\mathsf{q}, \mathsf{t}) \stackrel{?}{=} \frac{1}{(1-\mathsf{q})^b} \sum_c \mathsf{q}^c \chi(\overline{\mathcal{P}}(c)/\Gamma, \mathsf{q}^{\frac{1}{2}}\mathsf{t}) \tag{1.2}$$

specializes to (1.1). We will prove that

$$\mathsf{Quot}(\mathsf{q}, \mathsf{t}) \stackrel{?}{=} \frac{1}{(1-\mathsf{q})^b} \sum_c \mathsf{q}^c \chi(\overline{\mathcal{P}}(c)/\Gamma, \mathsf{t}), \tag{1.3}$$

thereby proving that Conjecture 1 is equivalent to (1.2).

1.3. In fact, we will propose a stronger conjecture than Conjecture 1, and prove a stronger statement than (1.3).

Weierstrass preparation shows that after changing coordinates, we may assume that $R = \mathbf{C}[\![x]\!][y]/(f)$, where $f(x, y) = 0$ defines a generically separable cover of the x-axis fully ramified at $(x, y) = (0, 0)$. Let n be the degree of the cover. Then R forms a free $\mathbf{C}[\![x]\!]$-module of rank n. In particular, if E is torsion-free of rank 1 over R and $M \in \mathcal{Quot}^\ell(E)(\mathbf{C})$, then $\bar{M} := M/xM$ is a vector space of dimension n on which y acts nilpotently.

For any partial flag $F = (0 = \bar{M}^0 \subsetneq \bar{M}^1 \subsetneq \cdots \subsetneq \bar{M}^k = \bar{M})$, the parabolic type of F is the integer composition of n formed by the sequence $(\dim \mathrm{gr}_i^F(\bar{M}))_{i=1}^k$, where $\mathrm{gr}_i^F(\bar{M}) = \bar{M}^i/\bar{M}^{i-1}$. For any fixed composition ν of n, there is a scheme of finite type $\mathcal{Quot}_\nu^\ell(E)$ whose $\mathbf{C}$-points parametrize pairs (M, F) in which $M \subseteq E$ corresponds to a $\mathbf{C}$-point of $\mathcal{Quot}^\ell(E)$ and F is a y-stable flag on $\bar{M}$ of parabolic type ν.

Let $\mathcal{H}_\nu^\ell, \mathcal{Q}_\nu^\ell, \mathsf{Hilb}_\nu, \mathsf{Quot}_\nu$ be the generalizations of $\mathcal{H}^\ell, \mathcal{Q}^\ell, \mathsf{Hilb}, \mathsf{Quot}$ in which $\mathcal{Quot}_\nu^\ell$ replaces $\mathcal{Quot}^\ell$. Then Conjecture 1 has the refinement below:

Conjecture 2 (Parabolic Hilb-vs-Quot)**.** *For any R and ν as above,*

$$\mathsf{Hilb}_\nu(\mathsf{q}, \mathsf{t}) = \mathsf{Quot}_\nu(\mathsf{q}, \mathsf{q}^{\frac{1}{2}}\mathsf{t}).$$

There is an analogous scheme $\overline{\mathcal{P}}_\nu$, whose points parametrize pairs (M, F) as before, except that $M \subseteq K$ now corresponds to a $\mathbf{C}$-point of $\overline{\mathcal{P}}$. Let $\overline{\mathcal{P}}_\nu(c) \subseteq \overline{\mathcal{P}}_\nu$ be the preimage of $\overline{\mathcal{P}}(c) \subseteq \overline{\mathcal{P}}$. Let

$$\mathsf{Pic}_\nu(\mathsf{q}, \mathsf{t}) = \sum_{0 \le c \le \delta} \mathsf{q}^c \chi(\overline{\mathcal{P}}_\nu(c)/\Gamma, \mathsf{t}).$$

The most obvious refinement of (1.3) involves Quot_ν and Pic_ν. In fact, we can make the following motivic improvement:

Let $\mathsf{Sch}^{\mathrm{fin}}_{\mathbf{C}}$ be the category of $\mathbf{C}$-schemes of finite type, and for any object X of $\mathsf{Sch}^{\mathrm{fin}}_{\mathbf{C}}$, let $[X]$ denote the class of X in the Grothendieck ring $K_0(\mathsf{Sch}^{\mathrm{fin}}_{\mathbf{C}})$. Thus the virtual weight polynomial $\chi(X, \mathsf{t})$ is the specialization of $[X]$ along a ring homomorphism $K_0(\mathsf{Sch}^{\mathrm{fin}}_{\mathbf{C}}) \to \mathbf{Z}[\mathsf{t}]$. Let

$$\mathsf{Quot}^{mot}_\nu(\mathsf{q}) = \sum_\ell \mathsf{q}^\ell [\mathcal{Q}^\ell(\nu)],$$
$$\mathsf{Pic}^{mot}_\nu(\mathsf{q}) = \sum_c \mathsf{q}^c [\overline{\mathcal{P}}_\nu(c)/\Gamma],$$

the motivic analogues of Quot_ν and Pic_ν. In Section 2, we prove:

Theorem 3. *In the notation above, and for any integer composition ν of n,*

$$\mathsf{Quot}^{mot}_\nu(\mathsf{q}) = \frac{1}{(1-\mathsf{q})^b} \mathsf{Pic}^{mot}_\nu(\mathsf{q}).$$

The main idea is to embed $\coprod_\ell \mathcal{Q}^\ell$ into $\overline{\mathcal{P}}$, then relate ℓ to c by way of a certain fundamental domain for the Γ-action.

In Section 2.7, we explain in detail how Conjecture 2 and Theorem 3 imply a virtual weight analogue of Cherednik's conjecture. In Section 2.8, we illustrate them for $f(x, y) = y^2 - x^2$ and $f(x, y) = y^2 - x^3$.

1.4. We explain in Section 3 that Conjecture 2 and Theorem 3 can be rephrased in terms of symmetric functions, without reference to a composition ν.

Let $\Lambda^n_{\mathsf{q},\mathsf{t}}$ be the vector space of degree-n symmetric functions in infinitely many variables over $\mathbf{Q}(\mathsf{q}, \mathsf{t})$, and let $\langle -, - \rangle$ be the $\mathbf{Q}(\mathsf{q}, \mathsf{t})$-linear Hall inner product on $\Lambda^n_{\mathsf{q},\mathsf{t}}$ [Mac]. Let $(h_\mu)_\mu$, where μ runs over integer partitions of n, denote the basis of $\Lambda^n_{\mathsf{q},\mathsf{t}}$ formed by the complete homogeneous symmetric functions. Springer theory, repackaged using the work of Frobenius, shows that there are unique symmetric functions $\mathcal{F}\mathsf{Hilb}, \mathcal{F}\mathsf{Quot}, \mathcal{F}\mathsf{Pic} \in \Lambda^n_{\mathsf{q},\mathsf{t}}$ determined by the identities

$$\left.\begin{aligned} \mathsf{Hilb}_\nu(\mathsf{q}, \mathsf{t}) &= \langle h_\mu, \mathcal{F}\mathsf{Hilb}(\mathsf{q}, \mathsf{t}) \rangle, \\ \mathsf{Quot}_\nu(\mathsf{q}, \mathsf{t}) &= \langle h_\mu, \mathcal{F}\mathsf{Quot}(\mathsf{q}, \mathsf{t}) \rangle, \\ \mathsf{Pic}_\nu(\mathsf{q}, \mathsf{t}) &= \langle h_\mu, \mathcal{F}\mathsf{Pic}(\mathsf{q}, \mathsf{t}) \rangle \end{aligned}\right\} \qquad \text{whenever } \nu \text{ is a re-ordering of } \mu.$$

Now Conjecture 2 and Theorem 3 can be written in terms of $\mathcal{F}\mathsf{Hilb}, \mathcal{F}\mathsf{Quot}, \mathcal{F}\mathsf{Pic}$: See (3.2) and (3.3) in Section 3, respectively.

1.5. Henceforth, suppose that $f(x, y)$ is a polynomial in x as well as y. Fix a 3-sphere around $(0, 0) \in \mathbf{C}^2$. The intersection of the zero locus $\{f(x, y) = 0\}$ with this 3-sphere is a topological link L_f, whose isotopy class depends only on f when the sphere is small enough. The number of branches b is precisely the number of connected components of L_f.

There is an isotopy invariant of links taking values in triply-graded vector spaces, known as HOMFLYPT or Khovanov–Rozansky (KhR) homology [DGR, KhR]. In [ORS], Oblomkov–Rasmussen–Shende conjectured an identity expressing the KhR homology of L_f in terms of the Hilbert schemes $\mathcal{H}^\ell$. The full statement requires either nested versions of these Hilbert schemes or certain strata within them.

For any link L, let $\bar{\mathcal{P}}_{L,\mathrm{ORS}}(a,q,t)$ be the graded dimension of the unreduced KhR homology of L in the conventions of [ORS], so that our $\bar{\mathcal{P}}_L$ is their $\bar{\mathcal{P}}(L)$. We will use a normalization $\bar{\mathsf{X}}_f(\mathsf{a},\mathsf{q},\mathsf{t}) \in \mathbf{Z}[\![\mathsf{q}]\!][\mathsf{a}^{\pm 1},\mathsf{t}^{\pm 1}]$ satisfying

$$\bar{\mathcal{P}}_{L_f,\mathrm{ORS}}(a,q,t) = (aq^{-1})^{2\delta-b}\,\bar{\mathsf{X}}_f(a^2t,q^2,q^2t^2). \tag{1.4}$$

For any integer m, let $\mathcal{H}^{\ell}_{m\text{-}nest} \subseteq \mathcal{H}^{\ell} \times \mathcal{H}^{\ell+m}$ be the closed subscheme whose points parametrize those pairs (I,J) of R such that $xI + yI \subseteq J \subseteq I$. With this notation, the ORS Conjecture [ORS, Conj. 2] states that

$$\bar{\mathsf{X}}_f(\mathsf{a},\mathsf{q},\mathsf{q}\mathsf{t}^2) = \sum_{\ell,m} \mathsf{q}^{\ell}\mathsf{a}^{2m}\mathsf{t}^{m(m-1)}\chi(\mathcal{H}^{\ell}_{m\text{-}nest},\mathsf{t}). \tag{1.5}$$

Note that this conjecture would imply that the virtual weight polynomials above contain only even powers of t.

It was essentially observed in [GORS] that once we fix the presentation of $f(x,y) = 0$ as a degree-n cover of the x-axis, the right-hand side of (1.5) can be written in terms of $\mathcal{F}\mathsf{Hilb}$. Namely, let $\Psi(\mathsf{a},-) : \Lambda^n_{\mathsf{q},\mathsf{t}} \to \mathbf{Q}(\mathsf{q},\mathsf{t})[\mathsf{a}]$ be the map

$$\Psi(\mathsf{a},-) = (1+\mathsf{a}) \sum_{0\leq k\leq n-1} \mathsf{a}^k \langle s_{(n-k,1^k)}, -\rangle,$$

where $s_\mu \in \Lambda^n_{\mathsf{q},\mathsf{t}}$ denotes the Schur function indexed by $\mu \vdash n$. This specialization map also appears in [H16, Ex. 4] and [W, Cor. 1]. In Section 3, we explain using [GORS, Lem. 9.3–9.4] that the right-hand side of (1.5) is $\Psi(\mathsf{a},\mathcal{F}\mathsf{Hilb}(\mathsf{q},\mathsf{t}))$. So the ORS Conjecture asserts that

$$\bar{\mathsf{X}}_f(\mathsf{a},\mathsf{q},\mathsf{q}\mathsf{t}^2) = \Psi(\mathsf{a},\mathcal{F}\mathsf{Hilb}(\mathsf{q},\mathsf{t})).$$

In particular, if Conjecture 2 (the Parabolic Hilb-vs-Quot Conjecture) holds, then (1.5) is equivalent to the following conjecture:

Conjecture 4 (KhR-vs-Quot)**.** *In the setup above,*

$$\bar{\mathsf{X}}_f(\mathsf{a},\mathsf{q},\mathsf{t}^2) = \Psi(\mathsf{a},\mathcal{F}\mathsf{Quot}(\mathsf{q},\mathsf{t})). \tag{1.6}$$

We expect Conjecture 4 to be significantly more tractable than the original ORS conjecture, as we demonstrate in what follows.

Remark 1.1. As a trade-off, the Quot schemes $\mathcal{Q}^{\ell}$ do not deform as nicely as the Hilbert schemes $\mathcal{H}^{\ell}$ as we vary R in families, because in any versal deformation of R, we can only deform S jointly with R in the stratum where δ is constant [T].

Remark 1.2. Note that $\bar{\mathcal{P}}_{L,\mathrm{ORS}}(a,q,-1)$ is the unreduced HOMFLYPT polynomial of L originally introduced in [HOMFLY]. In the analogous $\mathsf{t} \to -1$ limit, the right-hand side of (1.5) specializes to a generating function for the Euler characteristics of the schemes $\mathcal{H}^{\ell}_{m\text{-}nest}$. In this limit, the ORS conjecture specializes to an earlier conjecture of Oblomkov–Shende [OS], which was proved by Maulik [Mau] using the wall-crossing results of [DHS].

Conjecture 2 would imply that the $\mathsf{t} \to -\mathsf{q}^{\frac{1}{2}}$ limit of the right-hand side of (1.6) records the same numbers. By contrast, its $\mathsf{t} \to -1$ limit records the Euler characteristics of analogous but different schemes $\mathcal{Q}^{\ell}_{m\text{-}nest}$.

Remark 1.3. When L is the link closure of a braid β, the KhR homology of L can be computed from the Rouquier complex $\bar{\mathcal{T}}_\beta$ in the theory of Soergel bimodules, as we explain in Appendix A. There is a richer invariant of $\bar{\mathcal{T}}_\beta$: its (dg) horizontal trace $\mathsf{Tr}_{\mathrm{dg}}(\bar{\mathcal{T}}_\beta)$. In [GHW], Gorsky–Hogancamp–Wedrich show that when β has n strands, $\mathsf{Tr}_{\mathrm{dg}}(\bar{\mathcal{T}}_\beta)$ decategorifies to an element of $\Lambda^n_{\mathsf{q},\mathsf{t}}$, and the KhR homology of L can be obtained by specializing $\mathsf{Tr}_{\mathrm{dg}}(\bar{\mathcal{T}}_\beta)$ along a version of Ψ. It is natural to expect that Conjecture 4 has a further refinement, comparing $\mathcal{F}\mathsf{Quot}$ directly to $\mathsf{Tr}_{\mathrm{dg}}(\bar{\mathcal{T}}_\beta)$. In this setting, it is natural to take β to be the positive braid arising from the preimage in $f(x,y) = 0$ of a positive loop around $x = 0$.

In [Tr], for any positive braid β on n strands, the second author introduced a (derived) scheme $\mathcal{Z}(\beta)$ with an action of GL_n and a Springer-type action of S_n on its GL_n-equivariant compactly-supported cohomology. The S_n-action on the associated graded of the weight filtration recovers the (full) KhR homology of the link closure of β, and conjecturally recovers an underived trace $\mathsf{Tr}(\bar{\mathcal{T}}_\beta)$. But we do not know a direct geometric relationship between $[\mathcal{Z}(\beta)/G]$ and the $\mathcal{Q}^\ell$.

1.6. We will establish Conjecture 4 for two infinite families of plane curve germs, both of the form $y^n = x^d$. Note that for any integer $d > 0$, the link of this plane curve is the positive (n,d) torus link.

In what follows, we write $\mathcal{F}\mathsf{Quot}_{n,d}$ and $\mathcal{F}\mathsf{Pic}_{n,d}$ for the symmetric functions $\mathcal{F}\mathsf{Quot}$ and $\mathcal{F}\mathsf{Pic}$ arising from $f(x,y) = y^n - x^d$. Similarly, we write $\bar{\mathsf{X}}_{n,d} = \bar{\mathsf{X}}_{y^n - x^d}$.

Theorem 5. *In the setup above, suppose that either of the following holds:*

(1) *d is coprime to n.*
(2) *d is a multiple of n.*

Then $\bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}^2) = \Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}_{n,d}(\mathsf{q},\mathsf{t}))$. *That is, Conjecture 4 holds for* $f(x,y) = y^n - x^d$.

Note that by [GMV20, (2)], the exponent $2\delta - b$ in (1.4) equals $nd - n - d$ in both cases of Theorem 5.

We prove case (1) of Theorem 5, the coprime case, in Section 4. We actually give two independent proofs:

(A) The first extends the combinatorial commutative algebra used to prove [ORS, Cor. A.5], thereby relating $\Psi(\mathcal{F}\mathsf{Quot}_{n,d})$ to the formula for $\bar{\mathsf{X}}_{n,d}$ conjectured by Gorsky–Neguţ in [GN] and proved by Mellit in [M22].
(B) The second proof is more roundabout: We invoke Theorem 3, then relate $\Psi(\mathcal{F}\mathsf{Pic}_{n,d})$ to $\bar{\mathsf{X}}_{n,d}$ by work of Hikita [Hi], Mellit [M21], Hogancamp–Mellit [HM], and Wilson [W]. Our new contribution is to match the filtration of $\overline{\mathcal{P}}/\Gamma$ induced by the function c with Hikita's filtration on an isomorphic variety. It turns out that they only match up to an involution discussed in [GM14].

The Gorsky–Neguţ formula in (A) implicitly involves certain semigroup modules and their generators, while the Hogancamp–Mellit recursion in (B) implicitly yields a formula for $\bar{\mathsf{X}}_{n,d}$ in terms of the "cogenerators" of these semigroup modules, by work of Gorsky–Mazin–Vazirani [GMV20]. As we will discuss in Section 4, these formulas have the same lowest a-degree: essentially, the q,t-Catalan number for

(n,d) in [H08]. However, they look very different in higher a-degrees. Thus it is remarkable that they both compute $\bar{\mathsf{X}}_{n,d}$.

We prove case (2), the case where $d = nk$ for some integer k, in Section 6. Here, the key is to recognize that the tools we need were developed in settings with extra structure: y-ified link homology on the KhR side, which we review in Section 5, and torus-equivariant homology on the Quot side. Thus we access $\mathcal{F}\mathsf{Quot}_{n,d}$ by way of its $T(b)$-equivariant analogue, where the torus $T(b) := \mathbf{G}_m^b$ acts on $\mathcal{Q}_\nu^\ell$ by scaling the uniformization $S \simeq \prod_{i=1}^b \mathbf{C}[\![\varpi_i]\!]$. Via work of Gorsky–Hogancamp [GH] and more recent work of Carlsson–Mellit [CM21], we respectively relate $\bar{\mathsf{X}}_{n,nk}$ and $\Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}_{n,d}(\mathsf{q},\mathsf{t}))$ to the same expression $\Psi(\mathsf{a}, \nabla^k p_{(1^n)})$, where ∇ is Bergeron–Garsia's nabla operator on symmetric functions and p_λ is the power-sum symmetric function for $\lambda \vdash n$. Note that $\nabla^k p_{(1^n)}$ is related to, but different from, the operator expressions in the shuffle conjecture of [HHLRU] and its rational generalization in [BGLX].

Both proof (B) of case (1) and the proof of case (2) involve comparisons to the affine Springer fibers studied in representation theory. In the former, we match $\overline{\mathcal{P}}_{(1^n)}/\Gamma$ with an affine Springer fiber for SL_n, studied in [Hi]; in the latter, we match $\coprod_\ell \mathcal{Q}^\ell$ with the positive part of an affine Springer fiber for GL_n, studied in [CM21]. These steps are relegated to Section 7.

Remark 1.4. Recently, Turner has computed the Borel–Moore homologies of many affine Springer fibers of "unramified" type for GL_3, generalizing the GL_3 case of [Ki]. For the corresponding plane curve germs, he has verified the $(\mathsf{a},\mathsf{q}) \to (0,1)$ limit of Conjecture 4, up to a certain localization [Tu].

1.7. Despite the claims in [HM, §1.2] and [GKS, §6.2], we believe there is no proof of the original ORS Conjecture that covers either of the two cases in Theorem 5. As we explain in Section 4, there does exist a combinatorial formula for $\mathcal{F}\mathsf{Hilb}_{n,d}$ when n and d are coprime, but it is much harder to match with $\bar{\mathsf{X}}_{n,d}$ than the analogous formula for $\mathcal{F}\mathsf{Quot}_{n,d}$.

Oblomkov–Rasmussen–Shende did verify their full conjecture when $f(x,y) = y^2 - x^d$ with d odd. As the map Ψ loses no information for $n \leq 3$, this implies Conjecture 2 for such f via case (1) of Theorem 5. Remarkably, we can use case (1) of Theorem 5 to prove:

Theorem 6. *Conjecture 2 holds for $f(x,y) = y^3 - x^d$ with d coprime to 3. Hence the original ORS conjecture (1.5) also holds for these cases.*

We give the proof at the end of Section 4. First, we show that for d coprime to n, the functions $\Psi(\mathsf{a}, \mathcal{F}\mathsf{Hilb}_{n,d}(\mathsf{q},\mathsf{t}))$ and $\Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}_{n,d}(\mathsf{q}, \mathsf{q}^{\frac{1}{2}}\mathsf{t}))$ match in the limit where $d \to \infty$: See Proposition 4.17. As the original functions agree with their limits up to order d in q, we can use symmetry properties on both sides to recover the finite identity from the asymptotic one. On the $\mathcal{F}\mathsf{Quot}_{n,d}$ side, the necessary symmetry arises via Theorem 5(1) from KhR homology, where it was conjectured in [DGR] and proved in [OR, GHM].

At the end of Section 6, we verify the lowest a-degree part of the original ORS Conjecture for $f(x,y) = y^2 - x^4$ and $f(x,y) = y^3 - x^3$, using the calculations in [Ki]. This proves Conjecture 1 for those f via case (2) of Theorem 5.

Remark 1.5. In proving Theorem 6, we also establish a closed formula for the (reduced) KhR homology of $(3, d)$ torus knots that had been conjectured by Dunfield–Gukov–Rasmussen [DGR, Conj. 6.3]. See the discussion in [ORS, §4].

1.8. Our proof of case (2) of Theorem 5, together with unpublished work of the first author, suggests a refinement of Conjecture 4: one with no analogue in the Hilbert-scheme setting of [ORS, Conj. 2].

In general, if L is a link with b components, then $\mathbf{C}[\vec{x}] := \mathbf{C}[x_1, \ldots, x_b]$ acts on the KhR homology of L by [GKS, Cor. 5.4]. This can be viewed as an action of the homology of the b-component unlink. The y-ified KhR homology of L, introduced by Gorsky–Hogancamp in [GH], is a monodromic deformation of its KhR homology that extends scalars from $\mathbf{C}$ to $\mathbf{C}[\vec{y}] := \mathbf{C}[y_1, \ldots, y_b]$, thereby extending the $\mathbf{C}[\vec{x}]$-action to a $\mathbf{C}[\vec{x}, \vec{y}]$-action.

We conjecture that these polynomial actions match existing, similar actions on the homology of the Quot schemes $\mathcal{Q}_\nu^\ell$. Letting

$$\Gamma_{\geq 0} = \{\vec{\varpi}^{\vec{x}} \mid \vec{x} \in \mathbf{Z}_{\geq 0}^b\} \subseteq \Gamma,$$

we see that $\Gamma_{\geq 0}$ acts on $\coprod_\ell \mathcal{Q}_\nu^\ell$ just as Γ acts on $\overline{\mathcal{P}}_\nu$. By construction, the $\Gamma_{\geq 0}$-action commutes with the $T(b)$-action from earlier. After identifying $\mathbf{C}[\vec{x}]$ with $\mathbf{C}[\Gamma_{\geq 0}]$, and $\mathbf{C}[\vec{y}]$ with the equivariant cohomology $\mathrm{H}^*_{T(b)}(pt)$, we obtain a $\mathbf{C}[\vec{x}, \vec{y}]$-action on the associated graded of the weight filtration $\mathsf{W}_{\leq *}$ on equivariant Borel–Moore homology, the latter defined via the cohomology of the dualizing complex:

$$\mathbf{C}[\vec{x}, \vec{y}] \curvearrowright \bigoplus_\ell \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_*^{\mathrm{BM}, T(b)}(\mathcal{Q}_\nu^\ell),$$

where the x_i shift ℓ by 1 and preserve weights, and the y_i preserve ℓ and shift weights by -2. As ν varies, these $\mathbf{C}[\vec{x}, \vec{y}]$-modules can be packaged together into a bigraded $(\mathbf{C}[\vec{x}, \vec{y}] \times \mathbf{C}S_n)$-module $\widetilde{\mathsf{Quot}}^{\vec{x},\vec{y}}$. The map Ψ extends by linearity to a functor from such bigraded modules to triply-graded $\mathbf{C}[\vec{x}, \vec{y}]$-modules. Abusing notation, we again write Ψ to denote this functor. We can now state the following refinement of Conjecture 4, with more explicit details left to Section 5.

Conjecture 7 (y-ified KhR-vs-Quot)**.** *Above, the y-ified KhR homology of L_f is isomorphic as a triply-graded $\mathbf{C}[\vec{x}, \vec{y}]$-module to $\Psi(\widetilde{\mathsf{Quot}}^{\vec{x},\vec{y}})$ after an appropriate regrading.*

1.9. In Section 8, we review and compare various filtrations on the Borel–Moore homology $\mathrm{H}_*^{\mathrm{BM}}(\overline{\mathcal{P}}/\Gamma)$. In particular, we establish a result needed in proof (B) of case (1) of Theorem 5, and describe one more application of the Hilb-vs-Quot Conjecture.

By taking unions of the strata $\overline{\mathcal{P}}(c)/\Gamma$, we produce a filtration of $\overline{\mathcal{P}}/\Gamma$ by closed subvarieties. We show that for $f(x, y) = y^n - x^d$ with n, d coprime, this filtration almost matches that introduced by Hikita on the affine Springer fiber in [Hi], but not quite: They differ by an involution that, at the combinatorial level, appeared in [GM14], and at the Lie-theoretic level, is induced by an involution of SL_n. Crucially, the involution preserves enough structure for us to match $\mathcal{F}\mathsf{Pic}_{n,d}$ with Hikita's symmetric function. For general R, we write $\mathsf{Q}_{\leq *}$ for the *gap filtration* induced by c on the Borel–Moore homology of $\overline{\mathcal{P}}/\Gamma$.

At the same time, Maulik–Yun constructed a filtration $\mathsf{P}_{\leq *}$ on the cohomology of $\overline{\mathcal{P}}/\Gamma$, by embedding a global analogue of $\overline{\mathcal{P}}$ within a versal family and using the decomposition theorem from the theory of perverse sheaves. They proved a formula relating $\mathsf{P}_{\leq *}\,\mathrm{H}^*(\overline{\mathcal{P}}/\Gamma)$ to the series $\mathsf{Hilb}(\mathsf{q},\mathsf{t})$ [MY], whose global version was also obtained in [MS]. The global version of $\mathsf{P}_{\leq *}$, under the name of perverse filtration, appears in geometric representation theory and nonabelian Hodge theory.

The increasing filtration $\mathsf{Q}_{\leq *}$ on $\mathrm{H}^{\mathrm{BM}}_*(\overline{\mathcal{P}}/\Gamma)$ defines a decreasing filtration $\mathsf{Q}^{\geq *}$ on $\mathrm{H}^*(\overline{\mathcal{P}}/\Gamma)$. Conjecture 1 would imply that at the level of virtual weight polynomials, $\mathrm{gr}^{\mathsf{P}}_*$ and $\mathrm{gr}^*_{\mathsf{Q}}$ match up to a certain shift. It is natural to make the following stronger conjecture, which also extends a conjecture in unpublished notes of Zhiwei Yun beyond the unibranch case.

Conjecture 8. *The weight grading on $\mathrm{gr}^{\mathsf{W}}_*\,\mathrm{H}^*(\overline{\mathcal{P}}/\Gamma)$ is supported in even degrees, and for all j, k, we have*

$$\mathrm{gr}^{\mathsf{P}}_{j+k}\,\mathrm{gr}^{\mathsf{W}}_{2k}\,\mathrm{H}^*(\overline{\mathcal{P}}/\Gamma) \simeq \mathrm{gr}^{j}_{\mathsf{Q}}\,\mathrm{gr}^{\mathsf{W}}_{2k}\,\mathrm{H}^*(\overline{\mathcal{P}}/\Gamma).$$

This would provide a purely elementary and local definition of $\mathrm{gr}^{\mathsf{P}}_*$, avoiding either the machinery of constructible derived categories or the need to embed the singularity within a family of global curves.

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2. Quot and Picard Schemes

2.1. The main goal of this section is to prove Theorem 3. We keep the definitions of R, S, K, b from the introduction.

2.2. First, we review the formal definition of the *compactified Picard scheme* [MY, §3.10]. Let $\mathfrak{m}_R$ be the maximal ideal of R, and for any R-module E, let $(-)\mathbin{\hat{\otimes}} E$ be the tensor product with E completed in the $\mathfrak{m}_R$-adic topology on E. Let $\overline{\mathcal{P}}_\dagger$ be the functor from $\mathbf{C}$-algebras to sets defined by

$$\overline{\mathcal{P}}_\dagger(A) = \left\{ \begin{array}{l|l} (A \mathbin{\hat{\otimes}} R)\text{-submodules} & \exists\, i \text{ such that } A \mathbin{\hat{\otimes}} \mathfrak{m}_R^i \subseteq M \subseteq A \mathbin{\hat{\otimes}} \mathfrak{m}_R^{-i} \\ M \subseteq A \mathbin{\hat{\otimes}} K & \text{and } (A \mathbin{\hat{\otimes}} \mathfrak{m}_R^{-i})/M \text{ is locally free over } A \\ & \text{of finite rank} \end{array} \right\}$$

for any $\mathbf{C}$-algebra A. An argument in [Gö, §2] shows that $\overline{\mathcal{P}}_\dagger$ is representable by an ind-scheme. Let $\overline{\mathcal{P}} \subseteq \overline{\mathcal{P}}_\dagger$ be the underlying reduced ind-scheme. Taking $A = \mathbf{C}$ recovers

$$\overline{\mathcal{P}}(\mathbf{C}) = \overline{\mathcal{P}}_\dagger(\mathbf{C}) = \{\text{finitely-generated } A\text{-submodules } M \subseteq K \mid KM = K\},$$

as in the introduction.

Remark 2.1. Even though $\overline{\mathcal{P}}_\dagger, \overline{\mathcal{P}}$ have the same $\mathbf{C}$-points, it is only $\overline{\mathcal{P}}$ that forms a scheme locally of finite type. For instance, if $R = \mathbf{C}[\![x]\!]$, then $\overline{\mathcal{P}}_\dagger \simeq x^{\mathbf{Z}} \times \overline{\mathcal{P}}_\dagger^{nil}$, where $\overline{\mathcal{P}}_\dagger^{nil}(A)$ parametrizes Laurent tails in x where each coefficient is a nilpotent element of A; by contrast, $\overline{\mathcal{P}} \simeq x^{\mathbf{Z}}$.

2.3. For any integer c, let $\overline{\mathcal{P}}(c) \subseteq \overline{\mathcal{P}}$ be the sub-ind-scheme defined by

$$\overline{\mathcal{P}}(c)(A) = \{M \in \overline{\mathcal{P}}(A) \mid (SM)/M \text{ is locally free over } A \text{ of rank } c\}.$$

Proposition 2.2. *If $M \in \overline{\mathcal{P}}_\dagger(A)$, then $(SM)/M$ is locally free over A of rank at most $\delta := \dim_{\mathbf{C}}(S/R)$.*

Proof. Observe that $(S \otimes_R M)/M$ is free over A of rank δ because

$$(S \otimes_R M)/M \simeq ((A \mathbin{\hat{\otimes}} S) \otimes_{A \hat{\otimes} R} M)/M \simeq (A \mathbin{\hat{\otimes}} S)/(A \mathbin{\hat{\otimes}} R).$$

Hence it suffices to show that $(SM)/M$ is a direct summand of $(S \otimes_R M)/M$ as an A-module.

Let $s_1, \ldots, s_\delta$ be a non-redundant (full) set of coset representatives for R in S. Then $SM = \sum_j (s_j + R)M = \sum_j s_j M$, so we can pick some subset $J \subseteq \{1, \ldots, \delta\}$, and $m_j \in M$ for $j \in J$, such that $\{s_j m_j\}_{j \in J}$ is a non-redundant set of coset representatives for M in SM. The A-linear map $(SM)/M \to (S \otimes_R M)/M$ that sends $s_j m_j + M \mapsto s_j \otimes m_j + 1 \otimes M$ is an A-linear section of the natural surjective map $(S \otimes_R M)/M \to (SM)/M$, as desired. □

Corollary 2.3. *$\overline{\mathcal{P}}$ is the union of the locally closed sub-ind-schemes $\overline{\mathcal{P}}(c)$ for $0 \leq c \leq \delta$. In fact, the locally closed subsets $\overline{\mathcal{P}}(c)$ form a stratification of $\overline{\mathcal{P}}$.*

Proof. It remains to explain why the $\overline{\mathcal{P}}(c)$ are locally closed: This follows from the upper semicontinuity of rank. □

2.4. Recall that we fix once and for all a uniformization $S \xrightarrow{\sim} \prod_{i=1}^b \mathbf{C}[\![\varpi_i]\!]$, and set $\Gamma = \{\varpi^{\vec{x}} \mid \vec{x} \in \mathbf{Z}^b\}$, where $\varpi^{\vec{x}} = \varpi_1^{x_1} \cdots \varpi_b^{x_b}$. The group Γ acts on $\overline{\mathcal{P}}$ by scaling. Adapting the proof of [KL, Cor. 1], one can check that $\overline{\mathcal{P}}/\Gamma$ is a projective variety. For all c, we have

$$(M, \vec{x}) \in \overline{\mathcal{P}}(c) \times \mathbf{Z}^b \implies \varpi^{\vec{x}} M \in \overline{\mathcal{P}}(c),$$

which lets us form the locally-closed subvariety $\overline{\mathcal{P}}(c)/\Gamma \subseteq \overline{\mathcal{P}}/\Gamma$.

For any finitely-generated R-submodule $E \subseteq K$, let $\mathcal{Q}uot^\ell(E)$ be the Quot scheme parametrizing submodules of E of codimension ℓ. There is a tautological map $\mathcal{Q}uot^\ell(E) \to \overline{\mathcal{P}}_\dagger$. Since $\mathcal{Q}uot^\ell(E)$ is reduced, the map factors through $\overline{\mathcal{P}}$. It identifies $\mathcal{Q}uot^\ell(E)$ with the subscheme of $\overline{\mathcal{P}}$ whose A-points are modules $M \subseteq A \mathbin{\hat{\otimes}} E$ such that $(A \mathbin{\hat{\otimes}} E)/M$ is locally free over A of rank ℓ.

As in the introduction, set $\mathcal{Q}^\ell = \mathcal{Q}uot^\ell(S)$ for all ℓ and $\Gamma_{\geq 0} = \{\varpi^{\vec{x}} \mid \vec{x} \in \mathbf{Z}^b_{\geq 0}\}$. We find that the free action of Γ on $\overline{\mathcal{P}}$ by scaling restricts to a free action of $\Gamma_{\geq 0}$ on $\coprod_\ell \mathcal{Q}^\ell$. Moreover, for all ℓ, we have

$$(M, \vec{x}) \in \mathcal{Q}^\ell \times \mathbf{Z}^b_{\geq 0} \implies \varpi^{\vec{x}} M \in \mathcal{Q}uot^{\ell + \mathrm{sum}(\vec{x})}(S), \tag{2.1}$$

where $\mathrm{sum}(\vec{x}) = e_1 + \cdots + e_b$.

Lemma 2.4. *Let $\mathcal{D} \subseteq \coprod_\ell \mathcal{Q}^\ell$ be the subscheme defined by*

$$\mathcal{D}(A) = \{M \subseteq A \mathbin{\hat{\otimes}} S \mid M \cap (A \mathbin{\hat{\otimes}} S)^\times \neq \emptyset\}.$$

Then $\mathcal{D}$ is a fundamental domain for both the Γ-action on $\overline{\mathcal{P}}$ and the $\Gamma_{\geq 0}$-action on $\coprod_\ell \mathcal{Q}^\ell$.

Proof. If $u = u(\varpi_1, \dots, \varpi_b)$ belongs to $(A \mathbin{\hat{\otimes}} S)^\times$, then the constant term of u must belong to $A^\times$. We deduce that if $M \in \mathcal{D}(A)$, then $\varpi^{\vec{x}} M \in \mathcal{D}(A)$ occurs only when $\vec{x}$ is the zero vector. Therefore $\mathcal{D}$ is irredundant under the action of Γ on $\overline{\mathcal{P}}$.

It remains to show that every element $M \in \overline{\mathcal{P}}(A)$, *resp.* $M \in \coprod_\ell \mathcal{Q}^\ell(A)$, takes the form $\varpi^{\vec{x}} M'$ for some $M' \in \mathcal{D}(A)$ and $\vec{x} \in \mathbf{Z}^b$, *resp.* $\vec{x} \in \mathbf{Z}^b_{\geq 0}$. Observe that $KM = A \mathbin{\hat{\otimes}} K$, because once we pick $i \geq 0$ such that $M \supseteq A \mathbin{\hat{\otimes}} \mathfrak{m}_R^i$, we obtain $KM \supseteq A \mathbin{\hat{\otimes}} K\mathfrak{m}_R^i = A \mathbin{\hat{\otimes}} K$. Therefore, $KM \ni 1$, which means we can find some $u \in (A \mathbin{\hat{\otimes}} K)^\times$ and $m \in M$ such that $um = 1$. This in turn means $m = u^{-1} \in M \cap (A \mathbin{\hat{\otimes}} K)^\times = \prod_{i=1}^b A(\!(\varpi_i)\!)^\times$.

In the case of $\coprod_\ell \mathcal{Q}^\ell$, we conclude as follows: Since $m \in A \mathbin{\hat{\otimes}} S = \prod_{i=1}^b A[\![\varpi_i]\!]$ as well, we get $m = \varpi^{\vec{x}} m'$ for some $\vec{x} \in \mathbf{Z}^b_{\geq 0}$ and $m' \in (A \mathbin{\hat{\otimes}} S)^\times$ by factoring out the largest powers of the uniformizers ϖ_i from m.

In the case of $\overline{\mathcal{P}}$, we conclude as follows: Write $m = (m_i)_{i=1}^b$ with $m_i \in A(\!(\varpi_i)\!)^\times$. The fact that $\overline{\mathcal{P}}$ is the underlying reduced ind-scheme of $\overline{\mathcal{P}}_\dagger$ means that we can assume, by reduction to the $b = 1$ case in Remark 2.1, that for all i, the coefficient of the lowest-degree term of m_i is a unit, not a nilpotent element, of A. Now we get $m = \varpi^{\vec{x}} m'$ for some $\vec{x} \in \mathbf{Z}^b$ and $m' \in (A \mathbin{\hat{\otimes}} S)^\times$ by factoring, as before. □

Lemma 2.5. *For any $\mathbf{C}$-algebra A and $M \in \coprod_\ell \mathcal{Q}^\ell(A)$, we have*

$$M \in \mathcal{D}(A) \overset{(1)}{\iff} SM = A \mathbin{\hat{\otimes}} S \overset{(2)}{\iff} \mathrm{rk}_A((SM)/M) = \mathrm{rk}_A((A \mathbin{\hat{\otimes}} S)/M).$$

In particular, $\mathcal{D}(\mathbf{C}) = \coprod_\ell \{M \in \mathcal{Q}^\ell(\mathbf{C}) \mid c(M) = \ell\}$.

Proof. Equivalence (2) holds because $SM \subseteq A \mathbin{\hat{\otimes}} S$. As for equivalence (1),

$$\begin{aligned} SM = A \mathbin{\hat{\otimes}} S &\iff SM \ni 1 \\ &\iff sm = 1 \text{ for some } s \in S \text{ and } m \in M \\ &\iff s'm = 1 \text{ for some } s' \in (A \mathbin{\hat{\otimes}} S)^\times \text{ and } m \in M \\ &\iff M \in \mathcal{D}(A). \end{aligned}$$

□

2.5. Using the Weierstrass preparation theorem, we now fix an isomorphism

$$R \simeq \mathbf{C}[\![x, y]\!]/(f)$$

such that $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbf{C}[\![x]\!])$ is a generically separable cover of degree n, fully ramified at $(x, y) = (0, 0)$. More explicitly, we may assume that $f(x, y)$ is a squarefree polynomial of the form $f(x, y) = y^n + \sum_{i=1}^n a_i(x) y^{n-i}$ for some $a_i(x) \in \mathbf{C}[\![x]\!]$ with $a_i(0) = 0$ for all i.

For any $\mathbf{C}$-algebra A and $M \in \overline{\mathcal{P}}(A)$, we write $\bar{M} = M/xM$, as in the introduction. We define a y-stable partial flag on $\bar{M}$ to be an increasing sequence of $A[y]$-submodules $F = (0 \subseteq \bar{M}^0 \subsetneq \bar{M}^1 \subsetneq \cdots \subsetneq \bar{M}^k = \bar{M})$ such that $\mathrm{gr}_i^F(\bar{M}) = \bar{M}^i/\bar{M}^{i-1}$

is locally free over A for all i. The *parabolic type* of F is the integer composition ν of n in which $\nu_i = \mathrm{rk}_A(\mathrm{gr}_i^F(\bar{M}))$. For any such composition ν, let $\overline{\mathcal{P}}_\nu$ be the ind-scheme defined by

$$\overline{\mathcal{P}}_\nu(A) = \left\{ (M, F) \,\middle|\, \begin{array}{l} M \in \overline{\mathcal{P}}_\nu(A), \\ F \text{ is a } y\text{-stable partial flag on } \bar{M} \text{ of type } \nu \end{array} \right\}$$

We define $\overline{\mathcal{P}}_\nu(c), \mathcal{Q}uot^\ell_\nu(E), \mathcal{D}_\nu$ analogously. Now, Corollary 2.3 and Lemmas 2.4–2.5 imply analogues where $\overline{\mathcal{P}}_\nu$, $\overline{\mathcal{P}}_\nu(c)$, $\mathcal{Q}^\ell_\nu$, $\mathcal{D}_\nu$ replace $\overline{\mathcal{P}}$, $\overline{\mathcal{P}}(c)$, $\mathcal{Q}^\ell$, $\mathcal{D}$.

2.6. **Proof of Theorem 3.** Recall that we want to show

$$\mathsf{Quot}^{mot}_\nu(\mathsf{q}) = \frac{1}{(1-\mathsf{q})^b}\,\mathsf{Pic}^{mot}_\nu(\mathsf{q}), \quad \text{where} \left\{ \begin{array}{l} \mathsf{Quot}^{mot}_\nu(\mathsf{q}) = \sum_\ell \mathsf{q}^\ell[\mathcal{Q}^\ell_\nu], \\ \mathsf{Pic}^{mot}_\nu(\mathsf{q}) = \sum_c \mathsf{q}^c[\overline{\mathcal{P}}_\nu(c)/\Gamma]. \end{array}\right.$$

Lemma 2.4 and (2.1) together imply that

$$\mathcal{Q}^\ell_\nu = \coprod_{\substack{(c,\vec{x}) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}^b_{\geq 0} \\ c + \mathrm{sum}(\vec{x}) = \ell}} \varpi^{\vec{x}} \cdot (\mathcal{D} \cap \mathcal{Q}^c_\nu).$$

So in the Grothendieck group of $\mathsf{Sch}^{\mathrm{fin}}_{\mathbf{C}}$, we have

$$\mathsf{Quot}^{mot}_\nu(\mathsf{q}) = \frac{1}{(1-\mathsf{q})^b} \sum_c \mathsf{q}^c[\mathcal{D} \cap \mathcal{Q}^c_\nu].$$

But Lemma 2.5 implies that $\mathcal{D}_\nu \cap \mathcal{Q}^c_\nu = \mathcal{D}_\nu \cap \overline{\mathcal{P}}_\nu(c)$ for all c. So we also have

$$\mathsf{Pic}^{mot}_\nu(\mathsf{q}) = \sum_c \mathsf{q}^c[\mathcal{D} \cap \overline{\mathcal{P}}_\nu(c)] = \sum_c \mathsf{q}^c[\mathcal{D} \cap \mathcal{Q}^c_\nu]$$

in the Grothendieck group, as desired.

Remark 2.6. Zhiwei Yun has pointed out to us that Theorem 3 extends beyond the planar case to any curve germ where both sides are well-defined, *i.e.*, where the functors $\overline{\mathcal{P}}/\Gamma$ and $\mathcal{Q}^\ell$ for $\ell \geq 0$ are all schemes of finite type.

However, Conjecture 1 fails for non-planar germs. If $R = \mathbf{C}[\![x, y, z]\!]/(xy, xz, yz)$, the union of the coordinate axes in xyz-space, then $S = \mathbf{C}[\![x]\!] \times \mathbf{C}[\![y]\!] \times \mathbf{C}[\![z]\!]$. Using [BRV, Prop. 6.1], we find that

$$\mathsf{Hilb}(\mathsf{q}, \mathsf{t}) = \frac{1}{(1-\mathsf{q})^3}(1 - 2\mathsf{q} + \mathsf{q}^2(\mathsf{t}^4 + \mathsf{t}^2 + 1) + \mathsf{q}^3(\mathsf{t}^4 - 2\mathsf{t}^2)).$$

By contrast, [Y, Ex. 2.7–2.8], [Ka], and Theorem 3 together give

$$\mathsf{Quot}(\mathsf{q}, \mathsf{q}^{\frac{1}{2}}\mathsf{t}) = \frac{1}{(1-\mathsf{q})^3}(1 - 2\mathsf{q} + \mathsf{q}^2(\mathsf{t}^2 + 1) + \mathsf{q}^3(\mathsf{t}^4 - 2\mathsf{t}^2) + \mathsf{q}^4\mathsf{t}^4).$$

It would be interesting to understand why the difference remains small.

2.7. **Cherednik's Conjecture.** Below, we explain how Conjecture 1 is essentially equivalent to Conjecture 4.5 of [C] via Theorem 3.

For any integer e, let $\overline{\mathcal{P}}^e \subseteq \overline{\mathcal{P}}$ be the sub-ind-scheme defined by

$$\overline{\mathcal{P}}^e(A) = \{M \in \overline{\mathcal{P}}(A) \mid e = \mathrm{rk}_A((A \mathbin{\hat{\otimes}} \mathfrak{m}_R^{-i})/M) - \mathrm{rk}_{\mathbf{C}}(\mathfrak{m}_R^{-i}/R) \text{ for all } i \gg 0\}$$

for any $\mathbf{C}$-algebra A. The ind-schemes $\overline{\mathcal{P}}^e$ are precisely the connected components of $\overline{\mathcal{P}}$. By definition, $\overline{\mathcal{J}} = \overline{\mathcal{P}}^0$.

The discussion in the introduction explained how Conjecture 1 and Theorem 3 would together imply (1.2). To see how (1.2) implies (1.1), we must explain why, in the unibranch case, $\overline{\mathcal{J}} = \overline{\mathcal{P}}/\Gamma$. Indeed, $b = 1$ implies that $S \simeq \mathbf{C}[\![\varpi]\!]$ and $\Gamma = \varpi^{\mathbf{Z}}$, which means Γ acts simply transitively on the connected components $\overline{\mathcal{P}}^e$.

Next, we explain how (1.1) is related to a point-counting analogue. Following Katz [Ka], recall that a class $[X] \in K_0(\mathsf{Sch}_{\mathbf{C}}^{\mathrm{fin}})$ is strongly polynomial count if and only if, for some finitely-generated subring $B \subseteq \mathbf{C}$, spreading out of $[X]$ to a class $[\mathcal{X}] \in K_0(\mathsf{Sch}_B^{\mathrm{fin}})$, and polynomial $p(X, \mathsf{t}) \in \mathbf{Z}[\mathsf{t}]$, we have $|\mathcal{X}_{\mathbf{F}}(\mathbf{F})| = p(X, q)$ for any finite field $\mathbf{F} = \mathbf{F}_q$ and ring morphism $B \to \mathbf{F}$. In this case, Katz shows that $p(X, \mathsf{t}^2)$ is precisely the virtual weight polynomial $\chi(X, \mathsf{t})$. We deduce that if $[\mathcal{H}^\ell]$ and $[\overline{\mathcal{J}}(c)]$ are strongly polynomial count for all ℓ and c, then (1.1) is equivalent to the statement that

$$\sum_\ell t^\ell |\mathcal{H}^\ell_{\mathbf{F}}(\mathbf{F})||_{q \to qt} \stackrel{?}{=} \frac{1}{1-t} \sum_c t^c |(\overline{\mathcal{J}}(c))_{\mathbf{F}}(\mathbf{F})| \tag{2.2}$$

for infinitely many (equivalently, all) finite fields $\mathbf{F} = \mathbf{F}_q$, where we have abused notation by conflating $\mathcal{H}^\ell$ and $\overline{\mathcal{J}}(c)$ with their spreadings out.

Lastly, we relate (2.2) to [C, Conj. 4.5]. In *loc. cit.*, Cherednik's $\mathcal{R}$ and $\mathcal{O} = \mathbf{C}[\![z]\!]$ are the respective analogues of our R and $S = \mathbf{C}[\![\varpi]\!]$ over $\mathbf{F}$. In particular, if they arise from R and S by spreading out, then:

- His $J_{\mathcal{R}}(\mathbf{F})$ is our $\coprod_c \varpi^{-c}\overline{\mathcal{J}}(c)_{\mathbf{F}}(\mathbf{F})$. In fact, this is also $\mathcal{D}_{\mathbf{F}}(\mathbf{F})$.
- His $\mathcal{H}^0_{mot}(q, t)$ is our $\sum_c t^c |\overline{\mathcal{J}}(c)_{\mathbf{F}}(\mathbf{F})|$.
- His $Z(q, t)$ is our $\sum_\ell t^\ell |\mathcal{H}^\ell_{\mathbf{F}}(\mathbf{F})|$.

In this case, (2.2) coincides with [C, Conj. 4.5].

2.8. We illustrate Conjecture 2 and Theorem 3 in minimal nontrivial examples.

Example 2.7. Take $f(x, y) = y^2 - x^2$. Setting $\varpi = y + x$ and $\varrho = y - x$ lets us write $R = \mathbf{C}[\![\varpi, \varrho]\!]/(\varpi\varrho)$ and $S = \mathbf{C}[\![\varpi]\!] \times \mathbf{C}[\![\varrho]\!]$.

For all integers i, j and $\lambda \in \mathbf{C}^\times$, consider the R-submodules of S given by $M_{i,j,\lambda} = \langle(\varpi^i, \lambda\varrho^j)\rangle$ and $N_{i,j} = \langle(\varpi^i, 0), (0, \varrho^j)\rangle$. We compute:

$$\overline{\mathcal{P}}^e(\mathbf{C}) = \{M_{i,j,\lambda} \mid i + j = e\} \sqcup \{N_{i,j} \mid i + j = e + 1\}.$$

With more work, one can check that $\overline{\mathcal{P}}^e$ is a $\mathbf{Z}$-indexed chain of projective lines intersecting transversely, in which the sets $\{M_{i,j,\lambda} \mid \lambda \in \mathbf{C}^\times\}$ correspond to copies of $\mathbf{G}_m$ and the points $N_{i,j}$ are the points of intersection. Embedding $\mathcal{H}^\ell$ and $\mathcal{Q}^\ell$ into $\overline{\mathcal{P}}$, we compute:

$$\begin{aligned}
\mathcal{H}^0(\mathbf{C}) &= \{M_{0,0}\},\\
\mathcal{H}^\ell(\mathbf{C}) &= \left\{ M_{i,j,\lambda} \;\middle|\; \begin{array}{l} i + j = \ell, \\ i, j \geq 1 \end{array} \right\} \sqcup \left\{ N_{i,j} \;\middle|\; \begin{array}{l} i + j = \ell + 1, \\ i, j \geq 1 \end{array} \right\} \quad \text{for } \ell \geq 1,\\
\mathcal{Q}^\ell(\mathbf{C}) &= \left\{ M_{i,j,\lambda} \;\middle|\; \begin{array}{l} i + j = \ell - 1, \\ i, j \geq 0 \end{array} \right\} \sqcup \left\{ N_{i,j} \;\middle|\; \begin{array}{l} i + j = \ell, \\ i, j \geq 0 \end{array} \right\} \quad \text{for } \ell \geq 0.
\end{aligned}$$

We deduce that

$$\begin{aligned}\mathsf{Hilb}^{mot}_{(2)}(\mathsf{q}) &:= \sum_\ell \mathsf{q}^\ell[\mathcal{H}^\ell] = 1 + \sum_{\ell \geq 1} \mathsf{q}^\ell(\ell + (\ell-1)[\mathbf{G}_m]),\\ \mathsf{Quot}^{mot}_{(2)}(\mathsf{q}) &:= \sum_\ell \mathsf{q}^\ell[\mathcal{Q}^\ell] = \sum_{\ell \geq 0} \mathsf{q}^\ell(\ell + 1 + \ell[\mathbf{G}_m]).\end{aligned}$$

Conjecture 1 predicts that these series match upon replacing $[\mathbf{G}_m]$ with $\chi(\mathbf{G}_m, \mathsf{t}) = \mathsf{t}^2 - 1$. Meanwhile, we compute $\mathcal{D} = \{N_{0,0}\} \sqcup \{M_{0,0,\lambda}\}$, where $c(N_{0,0}) = 0$ and $c(M_{0,0,\lambda}) = 1$ for all λ. We deduce that

$$\mathsf{Pic}^{mot}_{(2)}(\mathsf{q}) := \sum_c \mathsf{q}^c[\overline{\mathcal{P}}(c)] = 1 + \mathsf{q}[\mathbf{G}_m].$$

So for $\nu = (2)$, Theorem 3 says that $\mathsf{Quot}^{mot}_{(2)}(\mathsf{q}) = \frac{1}{(1-\mathsf{q})^2}(1 + \mathsf{q}[\mathbf{G}_m])$.

Example 2.8. Take $f(x, y) = y^2 - x^3$. Setting $x = \varpi^2$ and $y = \varpi^3$ lets us write $R = \mathbf{C}[\![\varpi^2, \varpi^3]\!]$ and $S = \mathbf{C}[\![\varpi]\!]$.

For all integers i and $\lambda \in \mathbf{C}$, consider the R-submodules of S given by $M_{i,\lambda} = \langle \varpi^i + \lambda\varpi^{i+1}\rangle$ and $N_i = \langle \varpi^i, \varpi^{i+1}\rangle$. We compute:

$$\overline{\mathcal{P}}^e(\mathbf{C}) = \{M_{d-1,\lambda} \mid \lambda \in \mathbf{C}\} \sqcup \{N_d\}.$$

One can check that $\overline{\mathcal{P}}^e$ is a projective line in which $\{M_{e-1,\lambda} \mid \lambda \in \mathbf{C}\}$ corresponds to $\mathbf{A}^1$ and and N_e corresponds to ∞. Embedding $\mathcal{H}^\ell$ and $\mathcal{Q}^\ell$ into $\overline{\mathcal{P}}$, we compute:

$$\begin{aligned}\mathcal{H}^0(\mathbf{C}) &= \{M_0\},\\ \mathcal{H}^1(\mathbf{C}) &= \{N_2\},\\ \mathcal{H}^\ell(\mathbf{C}) &= \{M_{\ell,\lambda}\} \sqcup \{N_{\ell+1}\} && \text{for } \ell \geq 2,\\ \mathcal{Q}^0(\mathbf{C}) &= \{N_0\},\\ \mathcal{Q}^\ell(\mathbf{C}) &= \{M_{\ell-1,\lambda}\} \sqcup \{N_\ell\} && \text{for } \ell \geq 1.\end{aligned}$$

We deduce that

$$\begin{aligned}\mathsf{Hilb}^{mot}_{(2)}(\mathsf{q}) &:= \sum_\ell \mathsf{q}^\ell[\mathcal{H}^\ell] = 1 + \mathsf{q} + \sum_{\ell \geq 2} \mathsf{q}^\ell[\mathbf{P}^1],\\ \mathsf{Quot}^{mot}_{(2)}(\mathsf{q}) &:= \sum_\ell \mathsf{q}^\ell[\mathcal{Q}^\ell] = 1 + \sum_{\ell \geq 1} \mathsf{q}^\ell[\mathbf{P}^1].\end{aligned}$$

Conjecture 1 predicts that these series match upon replacing $[\mathbf{P}^1]$ with $\chi(\mathbf{P}^1, \mathsf{t}) = \mathsf{t}^2 + 1$. Meanwhile, $\mathcal{D} = \{N_0\} \sqcup \{M_{0,\lambda}\}$, where $c(N_0) = 0$ and $c(M_{0,\lambda}) = 1$ for all λ. We deduce that

$$\mathsf{Pic}^{mot}_{(2)}(\mathsf{q}) := \sum_c \mathsf{q}^c[\overline{\mathcal{P}}(c)] = 1 + \mathsf{q}[\mathbf{A}^1].$$

So for $\nu = (2)$, Theorem 3 says that $\mathsf{Quot}^{mot}_{(2)}(\mathsf{q}) = \frac{1}{1-\mathsf{q}}(1 + \mathsf{q}[\mathbf{A}^1])$.

3. Springer Actions

3.1. In this section, we explain how the collection of polynomials $\{\chi(X_\nu, \mathsf{t})\}_\nu$, where X_ν is one of $\overline{\mathcal{P}}_\nu/\Gamma$, $\overline{\mathcal{P}}_\nu(c)/\Gamma$, $\mathcal{Q}uot^\ell_\nu(E)$, *etc.*, can be packaged into a single symmetric function. We also introduce variants of these schemes that we will need in Sections 4–6.

Throughout this section, we use the formalism of quotient stacks in the fpqc topology, but keep our exposition self-contained beyond the definition of a stack via its functor of points.

3.2. Fix an integer $n > 0$. Let $\mathcal{N}$ be the conical variety of nilpotent matrices in $\mathfrak{gl}_n$. By definition, $[\mathcal{N}/\mathrm{GL}_n]$ is the algebraic stack whose A-points form the groupoid of pairs (V, θ), where V is a locally-free A-module of rank n and θ is a nilpotent endomorphism of V, and an isomorphism of pairs $(V, \theta) \xrightarrow{\sim} (V', \theta')$ is an isomorphism of A-modules $V \xrightarrow{\sim} V'$ that transports θ onto θ'.

Recall that the GL_n-orbits on $\mathcal{N}$ are indexed by the integer partitions of n via Jordan type. Let $\mathcal{O}_\lambda \subseteq \mathcal{N}$ be the orbit indexed by $\lambda \vdash n$.

For each integer composition ν of n, let $\mathcal{B}_\nu$ be the flag variety of parabolic type ν, whose $\mathbf{C}$-points parametrize partial flags of type ν on $\mathbf{C}^n$. Let

$$\widetilde{\mathcal{N}}_\nu = \{(\theta, F) \in \mathcal{N} \times \mathcal{B}_\nu \mid F \text{ is } \theta\text{-stable}\}.$$

The A-points of $[\widetilde{\mathcal{N}}_\nu/\mathrm{GL}_n]$ form the groupoid of tuples (V, θ, F), where $(V, \theta) \in [\mathcal{N}/\mathrm{GL}_n](A)$ and F is an ν-stable partial flag of type ν on V in the sense of §3.1. Let $\pi = \pi_\nu : [\widetilde{\mathcal{N}}_\nu/\mathrm{GL}_n] \to [\mathcal{N}/\mathrm{GL}_n]$ be the forgetful map. If λ is the underlying partition of ν, and λ^t is the transpose of λ, then the image of π_ν is $[\overline{\mathcal{O}}_{\lambda^t}/\mathrm{GL}_n]$, the stack quotient of the orbit closure $\overline{\mathcal{O}}_{\lambda^t}$. In particular, $\mathcal{B}_{(1^n)}$ is the full flag variety and $\pi_{(1^n)}$ is a stacky version of the Springer resolution of $\mathcal{N}$.

Let X be any stack over $\mathbf{C}$ and $p : X \to [\mathcal{N}/\mathrm{GL}_n]$ a morphism. For each ν, let X_ν, $\pi_X = \pi_{X,\nu}$, and p_ν be defined by the cartesian square:

$$\begin{array}{ccc} X_\nu & \xrightarrow{p_\nu} & [\widetilde{\mathcal{N}}_\nu/\mathrm{GL}_n] \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{p} & [\mathcal{N}/\mathrm{GL}_n] \end{array} \tag{3.1}$$

In particular, taking $X = \overline{\mathcal{P}}$ and $p(M) = (\bar{M}, y)$ yields $X_\nu = \overline{\mathcal{P}}_\nu$. Analogous statements hold for $\overline{\mathcal{P}}(c)$, $\mathcal{Q}uot^\ell(E)$, and $\mathcal{D}_\nu$, as well as the quotients $\overline{\mathcal{P}}/\Gamma$, $\overline{\mathcal{P}}(c)/\Gamma$ once we observe that the map p for $X = \overline{\mathcal{P}}$ is invariant under Γ.

3.3. Now suppose that X is a scheme of finite type. In this case we write $\mathrm{H}^*_c(X)$ to denote the compactly-supported cohomology of X with complex coefficients, and $\mathsf{W}_{\leq *}$ to denote its weight filtration. The *virtual weight polynomial* of X is

$$\chi(X, \mathsf{t}) = \sum_{i,j} (-1)^i \mathsf{t}^j \dim \mathrm{gr}^{\mathsf{W}}_j \mathrm{H}^i_c(X)$$

by definition.

For any finite group G, we write $K_0(G)$ to denote its representation ring. When there is a weight-preserving action of G on $\mathrm{H}^*_c(X)$, we may regard $\chi(X, \mathsf{t})$ as an element of $\mathbf{Z}[\mathsf{t}] \otimes K_0(G)$.

Let $\mathbf{K}$ be a field. As in the introduction, let $\Lambda^n_{\mathbf{K}} = \Lambda^n_{\mathbf{K}}[\vec{X}]$ be the vector space of degree-n symmetric functions in a family of variables $\vec{X} = (X_i)_{i=1}^\infty$ over $\mathbf{K}$.

Let $\{s_\lambda\}_{\lambda\vdash n}$, *resp.* $\{h_\mu\}_{\mu\vdash n}$, be the basis of $\Lambda^n_{\mathbf{K}}$ of Schur functions, *resp.* complete homogeneous symmetric functions [Mac]. Let $\langle -,-\rangle$ be the $\mathbf{K}$-linear *Hall inner product* on $\Lambda^n_{\mathbf{K}}$ defined by orthonormality of the Schur functions. When $\mathbf{K}\supseteq\mathbf{Q}$, there is a $\mathbf{K}$-linear isomorphism

$$\mathcal{F}:\mathbf{K}\otimes K_0(S_n)\xrightarrow{\sim}\Lambda^n_{\mathbf{K}},$$

known as the *Frobenius character*. It sends the irreducible character of S_n indexed by λ to the Schur function s_λ, and the character of the induced representation $\mathrm{Ind}_{S_\nu}^{S_n}(1)$ to the complete homogeneous function h_μ, where $S_\mu\subseteq S_n$ is the Young subgroup of type μ.

Proposition 3.1. *If X is of finite type, then there is a weight-preserving action of S_n on $X_{(1^n)}$ such that $\mathrm{H}^*_c(X_\nu)=\mathrm{H}^*_c(X_{(1^n)})^{S_\nu}$ for all ν. In particular,*

$$\chi(X_\nu,\mathsf{t})=\langle h_\mu,\mathcal{F}\chi(X_{(1^n)},\mathsf{t})\rangle,$$

where μ is the integer partition obtained by sorting ν. Moreover, as we run over ν, these identities uniquely determine $\chi(X_{(1^n)},\mathsf{t})$ as an element of $\mathbf{K}\otimes K_0(S_n)$.

In what follows, we freely use functors between bounded derived categories of mixed complexes of sheaves with constructible cohomology, where "mixed" means we either use mixed Hodge modules, or spread out and reduce to a finite field to use mixed complexes of ℓ-adic sheaves, fixing an isomorphism $\bar{\mathbf{Q}}_\ell\simeq\mathbf{C}$.

Proof. For all ν, let $\mathcal{S}_\nu=\pi_{\nu,*}\mathbf{C}$ and $\mathcal{S}_{X,\nu}=\pi_{X,\nu,*}\mathbf{C}$.

Note that $\mathcal{S}_{(1^n)}$ is the GL_n-equivariant Springer sheaf. By the work of Lusztig *et al.* [Tr, Ch. 4], the underived endomorphism ring $\mathrm{End}(\mathcal{S}_{(1^n)})$ is pure of weight 0 and isomorphic to $\mathbf{C}S_n$. This defines an S_n-action on $\mathcal{S}_{(1^n)}$. Since (3.1) is a cartesian square, base change lifts this action to $\mathcal{S}_{X,(1^n)}\simeq p^*\mathcal{S}_{(1^n)}$. Taking hypercohomology, we get an action of S_n on $\mathrm{H}^*_c(X_{(1^n)})$. Since $\mathrm{End}(\mathcal{S}_{(1^n)})$ is concentrated in weight zero, the last action preserves weights.

For general ν, we have $\mathcal{S}_\nu\simeq\mathcal{S}_{(1^n)}^{S_\nu}$ by [BM, §2.7]. So again by the cartesian square (3.1), $\mathcal{S}_{X,\nu}\simeq\mathcal{S}_{X,(1^n)}^{S_\nu}$. Therefore

$$\mathrm{H}^*_c(X,\mathcal{S}_{X,\nu})\simeq\mathrm{H}^*_c(X,\mathcal{S}_{X,(1^n)}^{S_\nu})\simeq\mathrm{H}^*_c(X,\mathcal{S}_{X,(1^n)})^{S_\nu},$$

where the second step uses the fact that the inclusion $\mathcal{S}_{X,(1^n)}^{S_\nu}\subseteq\mathcal{S}_{X,(1^n)}$ is split (say, via the isotypic decomposition of $\mathcal{S}_{X,(1^n)}$). Above, the first expression is $\mathrm{H}^*_c(X_\nu)$ and the last expression is $\mathrm{H}^*_c(X_{(1^n)})^{S_\nu}$.

The statements about the Hall inner product and uniqueness now follow from Frobenius reciprocity and the fact that the h_μ span $\Lambda^n_{\mathbf{K}}$. ☐

Remark 3.2. We write $\mathrm{H}^{\mathrm{BM}}_*(X)$ to denote the *Borel–Moore homology* of X with complex coefficients, defined via the hypercohomology of the dualizing sheaf on X. Verdier duality implies that $\mathrm{H}^i_c(X)$ and $\mathrm{H}^{\mathrm{BM}}_{-i}(X)$ are dual vector spaces for all i. Therefore, Proposition 3.1 also implies that $\mathrm{H}^{\mathrm{BM}}_*(X_\nu)=\mathrm{H}^{\mathrm{BM}}_*(X_{(1^n)})_{S_\nu}$ for all ν, where $(-)_G$ denotes the coinvariants of a G-action.

3.4. For each integer $r \geq 0$, let $\mathcal{N}_{r\text{-}len} \subseteq \mathcal{N}$ be the union of the orbits indexed by partitions of length r, *i.e.*, the subvariety of nilpotent matrices θ such that $\dim \ker(\theta) = r$. Let $X_{r\text{-}len} = p^{-1}(\mathcal{N}_{r\text{-}len}) \subseteq X$.

As in the introduction, let $\Psi(\mathsf{a}, -) : \Lambda^n_{\mathbf{K}} \to \mathbf{K}[\mathsf{a}]$ be the map

$$\Psi(\mathsf{a}, -) = (1+\mathsf{a}) \sum_{0 \leq k \leq n-1} \mathsf{a}^k \langle s_{(n-k,1^k)}, - \rangle.$$

The following statement is a reformulation of [GORS, Lem. 9.3–9.4]:

Lemma 3.3. *We have*

$$\Psi(\mathsf{a}, \mathcal{F}\chi(X_{(1^n)}, \mathsf{t})) = \sum_{0 \leq r \leq n} \chi(X_{r\text{-}len}, \mathsf{t}) \prod_{0 \leq j \leq r-1} (1 + \mathsf{a}\mathsf{t}^{2j})$$

in $\mathbf{Q}(\mathsf{t})[\mathsf{a}]$.

3.5. For each integer $m \geq 0$, let $P_{n-m,m} \subseteq \mathrm{GL}_n$ be a parabolic subgroup whose Levi quotient is isomorphic to $\mathrm{GL}_{n-m} \times \mathrm{GL}_m$: for instance, the appropriate subgroup of block-upper-triangular matrices. Let $X_{m\text{-}nest}$ and $\rho_X = \rho_{X,m}$ be defined by the cartesian square:

$$\begin{array}{ccccc} X_{m\text{-}nest} & & \longrightarrow & & [pt/P_{n-m,m}] \\ \downarrow{\scriptstyle \rho_X} & & & & \downarrow \\ X & \xrightarrow{p} & [\mathcal{N}/\mathrm{GL}_n] & \longrightarrow & [pt/\mathrm{GL}_n] \end{array}$$

In the tautological case where $X = [\mathcal{N}/\mathrm{GL}_n]$, we can check that $X_{m\text{-}nest}$ is the stack whose A-points form the groupoid of tuples (V, θ, V'), where $(V, \theta) \in [\mathcal{N}/\mathrm{GL}_n](A)$ and V' is an A-submodule of $\ker(\theta)$ such that $\ker(\theta)/V'$ is locally free over A of rank m.

For all r, the map $\rho_{X,m}^{-1}(X_{r\text{-}len}) \to X_{r\text{-}len}$ is an fpqc-locally trivial fibration whose fiber is the Grassmannian of codimension-m subspaces of $\mathbf{C}^r$. The virtual weight polynomials of Grassmannians can be computed via their Schubert stratifications, which show them to be q-binomial coefficients for $q = \mathsf{t}^2$. Generalizing the argument in [ORS, GORS], we deduce:

Lemma 3.4. *We have*

$$\sum_{0 \leq m \leq n} \mathsf{a}^m \mathsf{t}^{m(m-1)} \chi(X_{m\text{-}nest}, \mathsf{t}) = \sum_{0 \leq r \leq n} \chi(X_{r\text{-}len}, \mathsf{t}) \prod_{0 \leq j \leq r-1} (1 + \mathsf{a}\mathsf{t}^{2j})$$

in $\mathbf{Q}(\mathsf{t})[\mathsf{a}]$.

3.6. We now return to the choices for X_ν that we need in the rest of the paper. Taking $X_\nu = \overline{\mathcal{P}}_\nu(c)$ for varying c, we set

$$\mathcal{F}\mathsf{Pic}(\mathsf{q}, \mathsf{t}) = \sum_c \mathsf{q}^c \mathcal{F}\chi(\overline{\mathcal{P}}_{(1^n)}(c), \mathsf{t}).$$

Fixing a finitely-generated R-module $E \subseteq K$ and taking $X_\nu = \mathcal{Q}uot^\ell_\nu(E)$ for varying ℓ, we set

$$\mathcal{F}\mathsf{Quot}_E(\mathsf{q}, \mathsf{t}) = \sum_\ell \mathsf{q}^\ell \mathcal{F}\chi(\mathcal{Q}uot^\ell_{(1^n)}(E), \mathsf{t}),$$

The symmetric functions $\mathcal{F}\mathsf{Hilb}, \mathcal{F}\mathsf{Quot}$ from the introduction are now given by

$$\mathcal{F}\mathsf{Hilb}(\mathsf{q},\mathsf{t}) = \mathcal{F}\mathsf{Quot}_R(\mathsf{q},\mathsf{t}) \quad \text{and} \quad \mathcal{F}\mathsf{Quot}(\mathsf{q},\mathsf{t}) = \mathcal{F}\mathsf{Quot}_S(\mathsf{q},\mathsf{t}).$$

Conjecture 2 can be rewritten as the single identity:

$$\mathcal{F}\mathsf{Hilb}(\mathsf{q},\mathsf{t}) \stackrel{?}{=} \mathcal{F}\mathsf{Quot}(\mathsf{q},\mathsf{q}^{\frac{1}{2}}\mathsf{t}). \tag{3.2}$$

And Theorem 3 can be rewritten as the single identity:

$$\mathcal{F}\mathsf{Quot}(\mathsf{q},\mathsf{t}) = \frac{1}{(1-\mathsf{q})^b}\mathcal{F}\mathsf{Pic}(\mathsf{q},\mathsf{t}). \tag{3.3}$$

Example 3.5. Let $\mathcal{B}_{\nu,[V,\theta]} \subseteq \mathcal{B}_\nu$ denote the fiber of π_ν above $[V,\theta]$. The Springer theory of GL_2 can be summarized as follows:

$$\mathcal{F}\chi(\mathcal{B}_{(1^2),[V,\theta]},\mathsf{t}) = \begin{cases} s_{(2)} & \theta \neq 0, \textit{ i.e.}, (V,\theta) \text{ of Jordan type } (2), \\ s_{(2)} + \mathsf{t}^2 s_{(1^2)} & \theta = 0, \textit{ i.e.}, (V,\theta) \text{ of Jordan type } (1^2). \end{cases}$$

Now take $f(x,y) = y^2 - x^3$. In the notation of Example 2.8, $(\bar{M}_{d-1,\lambda}, y)$ has Jordan type (2), while $(\bar{N}_d, y)$ has Jordan type (1^2). Thus we get:

$$\begin{aligned} \mathcal{F}\mathsf{Pic}(\mathsf{q},\mathsf{t}) &= (1+\mathsf{q}\mathsf{t}^2)s_{(2)} + \mathsf{t}^2 s_{(1^2)}, \\ \mathcal{F}\mathsf{Hilb}(\mathsf{q},\mathsf{t}) &= \left(1 + \mathsf{q} + \frac{\mathsf{q}^2(\mathsf{t}^2+1)}{1-\mathsf{q}}\right) s_{(2)} + \frac{\mathsf{q}\mathsf{t}^2}{1-\mathsf{q}} s_{(1^2)}, \\ \mathcal{F}\mathsf{Quot}(\mathsf{q},\mathsf{t}) &= \left(1 + \frac{\mathsf{q}(\mathsf{t}^2+1)}{1-\mathsf{q}}\right) s_{(2)} + \frac{\mathsf{t}^2}{1-\mathsf{q}} s_{(1^2)}. \end{aligned}$$

Here we can verify (3.2) and (3.3) directly.

Finally, for any finitely-generated R-module $E \subseteq K$, we spell out the meaning of $X_{r\text{-}len}$ and $X_{m\text{-}nest}$ when $X_\nu = \mathcal{Q}uot^\ell_\nu(E)$:

(1) $X_{r\text{-}len}$ is the locally-closed subscheme of $X = \mathcal{Q}uot^\ell(E)$ whose A-points are those $M \in X(A)$ such that $M/(xM + yM) \simeq \bar{M}/y\bar{M} \simeq \ker(y \mid \bar{M})$ is locally free over A of rank r.

(2) $X_{m\text{-}nest}$ is the scheme of finite type whose A-points parametrize pairs (M,N), where $M \in \mathcal{Q}uot^\ell(E)(A)$ and $N \in \mathcal{Q}uot^{\ell+m}(E)(A)$ and

$$xM + yM \subseteq N \subseteq M.$$

Note that these containments are together equivalent to requiring that $N/(xN + yN)$ be a submodule of $M/(xM + yM)$.

We henceforth write $\mathcal{Q}uot^\ell_{r\text{-}len}(E)$ and $\mathcal{Q}uot^\ell_{m\text{-}nest}(E)$ in place of $\mathcal{Q}uot^\ell(E)_{r\text{-}len}$ and $\mathcal{Q}uot^\ell(E)_{m\text{-}nest}$, respectively. Lemmas 3.3–3.4 imply:

Corollary 3.6. *We have*

$$\Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}_E(\mathsf{q},\mathsf{t})) = \sum_{\ell,m} \mathsf{q}^\ell \mathsf{a}^m \mathsf{t}^{m(m-1)} \chi(\mathcal{Q}uot^\ell_{m\text{-}nest}(E), \mathsf{t})$$

for any finitely-generated R-module $E \subseteq K$.

We set $\mathcal{H}^\ell_{m\text{-}nest} = \mathcal{Q}uot^\ell_{m\text{-}nest}(R)$ as in the introduction, and similarly, $\mathcal{Q}^\ell_{m\text{-}nest} = \mathcal{Q}uot^\ell_{m\text{-}nest}(S)$. In [OS, ORS], the $\mathcal{H}^\ell_{m\text{-}nest}$ are called *nested Hilbert schemes*.

4. Torus Knots

4.1. In this section, we give two independent proofs of case (1) of Theorem 5, stating that

$$\bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}^2) = \Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}_{n,d}(\mathsf{q},\mathsf{t})) \quad \text{for } d \text{ coprime to } n.$$

The structure of our two proofs and their relation to one another are summarized by this commutative diagram:

$$\begin{array}{ccccccc}
\text{EHA} & \xleftrightarrow{\text{[M21]}} & \text{Hikita} & \xleftrightarrow{\text{[Hi]}} & \mathcal{F}\mathsf{Pic}_{n,d}(\mathsf{q},\mathsf{t}) & \overset{\text{Thm 3}}{\longleftrightarrow} & \mathcal{F}\mathsf{Quot}_{n,d}(\mathsf{q},\mathsf{t}) \\
 & & \Big\downarrow{\scriptstyle \text{[M21, W]}} & & & & \vdots\downarrow{\scriptstyle \text{Cor 3.6, Prop 4.4}} \\
 & & \text{Cogen} & \xleftrightarrow{\text{[HM]}} & \bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}^2) & \xleftrightarrow{\text{[M22]}} & \text{Gen}
\end{array} \tag{4.1}$$

The horizontal arrows indicate identities; the vertical arrows indicate specializations. The dotted arrows indicate new bridges. Our first proof, labeled (A) in the introduction, follows the lower/right path from $\mathcal{F}\mathsf{Quot}_{n,d}$ to $\bar{\mathsf{X}}_{n,d}$; our second proof, labeled (B), follows the upper/left path.

4.2. For the rest of this section, fix an integer $d > 0$ coprime to n. Let $R = \mathbf{C}[\![x,y]\!]/(y^n - x^d)$. Setting $x = \varpi^n$ and $y = \varpi^d$ lets us write $R = \mathbf{C}[\![\varpi^n, \varpi^d]\!]$ and $S = \mathbf{C}[\![\varpi]\!]$ and $K = \mathbf{C}(\!(\varpi)\!)$, generalizing Example 2.8. Note that here,

$$\delta = \frac{1}{2}(n-1)(d-1)$$

by a classical formula of Sylvester. The number of branches is $b = 1$, so the link of the singularity has one component: *i.e.*, it is a knot.

4.3. Let $\mathbf{G}_m$ act on $\mathrm{Spec}(R)$ according to $t \cdot (x,y) = (t^n x, t^d y)$, and on $\mathrm{Spec}(K)$ according to $t \cdot \varpi = t\varpi$. These actions are compatible. In particular, they induce a $\mathbf{G}_m$-action on $\overline{\mathcal{P}}_\dagger$: If A is a $\mathbf{C}$-algebra and $t \in A^\times = \mathbf{G}_m(A)$ and $M \subseteq A \mathbin{\hat{\otimes}} K$ is an $(A \mathbin{\hat{\otimes}} R)$-module corresponding to an A-point of $\overline{\mathcal{P}}_\dagger$, then we define $t \cdot M$ to be the rescaling tM. This action restricts to $\overline{\mathcal{P}}$.

Let E be a finitely-generated R-submodule of K fixed by $\mathbf{C}^\times = \mathbf{G}_m(\mathbf{C})$. The $\mathbf{G}_m$-action on $\overline{\mathcal{P}}$ restricts to $\mathcal{Q}uot^\ell(E)$ for all ℓ. We use this action to skeletonize $\mathcal{Q}uot^\ell(E)$ into combinatorics. Let

$$\begin{aligned}
\Gamma(E) &= \{\mathrm{val}_\varpi(s) \mid s \in E \setminus \{0\}\}, \\
I(E) &= \{\Delta \subseteq \Gamma(E) \mid \Delta + n \subseteq \Delta,\ \Delta + d \subseteq \Delta\}, \\
I^\ell(E) &= \{\Delta \in I(E) \mid |\Gamma(E) \setminus \Delta| = \ell\}.
\end{aligned}$$

Note that $\Gamma(R) = n\mathbf{Z}_{\geq 0} + d\mathbf{Z}_{\geq 0}$ and $\Gamma(S) = \mathbf{Z}_{\geq 0}$.

Remark 4.1. In general, additive submonoids of $\mathbf{Z}_{\geq 0}$ are also known as *numerical semigroups*. A subset of $\mathbf{Z}$ stable under addition with a numerical semigroup Γ is also known a *Γ-module*. Thus $\Gamma(R)$ is a numerical semigroup, $\Gamma(E)$ is a $\Gamma(R)$-module, and $I(E)$ is the set of $\Gamma(R)$-submodules of $\Gamma(E)$.

For all $\Delta \in I(E)$, let

$$\begin{aligned}\mathsf{Gen}_n(\Delta) &= \{k \in \Delta \mid k - n \notin \Delta\},\\ \mathsf{Gen}(\Delta) &= \{k \in \Delta \mid k - n \notin \Delta,\, k - d \notin \Delta\}.\end{aligned}$$

The elements of $\mathsf{Gen}_n(\Delta)$, *resp.* $\mathsf{Gen}(\Delta)$, are called the *n-generators* [GMV20], *resp.* *generators*, of Δ. The following lemma can be proved by arguments completely analogous to those of Piontkowski in [P, §3], by taking $\mathcal{Q}uot^\ell(E)$ in place of $\overline{\mathcal{J}}$.

Lemma 4.2. *In the setup above, the $\mathbf{G}_m$-action on $\mathcal{Q}uot^\ell(E)$ has isolated fixed points. We have a bijection from $I^\ell(E)$ to the set of fixed $\mathbf{C}$-points, given by*

$$\begin{array}{rcl} I^\ell(E) & \xrightarrow{\sim} & \mathcal{Q}uot^\ell(E)^{\mathbf{G}_m},\\ \Delta & \mapsto & M_\Delta := R\langle t^k \mid k \in \Delta\rangle. \end{array}$$

Moreover, $\mathcal{Q}uot^\ell(E)$ is partitioned by the subschemes

$$\mathbf{A}_\Delta = \{M \in \mathcal{Q}uot^\ell(E) \mid \lim_{t\to 0}\,(t \cdot M) = M_\Delta\},$$

and each $\mathbf{A}_\Delta$ forms an affine space.

Although the partition above is similar to a Białynicki-Birula decomposition, it does not follow from said theorem when $\mathcal{Q}uot^\ell(E)$ is singular.

4.4. Recall the nested Quot schemes $\mathcal{Q}uot^\ell_{m\text{-}nest}(E)$ that we reviewed at the end of Section 3. The diagonal $\mathbf{G}_m$-action on $\mathcal{Q}uot^\ell(E) \times \mathcal{Q}uot^{\ell+m}(E)$ restricts to an action on $\mathcal{Q}uot^\ell_{m\text{-}nest}(E)$. Let

$$I^\ell_{m\text{-}nest}(E) = \{(\Delta, \Delta') \in I^\ell(E) \times I^{\ell+m}(E) \mid \Delta \supseteq \Delta' \supseteq \Delta + \Gamma_{E,>0}\}.$$

The following lemma is proved in [ORS, §3.3] for $E = R$, and the proof for any other $E \subseteq K$ is analogous.

Lemma 4.3. *The $\mathbf{G}_m$-action on $\mathcal{Q}uot^\ell_{m\text{-}nest}(E)$ has isolated fixed points. Writing $\Gamma_{E,>0} = \Gamma(E) \setminus \{0\}$, we have a bijection*

$$\begin{array}{rcl} I^\ell_{m\text{-}nest}(E) & \xrightarrow{\sim} & \mathcal{Q}uot^\ell_{m\text{-}nest}(E)^{\mathbf{G}_m},\\ (\Delta, \Delta') & \mapsto & (M_\Delta, M_{\Delta'}). \end{array}$$

Moreover, $\mathcal{Q}uot^\ell_{m\text{-}nest}(E)$ is partitioned by the subschemes

$$\mathbf{A}_{\Delta,\Delta'} = \{(M, M') \in \mathcal{Q}uot^\ell_{m\text{-}nest}(E) \mid \lim_{t\to 0}(M, M') = (M_\Delta, M_{\Delta'}\},$$

and each $\mathbf{A}_{\Delta,\Delta'}$ forms an affine space.

4.5. Given $\Delta \in I(E)$, let

$$\begin{aligned}\xi_n(\Delta, k) &= \{j \in \mathsf{Gen}_n(\Delta) \mid k - d < j < k\} \qquad \text{for all } k \in \mathsf{Gen}(\Delta),\\ \Pi_n^{\mathsf{Gen}}(\Delta, \mathsf{a}, \mathsf{t}) &= \prod_{k \in \mathsf{Gen}(\Delta)} (1 + \mathsf{a}\mathsf{t}^{|\xi_n(\Delta,k)|}).\end{aligned}$$

For $E = R$, the following proposition is [ORS, Cor. A.5]. To translate into the notation of [ORS, §A.1], note that our a, t correspond to their $a^2 t, t^2$, and hence, our $|\xi_n(\Delta, k)|$ corresponds to their $\beta_k(\Delta) - 1$. In the proof below, we merely list the changes needed to extend the proof to any $E \in \overline{\mathcal{P}}(\mathbf{C})$.

Proposition 4.4. *Let $R = \mathbf{C}[\![\varpi^n, \varpi^d]\!]$ for coprime $n, d > 0$. For any finitely-generated R-submodule $E \subseteq \mathbf{C}(\!(\varpi)\!)$ fixed by the $\mathbf{C}^\times$-action rescaling ϖ, we have*

$$\Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}_E(\mathsf{q}, \mathsf{t})) = \sum_\ell \mathsf{q}^\ell \sum_{\Delta \in I^\ell(E)} \mathsf{t}^{2\dim(\mathbf{A}_\Delta)} \Pi_n^{\mathsf{Gen}}(\Delta, \mathsf{a}, \mathsf{t}^2)$$

in the notation of Section 3.

Proof. By Corollary 3.6, it suffices to show that

$$\sum_{\Delta \in I^\ell(E)} \mathsf{t}^{2\dim(\mathbf{A}_\Delta)} \Pi_n^{\mathsf{Gen}}(\Delta, \mathsf{a}, \mathsf{t}^2) = \sum_{0 \le m \le n} \mathsf{a}^m \mathsf{t}^{m(m-1)} \chi(\mathcal{Q}uot^\ell_{m\text{-}nest}(E), \mathsf{t}) \tag{4.2}$$

for all $\ell \geq 0$.

Theorems 13 and 14 of [ORS] give formulas for $\dim(\mathbf{A}_\Delta)$ and $\dim(\mathbf{A}_{\Delta,\Delta'})$ in the $E = R$ case. For general E, analogous proofs give the dimension formulas

$$\dim(\mathbf{A}_\Delta) = \sum_i (|\Gamma(E)_{>\gamma_i} \setminus \Delta| - |\Gamma(E)_{>\sigma_i} \setminus \Delta|), \tag{4.3}$$

$$\begin{aligned} \dim(\mathbf{A}_{\Delta,\Delta'}) = \sum_{\substack{i \\ \gamma_i \notin \Delta'}} |\Gamma(E)_{>\gamma_i} \setminus \Delta| + \sum_{\substack{i \\ \gamma_i \in \Delta'}} |\Gamma(E)_{>\gamma_i} \setminus \Delta'| \\ - \sum_i |\Gamma(E)_{>\sigma_i} \setminus \Delta'| \end{aligned} \tag{4.4}$$

for any $\Delta \in I^\ell(E)$ with generators $\gamma_1, \ldots \gamma_r$, syzygies $\sigma_1, \ldots, \sigma_r$, and subset $\Delta' \in I^{\ell+m}(E)$ such that $(\Delta, \Delta') \in I^\ell_{m\text{-}nest}(E)$, where $\Gamma(E)_{>k} = \Gamma(E) \cap \mathbf{Z}_{>k}$.

Next, Lemma A.4 of *ibid.* shows that in the $E = R$ case, if $k \in \mathsf{Gen}(\Delta)$, then

$$|\xi_n(\Delta, k)| = |\{i \mid \gamma_i < k\}| - |\{i \mid \sigma_i < k\}|, \tag{4.5}$$

with the same notation for generators and syzygies as before. Then Lemma A.1 and Theorem A.2 of *ibid.* show that for $E = R$ and any fixed Δ, formulas (4.3), (4.4), and (4.5) together imply that

$$\mathsf{t}^{2\dim(\mathbf{A}_\Delta)} \Pi_n^{\mathsf{Gen}}(\Delta, \mathsf{a}, \mathsf{t}^2) = \sum_{0 \le m \le n} \mathsf{a}^m \mathsf{t}^{m(m-1)} \sum_{\substack{\Delta' \\ (\Delta,\Delta') \in I^\ell_{m\text{-}nest}(E)}} \mathsf{t}^{2\dim(\mathbf{A}_{\Delta,\Delta'})}.$$

The proofs of these statements for general E are the same. By Lemma 4.3, summing the last identity over all $\Delta \in I^\ell(E)$ recovers (4.2). $\square$

4.6. **Proof (A) of Case (1) of Theorem 5.** For all nonnegative integers j, we observe that

$$\Delta \in I^\ell(S) \implies \Delta + j \in I^{\ell+j}(S),$$
$$k \in \mathsf{Gen}(\Delta), \textit{ resp. } \mathsf{Gen}_n(\Delta) \implies k + j \in \mathsf{Gen}(\Delta + j), \textit{ resp. } \mathsf{Gen}_n(\Delta + j),$$

and consequently, that:

(1) $\dim(\mathbf{A}_{\Delta+j}) = \dim(\mathbf{A}_\Delta)$.
(2) $\xi_n(\Delta + j, k + j) = \xi_n(\Delta, k) + j$ for all $k \in \mathsf{Gen}(\Delta)$.

(3) $\Pi_n^{\mathsf{Gen}}(\Delta + j) = \Pi_n^{\mathsf{Gen}}(\Delta)$.

Now consider this combinatorial version of the domain $\mathcal{D}$ in Section 2:

$$D_{n,d} = \{\Delta \in I(S) \mid \min(\Delta) = 0\}.$$

By observations (1)–(3), the formula for $\Psi(\mathcal{F}\mathsf{Quot}(\mathsf{q},\mathsf{t}),\mathsf{a})$ in Proposition 4.4 equals

$$\frac{1}{1-\mathsf{q}} \sum_{\Delta \in D_{n,d}} \mathsf{q}^{|\mathbf{Z}_{\geq 0} \setminus \Delta|} \mathsf{t}^{2\dim(\mathbf{A}_\Delta)} \Pi_n^{\mathsf{Gen}}(\Delta, \mathsf{a}, \mathsf{t}^2).$$

It remains to match this formula with the formula for $\bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}^2)$ for coprime n, d conjectured in [GN] and proved in [M22]. It will be convenient to replace t with $\mathsf{t}^{\frac{1}{2}}$ everywhere in what follows.

In [GM13], Gorsky–Mazin gave a bijection from $D_{n,d}$ to the set of $n \times d$ rational Dyck paths, under which $|\mathbf{Z}_{\geq 0} \setminus \Delta|$ and $\dim(\mathbf{A}_\Delta)$ correspond to the statistics on Dyck paths respectively denoted area and codinv in [GMV20]. Explicitly, form the semi-infinite grid of unit squares in the x, y-plane whose vertices are the lattice points with $0 \leq x \leq d$ and $y \geq 0$. Label the bottom left square, closest to the origin, with the integer $-d$; label the other squares with integers that *decrease* by d as we go across rows, and *increase* by n as we go up columns. For instance, the grid for $(n, d) = (4, 5)$ is shown to the right, with non-negative labels in blue. For any $\Delta \in D_{n,d}$, the boundary of the region of squares with labels in Δ must contain a lattice path $\pi(\Delta)$ from $(x, y) = (0, 0)$ to $(x, y) = (d, n)$ that stays above the line $y = \frac{d}{n}x$, since Δ contains 0 and every element of Δ is nonnegative. Gorsky–Mazin's bijection sends $\Delta \mapsto \pi(\Delta)$.

19	14	9	4
15	10	5	0
11	6	1	-4
7	2	-3	-8
3	-2	-7	-12
-1	-6	-11	-16
-5	-10	-15	-20

Remark 4.5. In [GM13], the set $D_{n,d}$ is described as indexing the fixed points of a $\mathbf{G}_m$-action on $\overline{\mathcal{J}}$, rather than $\mathcal{D}$. However, this indexing really factors through the decomposition $\mathcal{D} = \coprod_c \varpi^{-c}\overline{\mathcal{J}}(c)$, corresponding to the fact that the elements of $D_{n,d}$ are what Gorsky–Mazin call *0-normalized* modules for $\Gamma(R)$. Compare to Section 2.7, where a similar remark applies to Cherednik's notation.

Let $\pi = \pi(\Delta)$ in what follows. Let

$$v_*(\pi) = \{(x, y) \in \pi \mid (x-1, y), (x, y+1) \in \pi\},$$
$$v^*(\pi) = \{(x, y) \in \pi \mid (x+1, y), (x, y-1) \in \pi\}.$$

Following [M22], we refer to elements of $v_*(\pi)$, *resp.* $v^*(\pi)$, as *inner vertices*, *resp.* *outer vertices*, of π. That is, inner vertices are the bottom right corners of squares whose bottom and right edges are contained in π, while outer vertices are the top left corners of squares whose top and left edges are contained in π. (Note that outer vertices are called "internal vertices"(!) in [GN].)

The squares whose bottom edges are contained in π are precisely those labelled by elements of $\mathsf{Gen}_n(\Delta)$. Of these, those whose right edges are also contained in π

are those labelled by elements of $\mathsf{Gen}(\Delta) \setminus \{0\}$. Hence, there is a bijection

$$\mathsf{Gen}(\Delta) \setminus \{0\} \xrightarrow{\sim} v_*(\pi)$$

sending any generator of Δ to the bottom right corner of the square it labels.

For an arbitrary lattice point p, let $l_{d/n}(p)$ be the line of slope $\frac{d}{n}$ through p, and let $\kappa_\pi(p)$ be the set of horizontal unit steps of π that intersect $l_{d/n}(p)$ in their interiors. The following lemma is inspired by the constructions in [GM13] and [ORS, §A].

Lemma 4.6. *If $k \in \mathsf{Gen}(\Delta) \setminus \{0\}$ labels the square with bottom right corner $p \in v_*(\pi)$, then the map $\xi_n(\Delta, k) \to \kappa_\pi(p)$ that sends k to the bottom edge of the square labelled k is a bijection. Thus*

$$\frac{1}{1+\mathsf{a}} \Pi_n^{\mathsf{Gen}}(\Delta, \mathsf{a}, \mathsf{t}) = \prod_{p \in v_*(\pi)} (1 + \mathsf{a}\mathsf{t}^{|\kappa_\pi(p)|}).$$

Proof. We observe that if p is the bottom right corner of a square labelled k, then the line $l_{d/n}(p)$ intersects the bottom edge of a square labelled j if and only if $k - d < j < k$. Indeed, this is easiest to see when $p = (n, d)$ and $k = 0$, and the general case follows from translating $l_{d/n}(n, d)$ onto $l_{d/n}(p)$. □

In our notation, the formula for $\bar{\mathsf{X}}_{n,d}$ for coprime n, d in [GN, M22] is:

$$\bar{\mathsf{X}}_{n,d}(\mathsf{a}, \mathsf{q}, \mathsf{t}) = \frac{1}{1-\mathsf{q}} \sum_{\substack{n \times d \\ \text{Dyck paths } \pi}} \mathsf{q}^{\mathsf{area}(\pi)} \mathsf{t}^{\mathsf{codinv}(\pi)} \prod_{p \in v^*(\pi)} (1 + \mathsf{a}\mathsf{t}^{|\kappa_\pi(p)|}),$$

where, by [GM13], $\mathsf{area}(\pi) = |\mathbf{Z}_{\geq 0} \setminus \Delta|$ and $\mathsf{codinv}(\pi) = \dim(\mathbf{A}_\Delta)$ whenever $\pi = \pi(\Delta)$. See the end of Appendix A for the precise matching of grading conventions. So by Lemma 4.6, it remains to show:

Lemma 4.7. *For any $n \times d$ Dyck path π as above,*

$$\prod_{p \in v_*(\pi)} (1 + \mathsf{a}\mathsf{t}^{|\kappa_\pi(p)|}) = \frac{1}{1+\mathsf{a}} \prod_{p \in v^*(\pi)} (1 + \mathsf{a}\mathsf{t}^{|\kappa_\pi(p)|}).$$

Proof. Since d and n are coprime, no two elements of $v_*(\pi) \cup v^*(\pi)$ have the same perpendicular distance to the line $l := l_{d/n}(n, d)$. The one farthest from l must belong to $v^*(\pi)$. Let p_0 be this element, and let $p_1, p_2, \ldots, p_m$ be the remaining elements ordered by decreasing distance from l. For $1 \leq i \leq m$, let

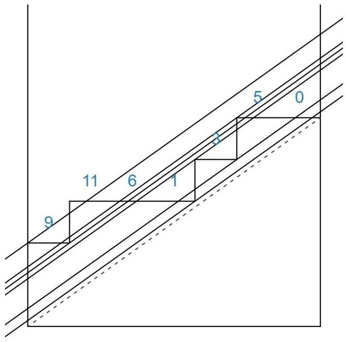
$$\epsilon_i = \begin{cases} -1 & (p_{i-1}, p_i) \in v_*(\pi) \times v_*(\pi), \\ 0 & (p_{i-1}, p_i) \in v_*(\pi) \times v^*(\pi) \cup v^*(\pi) \times v_*(\pi), \\ 1 & (p_{i-1}, p_i) \in v^*(\pi) \times v^*(\pi). \end{cases}$$

Let $\tau_i = \sum_{j \leq i} \epsilon_i$. Then for all i, we have $\tau_i = |\kappa_\pi(p_i)| \geq 0$.

If $m = 0$, then we are done; else, we must have $\tau_1 = \tau_m = 1$. It follows that every value attained by the sequence $\tau_1, \ldots, \tau_m$ must occur as many times for indices i with $p_i \in v_*(\pi)$ as for indices i with $p_i \in v^*(\pi) \setminus \{p_0\}$. □

Example 4.8. The figure below shows a 7×5 Dyck path π for which $|v_*(\pi)| = 3$ and $|v^*(\pi)| = 4$. The corresponding $\Delta \in D_{7,5}$ yields $\mathsf{Gen}_7(\Delta) = \{0, 5, 3, 1, 6, 11, 9\}$

and $\mathsf{Gen}(\Delta) = \{0, 3, 1, 9\}$.



In the notation of Lemma 4.7, $(\epsilon_i)_i = (1, 1, 0, 0, 0, -1)$ and $(\tau_i)_i = (1, 2, 2, 2, 2, 1)$.

Remark 4.9. Lemma 4.7 refines the last display on [M22, 60], which merely asserts that $\sum_{p \in v_*(\pi)} |\kappa_\pi(p)| = \sum_{p \in v^*(\pi)} |\kappa_\pi(p)|$.

4.7. **Proof (B) of Case (1) of Theorem 5.** We will explain each arrow in the left-hand portion of diagram (4.1).

For general f, the varieties $\overline{\mathcal{P}}_\nu$ are isomorphic to varieties that have been well-studied in representation theory: namely, parabolic affine Springer fibers for GL_n, where n is again the $\mathbf{C}[\![x]\!]$-rank of $R = \mathbf{C}[\![x]\!][y]/(f)$. In Proposition 7.4, we give the explicit isomorphisms for the case where $f(x, y) = y^n - x^d$ with n, d coprime, and show that for $\nu = (1^n)$, they match the Springer actions on the cohomologies of the two sides. Note that Hikita worked with SL_n, not GL_n, but we account for this difference by passing to $\overline{\mathcal{P}}^0 \simeq \overline{\mathcal{P}}/\Gamma$: See part (3) of the proposition.

For such f, both sides admit affine pavings induced by $\mathbf{G}_m$-actions, analogous to those in Lemmas 4.2–4.3. On the affine Springer fiber for $\nu = (1^n)$, Hikita introduced a q, t-symmetric function, jointly describing the dimensions of the strata and a certain filtration of the variety by unions of strata [Hi]. We review this filtration in Section 8.4.

Remark 4.10. Hikita's symmetric function is now known as the *Hikita polynomial* for (n, d). It was independently introduced by Armstrong at the 2012 AMS Joint Mathematics Meetings [A].

At the same time, there is a filtration of $\overline{\mathcal{P}}((1^n))/\Gamma$ by unions of the subvarieties $\overline{\mathcal{P}}(c)((1^n))/\Gamma$, which we review in Section 8.2. Theorem 8.3 says that it differs from Hikita's filtration by an involution ι on his affine Springer fiber. Lemma 8.2(2) says that on Borel–Moore homology, ι is Springer-equivariant and preserves weights. We deduce that the Hikita polynomial for (n, d) is unchanged by ι, and matches $\mathcal{F}\mathsf{Pic}_{n,d}$ once we invoke the duality between Borel–Moore homology and compactly-supported cohomology. His variables t, q correspond to our variables q, t^2.

The rational shuffle theorem, formulated for n, d coprime in [GN] and proved by Mellit in [M21], matches the Hikita polynomial with an expression denoted $\mathbf{Q}_{d,n} \cdot (-1)^n$ in [BGLX]. Here, $\mathbf{Q}_{d,n}$ is an element of the elliptic Hall algebra (EHA), and $(-1)^n$ is a vector in the Fock-space representation of the EHA on symmetric functions. Mellit's proof implicitly yields a recursive formula for $\mathbf{Q}_{d,n} \cdot (-1)^n$, and hence $\mathcal{F}\mathsf{Pic}_{n,d}$, in terms of the Dyck-path operators from his prior work with Carlsson [CM18]. This recursive formula is stated explicitly in [W, Thm. 2–3].

Remark 4.11. The shuffle conjecture of [HHLRU], proved by Carlsson–Mellit in [CM18], is the $d = n + 1$ case of the rational shuffle conjecture, except with an expression ∇e_n in place of $\mathbf{Q}_{n+1,n} \cdot (-1)^n$. It turns out that these symmetric functions coincide [BGLX, Thm. 7.4]. Haglund proved that ∇e_n specializes to the Gorsky–Neguţ formula for $\bar{\mathsf{X}}_{n,n+1}$, as a consequence of his formula for q, t-Schröder numbers [H04]. See the version in [H16, Thm. 2].

In [HM], Hogancamp–Mellit establish a recursive formula for the unreduced HOMFLYPT homology of the positive (n, d) torus link, for arbitrary n, d. In [W, Cor. 1], Wilson shows that for n, d coprime, Mellit's recursion for $\mathbf{Q}_{d,n} \cdot (-1)^n$ specializes under Ψ to Hogancamp–Mellit's recursion for the knot homology. This completes proof (B).

Remark 4.12. Gorsky–Mazin–Vazirani observed that the recursive formula of [HM] can be written in a closed form [GMV20]. It uses the same set of semigroup modules $D_{n,d}$ as in proof (A), but replaces $\Pi_n^{\mathsf{Gen}}(\Delta, \mathsf{a}, \mathsf{t})$ with

$$\Pi_n^{\mathsf{Cogen}}(\Delta, \mathsf{b}, \mathsf{t}) = \prod_{k \in \mathsf{Cogen}(\Delta)} (1 + \mathsf{b}\mathsf{t}^{\lambda(\Delta,k)}),$$

where the product runs over the set of *(nonnegative) cogenerators*

$$\mathsf{Cogen}(\Delta) = \{k \in \mathbf{Z}_{\geq 0} \setminus \Delta \mid k + n \in \Delta,\ k + d \in \Delta\},$$

and for any $k \in \mathsf{Cogen}(\Delta)$, we set

$$\begin{aligned} \lambda(\Delta, k) &= |\{j \in \mathsf{Gen}_n(\Delta) \mid k + n + 1 \leq j \leq k + n + d\}| \\ &= |\{j \in \mathsf{Gen}_n(\Delta) \mid k + n < j < k + n + d\}|. \end{aligned}$$

This explains why, in diagram (4.1), the bottom-left corner is labeled Cogen. To match the resulting formulas for $\bar{\mathsf{X}}_{n,d}$, set $\mathsf{b} = \mathsf{a}\mathsf{q}^{-1}$.

Remark 4.13. It is natural to ask how much of diagram (4.1) generalizes to integers n, d that are not coprime. We will address this question in a sequel paper. In Section 6, where we address the $d = nk$ case, our proof does *not* involve generalizing (4.1). For now, we mention that:

(1) The rational shuffle conjecture was generalized to arbitrary $n, d > 0$ in [BGLX]. This is the actual result proved by Mellit in [M21].
(2) Our Theorem 3 and the Cogen formula for $\bar{\mathsf{X}}_{n,d}$ extend to arbitrary n, d.
(3) In [W], Wilson introduces generalizations of $\mathbf{Q}_{d,n} \cdot (-1)^n$ and the Hikita polynomial to arbitrary n, d, which differ from those in [BGLX]. He has nonetheless shown that his Hikita polynomial specializes to the Cogen formula in (2), and hence, to $\bar{\mathsf{X}}_{n,d}$.

4.8. **Gen versus Cogen.** This subsection is a digression on Remark 4.12. As mentioned, the precise matching between the Gen and Cogen formulas is

$$\frac{1}{1+\mathsf{a}} \sum_{\Delta \in D_{n,d}} \mathsf{q}^{|\mathbf{Z}_{\geq 0} \setminus \Delta|} \mathsf{t}^{\dim(\mathbf{A}_\Delta)} \Pi_n^{\mathsf{Gen}}(\Delta, \mathsf{a}, \mathsf{t}) = \sum_{\Delta \in D_{n,d}} \mathsf{q}^{|\mathbf{Z}_{\geq 0} \setminus \Delta|} \mathsf{t}^{\dim(\mathbf{A}_\Delta)} \Pi_n^{\mathsf{Cogen}}(\Delta, \mathsf{a}\mathsf{q}^{-1}, \mathsf{t}). \tag{4.6}$$

It is remarkable because Gen and Cogen behave very differently. Note that at $\mathsf{a} \to 0$, the terms $\Pi_n^{\mathsf{Gen}}, \Pi_n^{\mathsf{Cogen}}$ disappear above, and both sides specialize to

$$\sum_{\Delta \in D_{n,d}} \mathsf{q}^{|\mathbf{Z}_{\geq 0} \setminus \Delta|} \mathsf{t}^{\dim(\mathbf{A}_\Delta)}.$$

Similarly, our proofs of case (1) of Theorem 5 simplify drastically in the $\mathsf{a} \to 0$ limit; almost all of their combinatorial complexity lies in the higher a-degrees.

Remark 4.14. Let $C_{n,d}(q,t) = C_{n,d}(t,q)$ be the q,t-rational Catalan number introduced in [H08]. Via their bijection from $D_{n,d}$ to the set of $n \times d$ Dyck paths, Gorsky–Mazin showed that the last polynomial above is $\mathsf{t}^\delta C_{n,d}(\mathsf{q}, \mathsf{t}^{-1})$ [GM13].

Below, we illustrate the contrast between Gen and Cogen in examples where $d = n+1$. Throughout, we label the elements of $D_{n,d}$ in the form $\Delta_{a_1,\ldots,a_n}$, where $\mathsf{Gen}_n = \{a_1, \ldots, a_n\}$ and $a_i + \delta - |\mathbf{Z}_{\geq 0} \setminus \Delta| \equiv i - 1 \pmod{n}$ for all i.

Example 4.15. Take $(n,d) = (2,3)$. Then $\delta = 1$ and $D_{2,3} = \{\Delta_{0,3}, \Delta_{1,0}\}$ with these statistics:

Δ	$\mathsf{q}^{\lvert\mathbf{Z}_{\geq 0} \setminus \Delta\rvert} \mathsf{t}^{\dim(\mathbf{A}_\Delta)}$	$\mathsf{Gen} \setminus \{0\}$	$\frac{1}{1+\mathsf{a}}\Pi_n^{\mathsf{Gen}}$	Cogen	Π_n^{Cogen}
$\Delta_{0,3}$	qt	$\emptyset$	1	$\{1\}$	$1+\mathsf{b}$
$\Delta_{1,0}$	1	$\{1\}$	$1+\mathsf{at}$	$\emptyset$	1

Here, (4.6) becomes $\mathsf{qt} + 1(1+\mathsf{at}) = \mathsf{qt}(1+\mathsf{aq}^{-1}) + 1$.

Example 4.16. Take $(n,d) = (3,4)$. Then $\delta = 3$ and

$$D_{3,4} = \{\Delta_{0,4,8}, \Delta_{5,0,4}, \Delta_{1,5,0}, \Delta_{4,2,0}, \Delta_{0,1,2}\}$$

with these statistics:

Δ	$\mathsf{q}^{\lvert\mathbf{Z}_{\geq 0} \setminus \Delta\rvert} \mathsf{t}^{\dim(\mathbf{A}_\Delta)}$	$\mathsf{Gen} \setminus \{0\}$	$\frac{1}{1+\mathsf{a}}\Pi_n^{\mathsf{Gen}}$	Cogen	Π_n^{Cogen}
$\Delta_{0,4,8}$	$\mathsf{q}^3\mathsf{t}^3$	$\emptyset$	1	$\{5\}$	$1+\mathsf{b}$
$\Delta_{5,0,4}$	$\mathsf{q}^2\mathsf{t}^2$	$\{5\}$	$1+\mathsf{at}$	$\{1,2\}$	$(1+\mathsf{b})(1+\mathsf{bt})$
$\Delta_{1,5,0}$	qt^2	$\{1\}$	$1+\mathsf{at}$	$\{2\}$	$1+\mathsf{b}$
$\Delta_{4,2,0}$	qt	$\{2\}$	$1+\mathsf{at}$	$\{1\}$	$1+\mathsf{b}$
$\Delta_{0,1,2}$	1	$\{1,2\}$	$(1+\mathsf{at})(1+\mathsf{at}^2)$	$\emptyset$	1

Here, (4.6) becomes

$$\mathsf{q}^3\mathsf{t}^3 + (\mathsf{q}^2\mathsf{t}^2 + \mathsf{qt}^2 + \mathsf{qt})(1+\mathsf{at}) + 1(1+\mathsf{at})(1+\mathsf{at}^2) = (\mathsf{q}^3\mathsf{t}^3 + \mathsf{qt}^2 + \mathsf{qt})(1+\mathsf{aq}^{-1}) + \mathsf{q}^2\mathsf{t}^2(1+\mathsf{aq}^{-1})(1+\mathsf{aq}^{-1}\mathsf{t}) + 1.$$

In general, one can check that there is a permutation $\mathsf{Row} : D_{n,d} \to D_{n,d}$ defined by $\mathsf{Cogen}(\mathsf{Row}(\Delta)) = \mathsf{Gen}(\Delta) \setminus \{0\}$. Nathan Williams has pointed out to us that Row ought to be an example of rowmotion, a certain operation on the order ideals of a finite poset [SW]. To see how, regard $\mathbf{Z}_{\geq 0} \setminus \Gamma(R)$ as a poset in which $j \leq k$ if and only if $k - j \in \Gamma(R)$, and the sets $\mathbf{Z}_{\geq 0} \setminus \Delta$ for $\Delta \in D_{n,d}$ as its order ideals. We would be curious to know whether rowmotion sheds any light on the relationship between the Gen and Cogen formulas.

4.9. **Proof of Theorem 6.** Recall that we want to show

$$\mathcal{F}\mathsf{Hilb}_{3,d}(\mathsf{q},\mathsf{t}) = \mathcal{F}\mathsf{Quot}_{3,d}(\mathsf{q},\mathsf{q}^{\frac{1}{2}}\mathsf{t})$$

for $d > 0$ coprime to 3. The first step is the asymptotic statement:

Proposition 4.17. *For any integer $n > 0$, we have*

$$\lim_{\substack{d\to\infty \\ d \text{ coprime to } n}} \Psi(\mathsf{a}, \mathcal{F}\mathsf{Hilb}_{n,d}(\mathsf{q},\mathsf{t})) = \prod_{1\leq k\leq n} \frac{1+\mathsf{a}\mathsf{q}^{k-1}\mathsf{t}^{2k-2}}{1-\mathsf{q}^{k}\mathsf{t}^{2k-2}},$$

$$\lim_{\substack{d\to\infty \\ d \text{ coprime to } n}} \Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}_{n,d}(\mathsf{q},\mathsf{t})) = \prod_{1\leq k\leq n} \frac{1+\mathsf{a}\mathsf{t}^{2k-2}}{1-\mathsf{q}\mathsf{t}^{2k-2}},$$

where the limits are taken in $\mathbf{Q}[\![\mathsf{q},\mathsf{t}]\!][\mathsf{a}]$.

Proof. Throughout, Corollary 3.6 allows us to replace the expressions $\Psi(\mathcal{F}\mathsf{Hilb}_{n,d})$ and $\Psi(\mathcal{F}\mathsf{Quot}_{n,d})$ with corresponding generating functions for nested pairs of R-modules, and Lemma 4.3 allows us to compute the latter using the combinatorics of the monomial R-modules.

The identity for $\mathcal{F}\mathsf{Hilb}_{n,d}$ was shown in [ORS]: See their Proposition 6. (Recall that our variables $\mathsf{a},\mathsf{q},\mathsf{t}$ correspond to their variables a^2t, q^2, t.) To prepare for the proof of the second identity, we briefly review their argument.

Using "staircase diagrams" [ORS, §3.2] to index monomial ideals, or equivalently elements $\Delta \in I(R)$, then invoking Lemma 4.2, it is not hard to show that the identity for $\mathcal{F}\mathsf{Hilb}_{n,d}$ holds when $\mathsf{a} = 0$. Indeed, as $d \to \infty$, the defining condition that staircase width be bounded by d disappears.

The formula that incorporates a can be bootstrapped from the $\mathsf{a} = 0$ formula by systematically replacing single elements Δ with collections of pairs (Δ'', Δ'). Namely, if Δ is fixed, then we consider all 2^n ways of choosing a subset of $\{1, \ldots, n\}$, and add a column of height h to the staircase of Δ for each h in the subset. This determines some new $\Delta' \in I(R)$. We get a larger $\Delta'' \supseteq \Delta$ by replacing each new column with a column that is one box shorter in height. We can then check that each Δ gives rise to 2^n pairs (Δ'', Δ'), that every possible pair arises this way, and that the total contribution of the pairs (Δ'', Δ') to the series in $\mathsf{a},\mathsf{q},\mathsf{t}$ is the contribution of Δ to the $\mathsf{a} = 0$ series multiplied by some binomial factor. This factor is precisely the numerator $\prod_{k=1}^{n} (1 + \mathsf{a}\mathsf{q}^{k-1}\mathsf{t}^{2k-2})$.

Now we turn to the identity for $\mathcal{F}\mathsf{Quot}_{n,d}$. In place of staircases, we index elements $\Delta \in I(S)$ by vectors $\vec{g} = (g_1, \ldots, g_n) \in \mathbf{Z}^n_{\geq 0}$, where g_i is the number of elements of $\Gamma(S) = \mathbf{Z}_{\geq 0}$ that are greater than exactly $i - 1$ of the elements of $\mathsf{Gen}_n(\Delta)$. Again, as $d \to \infty$, any constraints on the vector $\vec{g}$ disappear. If Δ

is indexed by $\vec{g}$, then its contribution to the $\mathsf{a} = 0$ series is $\mathsf{q}^{\sum_i g_i}\mathsf{t}^{2\sum_i (i-1)g_i}$ by Lemma 4.2.

To bootstrap the a variable, we send Δ to the collection of all pairs (Δ, Δ') where Δ is the same and $\Delta' \subseteq \Delta$ is obtained as follows: Pick a subset of $\{1, \dots, n\}$, then form $\mathsf{Gen}_n(\Delta')$ from $\mathsf{Gen}_n(\Delta)$ by shifting up by 1 those elements of $\mathsf{Gen}(\Delta)$ whose residue modulo n belongs to the subset. By Lemma 4.3, the total contribution of these pairs to the series in $\mathsf{a}, \mathsf{q}, \mathsf{t}$ is the contribution of the original Δ to the $\mathsf{a} = 0$ series multiplied by the binomial factor $\prod_{k=1}^n (1 + \mathsf{a}\mathsf{t}^{2k-2})$. □

Observe that $\Psi(\mathsf{a}, \mathcal{F}\mathsf{Hilb}_{n,d}(\mathsf{q}, \mathsf{t}))$ and $\Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}_{n,d}(\mathsf{q}, \mathsf{t}))$ agree with their $d \to \infty$ limits up to degree d in q. At the same time:

Proposition 4.18. *For any plane curve germ with complete local ring R, the series $\Psi(\mathsf{a}, \mathcal{F}\mathsf{Hilb}(\mathsf{q}, \mathsf{t}))$ is determined by its expansion up to degree δ in q. If $R \simeq \mathbf{C}[\![\varpi^n, \varpi^d]\!]$ for coprime $n, d > 0$, then the same holds for $\Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}(\mathsf{q}, \mathsf{t}))$.*

Proof. Observe that the expansion of a formal series $\Psi \in \mathbf{Z}[\![\mathsf{q}]\!][\mathsf{a}^{\pm 1}, \mathsf{q}^{-1}, \mathsf{t}^{\pm 1}]$ up to a given q-degree determines the expansion of $(1-\mathsf{q})^b\Psi$ up to that q-degree, for any integer $b > 0$.

Proposition 3 of [ORS] shows that if $\Psi = \Psi(\mathsf{a}, \mathcal{F}\mathsf{Hilb}(\mathsf{q}, \mathsf{t}))$ and b is the number of branches of R, then $\mathsf{q}^{-\delta}(1-\mathsf{q})^b\Psi$ is a Laurent polynomial in q-degrees $-\delta$ through δ, invariant under $\mathsf{q}^{-1} \mapsto \mathsf{q}\mathsf{t}^2$. (Again, our q is their q^2.) So in this case, the expansion of Ψ up to q-degree δ determines the entire series.

Now take $\Psi = \Psi(\mathsf{a}, \mathcal{F}\mathsf{Hilb}(\mathsf{q}, \mathsf{t}))$, supposing that $R \simeq \mathbf{C}[\![\varpi^n, \varpi^d]\!]$ for coprime $n, d > 0$. By case (1) of Theorem 5, Ψ matches the graded dimension of the unreduced KhR homology of the (n, d)-torus knot, up to certain grading shifts and substitutions. Hence, $(1-\mathsf{q})\Psi$ matches the corresponding series from *reduced* KhR homology, as defined in Appendix A. Corollary 1.0.2 of [OR] or Theorem 1.2 of [GHM] show that the latter, normalized with our conventions and shifted by $\mathsf{q}^{-\frac{\delta}{2}}$, is a Laurent polynomial in $\mathsf{q}^{\frac{1}{2}}$-degrees $-\delta$ through δ, invariant under $\mathsf{q}^{-\frac{1}{2}} \mapsto \mathsf{q}^{\frac{1}{2}}\mathsf{t}$. So again, the expansion of Ψ up to q-degree δ determines the entire series. □

Together, Proposition 4.17 and Proposition 4.18 imply that if $\delta \leq d - 1$, then

$$\Psi(\mathsf{a}, \mathcal{F}\mathsf{Hilb}(\mathsf{q}, \mathsf{t})) = \Psi(\mathsf{a}, \mathcal{F}\mathsf{Quot}(\mathsf{q}, \mathsf{q}^{\frac{1}{2}}\mathsf{t})).$$

But $\delta = \frac{1}{2}(n-1)(d-1)$. So the hypothesis can be simplified to $n \leq 3$. Finally, when $n \leq 3$, the map Ψ loses no information, so we can omit it from both sides. This proves Theorem 6.

5. Polynomial Actions and y-ification

5.1. In this section, we review the precise definition of y-ified Khovanov–Rozansky homology, then give a precise statement of Conjecture 7, spelling out all of the gradings involved. This also serves as preparation for Section 6.

5.2. We freely assume the notation of Appendix A. Thus, $T = \mathbf{G}_m^n$ and SBim is the category of Soergel bimodules over $\mathbf{S} = \mathrm{H}_T^*(pt)$. We explain in Appendix A that for any braid β on n strands, the Khovanov–Rozansky homology of the link closure of β can be computed from Hochschild cohomology of the Rouquier complex $\bar{\mathcal{T}}_\beta$, an object of $\mathsf{K}^b(\mathbf{S}\mathsf{Bim})$.

In [GKS, §5.1], the authors explain that the term-by-term action of $\mathbf{S} \otimes \mathbf{S}^{\mathrm{op}}$ on $\bar{\mathcal{T}}_\beta$ factors through that of a smaller quotient. Fix matching coordinates

$$\mathbf{S} = \mathbf{C}[t_1, \ldots, t_n] \quad \text{and} \quad \mathbf{S}^{\mathrm{op}} = \mathbf{C}[t_1^{\mathrm{op}}, \ldots, t_n^{\mathrm{op}}].$$

Let $w \in S_n$ be the underlying permutation of β. Then the actions of t_i and $t_{w(i)}^{\mathrm{op}}$ on $\bar{\mathcal{T}}_\beta$ are homotopic for all i. So up to homotopy, the $(\mathbf{S} \otimes \mathbf{S}^{\mathrm{op}})$-action on $\bar{\mathcal{T}}_\beta$ factors through the quotient of $\mathbf{S} \otimes \mathbf{S}^{\mathrm{op}}$ by the ideal $\langle (t_i - t_{w(i)}^{\mathrm{op}})_i \rangle$.

At the same time, the actions of t_i and t_i^{op} on $\mathbf{S}$ coincide for all i. So under the Hochschild cohomology functor $\overline{\mathsf{HH}} = \bigoplus_{i,j} \mathrm{Ext}^i_{\mathbf{S}\otimes\mathbf{S}^{\mathrm{op}}}(\mathbf{S}, (-)(j))$, the $(\mathbf{S} \otimes \mathbf{S}^{\mathrm{op}})$-action on $\bar{\mathcal{T}}_\beta$ is transported to an action that also factors through the quotient of $\mathbf{S} \otimes \mathbf{S}^{\mathrm{op}}$ by the ideal $\langle (t_i - t_i^{\mathrm{op}})_i \rangle$.

Thus, $\overline{\mathsf{HH}}(\bar{\mathcal{T}}_\beta)$ inherits an action of the quotient ring of w-coinvariants

$$\mathbf{S}_w := \mathbf{S}/\langle (t_i - t_{w(i)})_i \rangle.$$

This is a polynomial ring on b variables, where b is the number of components of the link closure of β. It will be convenient to fix coordinates

$$\mathbf{S}_w = \mathbf{C}[\vec{x}] := \mathbf{C}[x_1, \ldots, x_b]$$

so that each x_j is the image of some t_i. Recalling that Soergel bimodules are graded so that $\deg(t_i) = 2$, we see that $\vec{x}$ acts on $\overline{\mathsf{HH}}(\bar{\mathcal{T}}_\beta)$ with bidegree $(0, 2)$. Hence, $\vec{x}$ acts on $\overline{\mathsf{HHH}}(\bar{\mathcal{T}}_\beta) = \bigoplus_{I,J,K} \mathrm{H}^K(\overline{\mathsf{HH}}^{I,J}(\bar{\mathcal{T}}_\beta))$ with tridegree $(0, 2, 0)$.

5.3. In [GH], Gorsky–Hogancamp introduced a deformation of $\overline{\mathsf{HHH}}$ called *y-ified Khovanov–Rozansky homology*, which we will denote $\overline{\mathsf{HY}}$ and review below.

We write d for the differential on $\bar{\mathcal{T}}_\beta$. Let h_i be a homotopy from the t_i-action on T_β to the $t_{w(i)}^{\mathrm{op}}$-action, so that $[d, h_i] = t_i - t_{w(i)}^{\mathrm{op}}$ as operators. We may choose the h_i so that they square to zero and anticommute. Let $\mathbf{S}' = \mathbf{C}[u_1, \ldots, u_n]$ be another copy of $\mathbf{S}$, and let $d' = d \otimes \mathrm{id} + \sum_i h_i \otimes u_i$ as an operator on $\bar{\mathcal{T}}_\beta \otimes \mathbf{S}'$. We compute that $(d')^2 = \sum_i (t_i - t_{w(i)}^{\mathrm{op}}) \otimes u_i$. We deduce that the induced action of $(d')^2$ on $\overline{\mathsf{HH}}(\bar{\mathcal{T}}_\beta) \otimes \mathbf{S}'_w$ vanishes, where

$$\mathbf{S}'_w := \mathbf{S}'/\langle (u_i - u_{w(i)})_i \rangle,$$

like before. By definition, $\overline{\mathsf{HY}}(\bar{\mathcal{T}}_\beta) = \bigoplus_{I,J,K} \overline{\mathsf{HY}}^{I,J,K}(\bar{\mathcal{T}}_\beta)$, where

$$\overline{\mathsf{HY}}^{I,J,K}(\bar{\mathcal{T}}_\beta) = \mathrm{H}^K(\overline{\mathsf{HH}}^{I,J}(\bar{\mathcal{T}}_\beta) \otimes \mathbf{S}'_w, d').$$

We again fix coordinates

$$\mathbf{S}'_w = \mathbf{C}[\vec{y}] := \mathbf{C}[y_1, \ldots, y_b]$$

so that each y_j is the image of some u_i. From the definition of d', we see that $\vec{y}$ acts on the complex $(\overline{\mathsf{HH}}(\bar{\mathcal{T}}_\beta) \otimes \mathbf{S}'_w, d')$ with bidegree $(0, -2)$ on the first factor and cohomological degree 2. Hence, $\vec{y}$ acts on $\overline{\mathsf{HY}}(\bar{\mathcal{T}}_\beta)$ with tridegree $(0, -2, 2)$.

Altogether, the y-ified homology of β is a triply-graded vector space $\overline{\mathsf{HY}}(\bar{\mathcal{T}}_\beta)$ equipped with a bigraded $\mathbf{C}[\vec{x}, \vec{y}]$-module structure, which recovers $\overline{\mathsf{HHH}}(\bar{\mathcal{T}}_\beta)$ upon passing from $\mathbf{C}[\vec{x}, \vec{y}]$ to $\mathbf{C}[\vec{x}, \vec{y}]/\langle \vec{y} \rangle = \mathbf{C}[\vec{x}]$.

5.4. Writing e for the writhe of β, as in Appendix A, let $\bar{\mathsf{Y}}_\beta := \bigoplus_{i,j,k \in \mathbf{Z}} \bar{\mathsf{Y}}_\beta^{i, \frac{j}{2}, \frac{k}{2}}$ be the $(\mathbf{Z} \times \frac{1}{2}\mathbf{Z} \times \frac{1}{2}\mathbf{Z})$-graded $\mathbf{C}[\vec{x}, \vec{y}]$-module defined by

$$\bar{\mathsf{Y}}_\beta^{i, \frac{j}{2}, \frac{k}{2}} = \overline{\mathsf{HY}}^{i, e-2i+j-k, e-k}(\bar{\mathcal{T}}_\beta).$$

From the formula

$$\overline{\mathsf{HY}}^{I,J,K}(\bar{\mathcal{T}}_\beta) = \bar{\mathsf{Y}}_\beta^{I, I+\frac{J}{2}-\frac{K}{2}, e-\frac{K}{2}},$$

we see that

$$\bar{\mathsf{X}}_\beta(\mathsf{a}, \mathsf{q}, \mathsf{t}) = \sum_{i,j,k \in \mathbf{Z}} \mathsf{a}^i \mathsf{q}^{\frac{j}{2}} \mathsf{t}^{\frac{k}{2}} \dim(\bar{\mathsf{Y}}_\beta^{i, \frac{j}{2}, \frac{k}{2}} \otimes_{\mathbf{Z}[\vec{x}, \vec{y}]} \mathbf{Z}[\vec{x}])$$

in the notation of Appendix A. Moreover, we see that $\vec{x}$ and $\vec{y}$ respectively act on each summand $\bar{\mathsf{Y}}_\beta^i := \bigoplus_{j,k \in \mathbf{Z}} \bar{\mathsf{Y}}_\beta^{i, \frac{j}{2}, \frac{k}{2}}$ with bidegrees $(1, 0)$ and $(0, -1)$.

We return to our setup where $f(x, y) = 0$ is a generically separable degree-n cover of the x-axis, embedded in the x, y-plane. The preimage in the cover of a positively-oriented loop around $x = 0$ is a braid β_f on n strands such that the number b of branches of f is also the number of components of the link closure of β, and such that $\bar{\mathsf{X}}_f = \bar{\mathsf{X}}_{\beta_f}$. We similarly set $\bar{\mathsf{Y}}_f = \bar{\mathsf{Y}}_{\beta_f}$.

Let $T(b) = \mathbf{G}_m^b$. As explained in the introduction, once we fix identifications

$$\mathbf{C}[\vec{x}] \simeq \mathbf{C}[\Gamma_{\geq 0}] \quad \text{and} \quad \mathbf{C}[\vec{y}] \simeq \mathrm{H}^*_{T(b)}(pt),$$

the commuting actions of $\Gamma_{\geq 0}$ and $T(b)$ on $\coprod_\ell \mathcal{Q}_\nu^\ell$ together produce a $\mathbf{C}[\vec{x}, \vec{y}]$-module structure on $\bigoplus_\ell \mathrm{H}_*^{\mathrm{BM}, T(b)}(\mathcal{Q}_\nu^\ell)$ for all compositions ν of n. The variables x_j and y_j respectively act by 1 and 0 on the length ℓ, by 0 and -2 on the cohomological degree, and by 0 and -2 on the weight filtration $\mathsf{W}_{\leq *}$.

Let $\mathsf{Q}_{S,\nu}^{\vec{x}, \vec{y}} := \bigoplus_{\ell, k} \mathsf{Q}_{S,\nu}^{\vec{x}, \vec{y}, \ell, k}$ be the $\mathbf{Z}^2$-graded $\mathbf{C}[\vec{x}, \vec{y}]$-module defined by

$$\mathsf{Q}_{S,\nu}^{\vec{x}, \vec{y}, \ell, k} = \mathrm{gr}_k^{\mathsf{W}} \mathrm{H}_*^{\mathrm{BM}, T(b)}(\mathcal{Q}_\nu^\ell).$$

We abbreviate by writing $\tilde{\mathsf{Q}}_S^{\vec{x}, \vec{y}} = \mathsf{Q}_{S,(1^n)}^{\vec{x}, \vec{y}}$. The Springer action of S_n on the Borel–Moore homology of $\coprod_\ell \mathcal{Q}_{(1^n)}^\ell$ lifts to its equivariant Borel–Moore homology and commutes with the $\mathbf{C}[\vec{x}, \vec{y}]$-action above. So by Proposition 3.1, we can use the bigraded $(\mathbf{C}[\vec{x}, \vec{y}] \times \mathbf{C}S_n)$-module formed by $\tilde{\mathsf{Q}}_S^{\vec{x}, \vec{y}}$ to recover the bigraded $\mathbf{C}[\vec{x}, \vec{y}]$-modules $\mathsf{Q}_{S,\nu}^{\vec{x}, \vec{y}}$ for all ν.

Abusing notation, let Ψ be the functor from bigraded $\mathbf{C}S_n$-modules to triply-graded vector spaces given by

$$\Psi(M)^{i,j,k} = \bigoplus_{j,k} \mathrm{Hom}_{S_n}(V_{(n-i+1, 1^{i-1})} \oplus V_{(n-i, 1^i)}, M^{j,k}),$$

where in general, V_λ is the irreducible representation of S_n indexed by $\lambda \vdash n$. Altogether, the most precise version of Conjecture 7 is:

Conjecture 5.1. *In the setup above,*

(1) $\bar{\mathsf{Y}}_f$ *is supported in integral tridegrees.*

(2) *There is an isomorphism of* $\mathbf{C}[\vec{x},\vec{y}]$*-modules* $\bar{\mathsf{Y}}_f \xrightarrow{\sim} \Psi(\tilde{\mathsf{Q}}_S^{\vec{x},\vec{y}})$ *that sends degree* (i,j,k) *onto degree* $(i,j,2k)$*. In particular,* $\Psi(\tilde{\mathsf{Q}}_S^{\vec{x},\vec{y}})$ *is supported in even cohomological degrees.*

Remark 5.2. In the definition of $\tilde{\mathsf{Q}}_S^{\vec{x},\vec{y}}$, we did not collapse the cohomological degree to an Euler characteristic, as in the definition of $\mathsf{Quot}(\mathsf{q},\mathsf{t})$. Thus, the statement that $\Psi(\tilde{\mathsf{Q}}_S^{\vec{x},\vec{y}})$ is supported in even cohomological degrees is needed to ensure that Conjecture 5.1 specializes to Conjecture 4 upon base change from $\mathbf{C}[\vec{x},\vec{y}]$ to $\mathbf{C}[\vec{x}]$. An analogous statement about the cohomology of $\overline{\mathcal{P}}/\Gamma$ was shown in [GMO] for certain unibranch plane curve germs, called "generic" germs in *ibid.*

6. (n,nk) Torus Links

6.1. In this section, we prove case (2) of Theorem 5, stating that

$$\bar{\mathsf{X}}_{n,nk}(\mathsf{a},\mathsf{q},\mathsf{t}^2) = \Psi(\mathsf{a},\mathcal{F}\mathsf{Quot}_{n,nk}(\mathsf{q},\mathsf{t})) \quad \text{for any integer } k>0.$$

Throughout, we set $f(x,y) = y^n - x^{nk}$. For such f, our argument will implicitly prove Conjecture 5.1(1), as well as the matching of trigradings in Conjecture 5.1(2). The strategy is to relate both sides to $\nabla^k p_{(1^n)} \in \Lambda^n_{\mathsf{q},\mathsf{t}}$, where in general, p_λ is the power-sum symmetric function indexed by $\lambda \vdash n$, and ∇ is the Bergeron–Garsia operator on $\Lambda^n_{\mathsf{q},\mathsf{t}}$ [HHLRU]. We will use the theory of symmetric functions quite freely; for more background on our tools, we refer to [Ha, Mac].

6.2. In [CM21], Carlsson–Mellit computed a version of the underlying bigraded $\mathbf{C}S_n$-module of $\tilde{\mathsf{Q}}_S^{\vec{x},\vec{y}}$ for the chosen f. To make this precise, let

$$\tilde{\mathsf{Q}}^{\mathrm{BM},T(n)}_{S,n,nk}(\mathsf{q},\mathsf{t}) = \sum_{\ell,k} \mathsf{q}^\ell \mathsf{t}^k \mathrm{H}^{\mathrm{BM},T(n)}_k(\mathcal{Q}^\ell_{(1^n)}) \in \mathbf{Q}(\mathsf{q},\mathsf{t}) \otimes K_0(S_n).$$

Recall the Frobenius character $\mathcal{F} : \mathbf{Q}(\mathsf{q},\mathsf{t}) \otimes K_0(S_n) \to \Lambda^n_{\mathsf{q},\mathsf{t}}$ from Section 3.

Proposition 6.1. *For all integers* $n,k>0$*, we have*

$$\mathcal{F}\tilde{\mathsf{Q}}^{\mathrm{BM},T(n)}_{S,n,nk}(\mathsf{q},\mathsf{t}) = \frac{1}{(1-\mathsf{q})(1-\mathsf{t}^2)} \nabla^k p_n.$$

Proof. Just as the ind-schemes $\overline{\mathcal{P}}_\nu$ are isomorphic to parabolic affine Springer fibers for GL_n, so the ind-schemes $\coprod_\ell \mathcal{Q}^\ell_\nu$ are isomorphic to the *positive* parts of certain affine Springer fibers, in the terminology of [GK, CM21]. This can be shown by adapting the proof of [GK, Thm. 1.1]. In Proposition 7.2, we give the explicit isomorphisms for the case where $f(x,y) = y^n - x^{nk}$, and show that for $\nu = (1^n)$, they match the Springer actions on the two sides. In particular, we match $\coprod_\ell \mathcal{Q}^\ell_{(1^n)}$ for this choice of f with the ind-scheme denoted Z_k in [CM21].

There is an extra S_n-action on the $T(n)$-equivariant Borel–Moore homology of Z_k called the *dot action*, induced by the S_n-action on the homotopy type of the curve $y^n = x^{kn}$ that permutes its branches. The dot action commutes with the Springer action. In this way, we can upgrade $\mathcal{F}\tilde{\mathsf{Q}}^{\mathrm{BM},T(n)}_{S,n,nk}(\mathsf{q},\mathsf{t})$ to an element

$$\mathcal{F}_{\vec{X},\vec{Y}}\tilde{\mathsf{Q}}^{\mathrm{BM},T(n)}_{S,n,nk}(\mathsf{q},\mathsf{t}) \in \Lambda^n_{\mathsf{q},\mathsf{t}}[\vec{X},\vec{Y}],$$

where $\Lambda_{\mathsf{q},\mathsf{t}}[\vec{X}, \vec{Y}] = \Lambda_{\mathsf{q},\mathsf{t}}[\vec{X}] \otimes_{\mathbf{Q}(\mathsf{q},\mathsf{t})} \Lambda_{\mathsf{q},\mathsf{t}}[\vec{Y}]$. Above, $\vec{X}$ and $\vec{Y}$ respectively record the Springer and dot actions. The actual statement of [CM21, Thm. A] is

$$\mathcal{F}_{\vec{X},\vec{Y}} \tilde{\mathsf{Q}}^{\mathrm{BM},T(n)}_{S,n,nk}(\mathsf{q}, \mathsf{t}^{\frac{1}{2}}) = \nabla^k e_n \left[\frac{\vec{X}\vec{Y}}{(1-\mathsf{q})(1-\mathsf{t})}\right],$$

in plethystic notation.

We want to recover the Frobenius character in $\vec{X}$ alone. To this end, it suffices to pair the right-hand side with $p_{(1^n)}[\vec{Y}]$ under the Hall inner product: Indeed, under $\mathcal{F}$, pairing with $p_{(1^n)}$ corresponds to evaluating a character of S_n at the identity element. Note that $(g, h) \mapsto \langle g[\frac{\vec{X}\vec{Y}}{(1-\mathsf{q})(1-\mathsf{t})}], h\rangle$ is a version of the Macdonald q,t-inner product [Ha, §3.5], with respect to which the power-sum symmetric functions form an orthogonal basis of $\Lambda^n_{\mathsf{q},\mathsf{t}}$. Therefore

$$\begin{aligned}\left\langle \nabla^k e_n \left[\frac{\vec{X}\vec{Y}}{(1-\mathsf{q})(1-\mathsf{t})}\right], p_{(1^n)}[\vec{Y}] \right\rangle &= \nabla^k p_{(1^n)} \left[\frac{\vec{X}}{(1-\mathsf{q})(1-\mathsf{t})}\right] \\ &= \frac{1}{(1-\mathsf{q})^n(1-\mathsf{t})^n} \nabla^k p_{(1^n)}[\vec{X}],\end{aligned}$$

where the second equality used $p_{(1^n)} = p_1^n$. Finally, substituting t^2 for t everywhere gives the statement in the proposition. □

Remark 6.2. Interestingly, the fundamental domain $\mathcal{D}_{(1^n)}$ from Lemma 2.4 and its ensuing discussion appears implicitly in [CM21]: Its complement is an open sub-ind-scheme of Z_k that features heavily in the proof of [CM21, Thm. A].

Corollary 6.3. *For all integers $n, k > 0$, we have*

$$\mathcal{F}\mathsf{Quot}_{n,nk}(\mathsf{q}, \mathsf{t}) = \frac{1}{(1-\mathsf{q})^n} \nabla^k p_{(1^n)}.$$

Proof. Since the homology of Z_k is pure [GKM04, GKM06], it is $T(n)$-equivariantly formal [GKM04, Lem. 2.2]. We deduce that if $\mathsf{Q}^{\mathrm{BM}}_{S,n,nk}$ is the analogue of $\tilde{\mathsf{Q}}^{\mathrm{BM},T(n)}_{S,n,nk}$ for non-equivariant Borel–Moore homology, then

$$\mathcal{F}\mathsf{Q}^{\mathrm{BM}}_{S,n,nk}(\mathsf{q}, \mathsf{t}) = (1-\mathsf{t}^2)^n \mathcal{F}\tilde{\mathsf{Q}}^{\mathrm{BM},T(n)}_{S,n,nk}(\mathsf{q}, \mathsf{t}) = \frac{1}{(1-\mathsf{q})^n} \nabla^k p_{(1^n)}.$$

Next, recall that Borel–Moore homology and compactly-supported cohomology with complex coefficients are dual to each other. Finally, since both are supported in even degrees [CM21, 38], and in degree i, pure of weight i [GKM06, Cor. 1.3], we know that $\sum_k \mathsf{t}^k \dim \mathrm{H}^*_c(Z_k) = \chi(Z_k, \mathsf{t})$. □

6.3. Turning to the KhR side, observe that Gorsky–Hogancamp computed the y-ified KhR homology of the (n, nk) torus link in [GH], obtaining its usual KhR homology as a corollary.

We need to fix a typo there. In the proofs of [GH, Thm. 7.13–14], which use coherent sheaves on the Hilbert scheme of n points on $\mathbf{A}^2$ and its isospectral variant, the authors should be tracking the equivariance parameters coming from the scaling action of $\mathbf{G}_m^2$ along the axes. These contribute denominators of the form $(1-\mathsf{q})^n$

or $(1-\mathsf{t})^n$ in various places. After correction, [GH, Thm. 7.13] says

$$\bar{\mathsf{Y}}_{n,nk}(\mathsf{a},\mathsf{q},\mathsf{t}) := \sum_{i,j,k} \mathsf{a}^i \mathsf{q}^{\frac{j}{2}} \mathsf{t}^{\frac{k}{2}} \dim(\bar{\mathsf{Y}}_f^{i,\frac{j}{2},\frac{k}{2}}) = \frac{1}{(1-\mathsf{q})^n(1-\mathsf{t})^n} \Psi(\nabla^k p_{(1^n)}, \mathsf{a}).$$

Similarly, after correction, [GH, Thm. 7.14] says

$$\bar{\mathsf{X}}_{n,nk}(\mathsf{a},\mathsf{q},\mathsf{t}) = (1-\mathsf{t})^n \bar{\mathsf{Y}}_{n,nk}(\mathsf{a},\mathsf{q},\mathsf{t}) = \frac{1}{(1-\mathsf{q})^n} \Psi(\nabla^k p_{(1^n)}, \mathsf{a}).$$

Again, we refer to Section 5 and Appendix A to match our grading conventions with those in [GH]. This concludes the proof of case (2) of Theorem 5.

6.4. To conclude this section, we verify the $\mathsf{a} = 0$ limit of [ORS, Conj. 2] for two plane curve germs of the form $y^n = x^{nk}$. By way of case (2) of Theorem 5, this also verifies Conjecture 1 in these cases.

Example 6.4. Take $n = 2$ and $k = 2$. By [Ki, Ex. 6.18],

$$\mathsf{Hilb}(\mathsf{q},\mathsf{t}) = \frac{1}{(1-\mathsf{q})^2}(1 - \mathsf{q} + \mathsf{q}^2\mathsf{t}^2 - \mathsf{q}^3\mathsf{t}^2 + \mathsf{q}^4\mathsf{t}^4).$$

At the same time, the recursion of [HM, GMV20] gives

$$\bar{\mathsf{X}}_{2,4}(\mathsf{a},\mathsf{q},\mathsf{t}) = \frac{1}{(1-\mathsf{q})^2}(1 + \mathsf{q}(\mathsf{t}-1) + \mathsf{q}^2(\mathsf{t}^2 - \mathsf{t})).$$

These series agree under $(\mathsf{q},\mathsf{t}) \mapsto (\mathsf{q},\mathsf{q}\mathsf{t}^2)$.

Example 6.5. Take $n = 3$ and $k = 1$. By [Ki, Ex. 6.17],

$$\mathsf{Hilb}(\mathsf{q},\mathsf{t}) = \frac{1}{(1-\mathsf{q})^3}\left(\begin{array}{c} 1 - 2\mathsf{q} + \mathsf{q}^2(\mathsf{t}^2+1) + \mathsf{q}^3(\mathsf{t}^4 - 2\mathsf{t}^2) + \mathsf{q}^4(\mathsf{t}^4+\mathsf{t}^2) \\ -2\mathsf{q}^5\mathsf{t}^4 + \mathsf{q}^6\mathsf{t}^6 \end{array}\right).$$

At the same time, by [GMV20, Ex. 32],

$$\bar{\mathsf{X}}_{3,3}(\mathsf{a},\mathsf{q},\mathsf{t}) = \frac{1+\mathsf{q}\mathsf{t}}{1-\mathsf{q}} + \frac{\mathsf{q}\mathsf{t}^2 + 2\mathsf{q}^2\mathsf{t}^2}{(1-\mathsf{q})^2} + \frac{\mathsf{q}^3\mathsf{t}^3}{(1-\mathsf{q})^3}.$$

Again these agree under $(\mathsf{q},\mathsf{t}) \mapsto (\mathsf{q},\mathsf{q}\mathsf{t}^2)$.

7. Affine Springer Fibers

7.1. In this section, we establish the comparisons to affine Springer fibers needed in Section 4 and Section 6. For the convenience of readers unfamiliar with affine Lie theory, we keep our exposition self-contained beyond the definitions appearing in finite Lie theory. For the general relationship between local compactified Jacobians and affine Springer fibers, we refer to [L].

7.2. Suppose that G is a complex reductive algebraic group. Its *loop group* is the ind-group scheme $\hat{G}$ defined by $\hat{G}(A) = G(A(\!(x)\!))$ for all $\mathbf{C}$-algebras A, where $A(\!(x)\!) := A[\![x]\!][x^{-1}]$. Its *arc group* is the ind-group scheme $\hat{K}$ defined by $\hat{K}(A) = G(A[\![x]\!])$. Thus there is a projection map $\hat{K} \to G$ that sends $g(x) \mapsto g(0)$.

Henceforth, let $G = \mathrm{GL}_n$ and $\mathfrak{g} = \mathfrak{gl}_n$. Each integer composition ν of n defines a block-upper-triangular parabolic subgroup $P_\nu \subseteq G$. Its preimage $\hat{K}_\nu \subseteq \hat{K}$ is called the corresponding *parahoric subgroup*. The *partial affine flag variety of G* *of parabolic type* ν is the fpqc quotient $\hat{\mathcal{B}}_\nu = \hat{G}/\hat{K}_\nu$, which turns out to be an ind-scheme. For any $\gamma \in \mathfrak{g}(\mathbf{C}[\![x]\!])$, let

$$\hat{\mathcal{B}}^\gamma_{\nu,\dagger} = \{g\hat{K}_\nu \in \hat{\mathcal{B}}_\nu \mid \mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}(\hat{K}_\nu)\}.$$

The underlying reduced ind-scheme $\hat{\mathcal{B}}^\gamma_\nu \subseteq \hat{\mathcal{B}}^\gamma_{\nu,\dagger}$ is called the *affine Springer fiber* over γ *of parabolic type* ν.

Since $\hat{G}/\hat{K}$ is also known as the *affine Grassmannian*, we set $\hat{\mathcal{G}} = \hat{G}/\hat{K} = \hat{\mathcal{B}}_{(n)}$ and $\hat{\mathcal{G}}^\gamma = \hat{\mathcal{B}}^\gamma_{(n)}$.

7.3. **The Functor $\mathcal{L}$, and $y^n = x^{nk}$.** There is a well-known description of the affine Grassmannian as a space of lattices in $\mathbf{C}(\!(x)\!)^n$, and more generally, of the partial affine flag varieties of G as spaces of lattices equipped with partial flags. Namely, let $\mathcal{L}$ be the functor from $\mathbf{C}$-algebras to sets defined by

$$\mathcal{L}(A) = \left\{ \begin{array}{l|l} A[\![x]\!]\text{-submodules} & \exists\, i \text{ such that } x^i A[\![x]\!]^n \subseteq L \subseteq x^{-i}A[\![x]\!]^n \\ L \subseteq A(\!(x)\!)^n & \text{and } (x^{-i}A[\![x]\!]^n)/L \text{ is locally free over } A \\ & \text{of finite rank} \end{array} \right\}$$

for any $\mathbf{C}$-algebra A. For any ν, let $\mathcal{L}_\nu$ be the functor defined by

$$\mathcal{L}_\nu(A) = \left\{ (L, F) \;\middle|\; \begin{array}{l} L \in \mathcal{L}(A), \\ F \text{ is a partial flag on } \bar{L} := L/xL \text{ of type } \nu \end{array} \right\}.$$

Let $(\mathsf{v}_i)_{i=0}^{n-1}$ be the standard ordered basis of $\mathbf{C}^n$, but numbered from 0 through $n-1$. Let F^{std} be the unique partial flag on $\mathbf{C}^n$ of type ν with stabilizer P_ν under *right* multiplication by G, so that the ith subspace of F^{std} is that spanned by v_i for $n - \nu_n - \cdots - \nu_i \leq i \leq n-1$. The following result is explained in [Gö, §2.4] for $\nu = (n), (1^n)$; the argument for other ν is similar.

Lemma 7.1. *For each integer composition ν of n, there is an isomorphism of fpqc sheaves $\hat{\mathcal{B}}_\nu \xrightarrow{\sim} \mathcal{L}_\nu$ that sends*

$$g\hat{K}_\nu \mapsto (L_g, F_g) := (\mathbf{C}[\![x]\!]^n \cdot g^{-1}, F^{std} \cdot g^{-1}) \tag{7.1}$$

for all $g\hat{K}_\nu \in \hat{\mathcal{B}}_\nu(\mathbf{C})$. In particular, $\mathcal{L}$ is representable by an ind-scheme.

Let $\mathcal{L}_+ \subseteq \mathcal{L}$ be the sub-ind-scheme defined by

$$\mathcal{L}_+(A) = \{L \in \mathcal{L}(A) \mid L \subseteq A[\![x]\!]^n\}.$$

We define the *positive part* of $\hat{\mathcal{B}}_\nu$ to be the corresponding sub-ind-scheme $\hat{\mathcal{B}}_{\nu,+} \subseteq \hat{\mathcal{B}}_\nu$. Similarly, we define the *positive part* of $\hat{\mathcal{B}}^\gamma_\nu$ to be $\hat{\mathcal{B}}^\gamma_{\nu,+} = \hat{\mathcal{B}}^\gamma_\nu \cap \hat{\mathcal{B}}_{\nu,+}$. We set $\hat{\mathcal{G}}_+ = \hat{\mathcal{B}}_{(n),+}$ and $\hat{\mathcal{G}}^\gamma_+ = \hat{\mathcal{B}}^\gamma_{(n),+}$.

Fix a primitive nth root of unity $\zeta \in \mathbf{C}^\times$. For any integer $k > 0$, let

$$\gamma(k) = \mathrm{diag}(x^k, \zeta x^k, \ldots, \zeta^{n-1}x^k) \in \mathfrak{g}(\mathbf{C}[\![x]\!]).$$

We see that the centralizer of $\gamma(k)$ in $\hat{G}$ is precisely $\hat{T} \subseteq \hat{G}$, where $T \subseteq G$ is the maximal torus of diagonal matrices. The $\hat{T}$-action on $\hat{\mathcal{B}}^{\gamma(k)}_\nu$ by left multiplication restricts to a T-action on $\hat{\mathcal{B}}^{\gamma(k)}_{\nu,+}$. We note that the ind-scheme $\hat{\mathcal{B}}^{\gamma(k)}_{(1^n),+}$ is denoted Z_k in [CM21].

Proposition 7.2. *Suppose that*

$$R = \mathbf{C}[\![x, y]\!]/(y^n - x^{nk}) \quad \textit{for some integer } k > 0.$$

Fix an identification $S = \mathbf{C}[\![x]\!]^n$, hence an identification $T(n) = T$. Then:

(1) *The map (7.1) restricts to isomorphisms $\hat{\mathcal{B}}_\nu^{\gamma(k)} \xrightarrow{\sim} \overline{\mathcal{P}}_\nu$ and $\hat{\mathcal{B}}_{\nu,+}^{\gamma(k)} \xrightarrow{\sim} \coprod_\ell \mathcal{Q}_\nu^\ell$.*

Let $\hat{\mathcal{B}}_{\nu,+}^{\gamma(k),\ell} \subseteq \hat{\mathcal{B}}_{\nu,+}^{\gamma(k)}$ correspond to $\mathcal{Q}_\nu^\ell \subseteq \coprod_\ell \mathcal{Q}_\nu^\ell$ under the isomorphism in (1). Then the isomorphism matches:

(2) *The T-actions on $\hat{\mathcal{B}}_{\nu,+}^{\gamma(k),\ell}$ and $\mathcal{Q}_\nu^\ell$.*
(3) *The Springer actions of S_n on the T-equivariant Borel–Moore homologies of $\mathcal{B}_{(1^n),+}^{\gamma(k),\ell}$ and $\mathcal{Q}_{(1^n)}^\ell$, for all ℓ.*

Proof. Parts (1) and (2) follow from the definitions: Compare to [GK, Thm. 1.1]. To prove part (3), observe that the usual Springer action on the Borel–Moore homology of $\hat{\mathcal{B}}_{(1^n),+}^{\gamma(k),\ell}$ arises from Proposition 3.1 and Remark 3.2 via the outer rectangle in the following diagram, where every square is cartesian:

$$\begin{array}{ccccc}
[\hat{\mathcal{B}}_{(1^n),+}^{\gamma(k),\ell}/T] & \xrightarrow{\sim} & [\mathcal{Q}_{(1^n)}^\ell/T] & \longrightarrow & [\widetilde{\mathcal{N}}_{(1^n)}/\mathrm{GL}_n] \\
\downarrow & & \downarrow & & \downarrow \pi \\
[\hat{\mathcal{G}}_+^{\gamma(k),\ell}/T] & \xrightarrow{\sim} & [\mathcal{Q}^\ell/T] & \longrightarrow & [\mathcal{N}/\mathrm{GL}_n]
\end{array}$$

(Above, $\hat{\mathcal{G}}_+^{\gamma(k),\ell} := \hat{\mathcal{B}}_{(n),+}^{\gamma(k),\ell}$.) ∎

Remark 7.3. In [BL], Boixeda Alvarez–Losev construct commuting actions of two trigonometric double affine Hecke algebras (DAHAs) on the T-equivariant Borel–Moore homology of certain equivalued affine Springer fibers, for a certain torus T. One of their DAHA actions is a generalized Springer action; the other arises from combining a monodromic action of the affine Weyl group with the action of the equivariant cohomology $\mathrm{H}_T^*(pt)$.

In the GL_n case, their affine Springer fibers are precisely our $\hat{\mathcal{B}}_{(1^n)}^{\gamma(k)}$, and their T is our T. Via Proposition 7.2, the monodromic action of the cocharacter lattice and the action of equivariant cohomology in [BL] respectively correspond to the $\Gamma_{\geq 0}$- and $\mathrm{H}_T^*(pt)$-actions on the T-equivariant Borel–Moore homology of $\mathcal{Q}_{(1^n)}^\ell$ in Section 5. The monodromic action of the finite Weyl group corresponds to the dot action in Section 6.

7.4. **The Functor $\mathcal{M}$, and $y^n = x^d$.** Writing $x = \varpi^n$, let $\mathsf{ab} : \mathbf{C}(\!(x)\!)^n \xrightarrow{\sim} \mathbf{C}(\!(\varpi)\!)$ be the isomorphism of $\mathbf{C}(\!(x)\!)$-vector spaces defined by

$$\mathsf{ab}(\mathsf{v}_i) = \varpi^i,$$

where we have implicitly used $\mathbf{C}(\!(x)\!)^n = \mathbf{C}(\!(x)\!) \otimes \mathbf{C}^n$. Let $\mathcal{M}$ be the functor from $\mathbf{C}$-algebras to sets defined by

$$\mathcal{M}(A) = \left\{ \begin{array}{l|l} A[\![\varpi^n]\!]\text{-submodules} & \exists\, i \text{ such that } \varpi^i A[\![\varpi]\!] \subseteq M \subseteq \varpi^{-i} A[\![\varpi]\!] \\ M \subseteq A(\!(\varpi)\!) & \text{and } (\varpi^{-i} A[\![\varpi]\!])/M \text{ is locally free over } A \\ & \text{of finite rank} \end{array} \right\}$$

for any $\mathbf{C}$-algebra A. Thus $\mathcal{M}$ is the analogue of $\overline{\mathcal{P}}_\dagger$ with $\mathbf{C}[\![\varpi]\!]$ in place of R. For any ν, let $\mathcal{M}_\nu$ be the functor defined by

$$\mathcal{M}_\nu(A) = \left\{ (M, F) \,\middle|\, \begin{array}{l} M \in \mathcal{M}(A), \\ F \text{ is a partial flag on } \bar{M} := M/\varpi^n M \text{ of type } \nu \end{array} \right\}.$$

Then ab induces an isomorphism of fpqc sheaves

$$\mathcal{L}_\nu \xrightarrow{\sim} \mathcal{M}_\nu, \tag{7.2}$$

which we again denote by ab.

We now define the element of $\mathfrak{g}(\mathbf{C}[\![x]\!])$ studied in [Hi]. Let $(X^\bullet, \Phi, X_\bullet, \Phi^\vee)$ be the root datum of G with respect to the maximal torus of diagonal matrices. Let $\alpha_1, \ldots, \alpha_{n-1} \in \Phi$ be the simple roots with respect to the upper-triangular Borel subgroup $P_{(1^n)} \subseteq G$, and let $\rho^\vee = \frac{1}{2}\sum_i \alpha_i^\vee \in \Phi^\vee$, where $\alpha_i^\vee$ is the coroot corresponding to α_i. For any $d > 0$ coprime to n, let m, b be the integers such that

$$d = mn + b \quad \text{and} \quad 0 < b < n,$$

as in [Hi]. For each root α, let $e_\alpha \in \mathfrak{g}(\mathbf{C})$ be the zero-one matrix that generates the root subspace $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$, and for each integer j, let $e_j = \sum_{\alpha \mid \langle \alpha, \rho^\vee \rangle = j} e_\alpha$. Finally, let

$$\psi(d) = x^m e_b + x^{m+1} e_{b-n}. \tag{7.3}$$

In what follows, we will need the composition of isomorphisms

$$\hat{\mathcal{B}}_\nu \xrightarrow{\text{Lem 7.1}} \mathcal{L}_\nu \xrightarrow{\mathsf{ab}} \mathcal{M}_\nu \xrightarrow{\varpi^\delta} \mathcal{M}_\nu, \tag{7.4}$$

where the last label means means multiplication by ϖ^δ, and $\delta = \frac{1}{2}(n-1)(d-1)$, as in Section 4. We write the map on $\mathbf{C}$-points as $g\hat{K}_\nu \mapsto (M_g, F_g)$.

Proposition 7.4. *Suppose that*

$$R = \mathbf{C}[\![\varpi^n, \varpi^d]\!] \quad \textit{for some } d > 0 \textit{ coprime to } n.$$

Then:

(1) *The map (7.4) restricts to an isomorphism* $\hat{\mathcal{B}}_\nu^{\psi(d)} \xrightarrow{\sim} \overline{\mathcal{P}}_\nu$.

Let $\overline{\mathcal{P}}_\nu^e \subseteq \overline{\mathcal{P}}_\nu$ *be the preimage of* $\overline{\mathcal{P}}^e \subseteq \overline{\mathcal{P}}$*, and let* $\hat{\mathcal{B}}_\nu^{\psi(d),e} \subseteq \hat{\mathcal{B}}_\nu^{\psi(d)}$ *correspond to* $\overline{\mathcal{P}}_\nu^e \subseteq \overline{\mathcal{P}}_\nu$ *under the isomorphism in (1). Then:*

(2) *The isomorphism in (1) matches the Springer action of* S_n *on the Borel–Moore homologies of* $\mathcal{B}_{(1^n)}^{\psi(d),e}$ *and* $\overline{\mathcal{P}}_{(1^n)}^e$*, for all* e.

(3) $\hat{\mathcal{B}}_\nu^{\psi(d),0}$ *is the affine Springer fiber studied by Hikita in [Hi].*

Proof. Part (1): It suffices to work on $\mathbf{C}$-points. By checking on the basis $(\mathsf{v}_i)_i$, we find that ab transports the action of γ on $\mathbf{C}(\!(x)\!)^n$ by right multiplication onto the action of ϖ^d on $\mathbf{C}(\!(\varpi)\!)$ by multiplication. Therefore,

$$\begin{aligned} g\hat{K}_\nu \in \mathcal{B}_\nu^{\psi(d)}(\mathbf{C}) &\iff (\mathbf{C}[\![x]\!]^n \cdot g^{-1}, F^{std} \cdot g^{-1}) \text{ is } \gamma\text{-stable} \\ &\iff (M_g, F_g) \text{ is } R\text{-stable} \end{aligned}$$

for all $g\hat{K}_\nu \in \hat{\mathcal{B}}_\nu(\mathbf{C})$ and fixed $e \in \mathbf{Z}$.

Part (2): Similar to the proof of part (3) of Proposition 7.2, but replacing the diagram there with this one:

$$\begin{array}{ccccc} \hat{\mathcal{B}}^{\psi(d),e}_{(1^n)} & \xrightarrow{\sim} & \overline{\mathcal{P}}^{e}_{(1^n)} & \longrightarrow & [\widetilde{\mathcal{N}}_{(1^n)}/\mathrm{GL}_n] \\ \downarrow & & \downarrow & & \downarrow \pi \\ \hat{\mathcal{G}}^{\psi(d),e} & \xrightarrow{\sim} & \overline{\mathcal{P}}^{e} & \longrightarrow & [\mathcal{N}/\mathrm{GL}_n] \end{array} \tag{7.5}$$

(Above, $\hat{\mathcal{G}}^{\psi(d),e} := \hat{\mathcal{B}}^{\psi(d),e}_{(n)}$.)

Part (3): The multiplication by ϖ^δ in the last arrow of (7.4) ensures that $\hat{\mathcal{B}}^{\psi(d),0}_\nu$ contains the identity coset $\hat{K}_\nu \in \hat{\mathcal{B}}_\nu$. As a consequence, $\hat{\mathcal{B}}^{\psi(d),0}_\nu$ belongs to the connected component of $\hat{\mathcal{B}}_\nu$ that corresponds to the partial affine flag variety of SL_n of parabolic type ν. The latter is defined analogously to the partial affine flag variety of $G = \mathrm{GL}_n$, which means that $\hat{\mathcal{B}}^{\psi(d),0}_\nu$ is precisely the affine Springer fiber over $\psi(d)$ with structure group SL_n. □

8. Filtrations on $\mathrm{H}^*(\overline{\mathcal{P}}/\Gamma)$

8.1. In this section, we discuss the following filtrations on the variety $\overline{\mathcal{P}}/\Gamma$ or its cohomology:

(1) The gap filtration on the variety, defined in terms of the function $c(M) = \dim_{\mathbf{C}}(SM/M)$ from the introduction.
(2) The Hikita filtration [Hi], defined on the variety for $R = \mathbf{C}[\![\varpi^n, \varpi^d]\!]$ with n, d coprime, by intersecting the affine Springer fiber from Section 7.4 with increasing unions of affine Schubert cells.
(3) The perverse filtration on cohomology, defined in terms of a versal deformation of a global curve C into which $\mathrm{Spec}(R)$ embeds.

First, in Theorem 8.3, we relate (1) and (2) by way of an involution ι, as needed in Section 4.7. The involution ι is related to a duality studied by Gorsky–Mazin [GM14], but to our knowledge, our work is the first time it has been used to relate the filtrations above. Next, we discuss Conjecture 8 relating (1) and (3). For completeness, we also review the splitting of (3) constructed in [R]. Finally, we discuss related filtrations in [GORS, OY17, CO].

8.2. **The Gap Filtration.** Let $R = \mathbf{C}[\![x]\!][y]/(f)$ be an arbitrary generically separable degree-n cover of the x-axis, fully ramified at $(x, y) = (0, 0)$. For any integer composition ν of n, we define the *gap filtration* on $\overline{\mathcal{P}}_\nu$ to be its increasing filtration by the subvarieties

$$\overline{\mathcal{P}}_{\nu,\leq c} = \bigcup_{c' \leq c} \overline{\mathcal{P}}_\nu(c').$$

It descends to a filtration of $\overline{\mathcal{P}}_\nu/\Gamma$ by subvarieties $\overline{\mathcal{P}}_{\nu,\leq c}/\Gamma$. We define $\mathsf{Q}_{\leq *}$ to be the increasing filtration on the Borel–Moore homology of $\overline{\mathcal{P}}_\nu/\Gamma$ where

$$\mathsf{Q}_{\leq c}\,\mathrm{H}^{\mathrm{BM}}_*(\overline{\mathcal{P}}_\nu/\Gamma) = \mathrm{im}(\mathrm{H}^{\mathrm{BM}}_*(\overline{\mathcal{P}}_{\nu,\leq c}/\Gamma) \to \mathrm{H}^{\mathrm{BM}}_*(\overline{\mathcal{P}}_\nu/\Gamma)).$$

We define $\mathsf{Q}^{\geq *}$ to be the decreasing filtration on the cohomology of $\overline{\mathcal{P}}_\nu/\Gamma$ where

$$\mathsf{Q}^{\geq c}\,\mathrm{H}^*(\overline{\mathcal{P}}_\nu/\Gamma) = \ker(\mathrm{H}^*(\overline{\mathcal{P}}_\nu/\Gamma) \to \mathrm{H}^*(\overline{\mathcal{P}}_{\nu,\leq c}/\Gamma)).$$

Since compactly-supported cohomology is dual to Borel–Moore homology, and $\overline{\mathcal{P}}_\nu/\Gamma$ is proper, $\mathsf{Q}^{\geq c}$ is orthogonal to $\mathsf{Q}_{\leq c}$ for all c. We note in passing that these definitions still make sense for non-planar R.

As in Proposition 7.4, let $\overline{\mathcal{P}}^e_\nu \subseteq \overline{\mathcal{P}}_\nu$ be the preimage of $\overline{\mathcal{P}}^e \subseteq \overline{\mathcal{P}}$. We define $\overline{\mathcal{P}}^e_{\nu,\leq c}, \overline{\mathcal{P}}^e_{\leq c}, \overline{\mathcal{J}}_{\leq c}$ analogously to $\overline{\mathcal{P}}_{\nu,\leq c}$. Then $\overline{\mathcal{P}}^e_{\nu,\leq c}$ is the preimage of $\overline{\mathcal{P}}^e_{\leq c}$ along the projection $\overline{\mathcal{P}}_\nu \to \overline{\mathcal{P}}$, because $\overline{\mathcal{P}}_\nu(c')$ is the preimage of $\overline{\mathcal{P}}(c')$. Moreover, any isomorphism $\overline{\mathcal{J}} = \overline{\mathcal{P}}^0 \xrightarrow{\sim} \overline{\mathcal{P}}^e$ induced by multiplication by a uniformizer will preserve c, hence restrict to an isomorphism $\overline{\mathcal{J}}_{\leq c} \xrightarrow{\sim} \overline{\mathcal{P}}^e_{\leq c}$. This largely reduces the study of the gap filtration at the level of the varieties to the study of $\overline{\mathcal{J}}_{\leq c}$. Recall that in the unibranch case, $\overline{\mathcal{J}} \simeq \overline{\mathcal{P}}/\Gamma$.

8.3. **The Gap Filtration for $y^n = x^d$.** Suppose that $R = \mathbf{C}[\![\varpi^n, \varpi^d]\!]$ with n, d coprime. Recall that in this case, there is a $\mathbf{G}_m$-action on $\overline{\mathcal{P}}$ induced by scaling ϖ, which necessarily stabilizes the connected component $\overline{\mathcal{J}}$. As in Section 4, let

$$I^\delta(S) := \{\Delta \subseteq \mathbf{Z}_{\geq 0} \mid \Delta + n \subseteq \Delta,\ \Delta + d \subseteq \Delta,\ |\mathbf{Z}_{\geq 0} \setminus \Delta| = \delta\}.$$

By [P, §3], the setup of Lemma 4.2 restricts to a bijection

$$\begin{array}{ccc} I^\delta(S) & \xrightarrow{\sim} & \overline{\mathcal{J}}^{\mathbf{G}_m}, \\ \Delta & \mapsto & M_\Delta \end{array}$$

that partitions $\overline{\mathcal{J}}$ into the affine spaces $\mathbf{A}_\Delta$ for $\Delta \in I^\delta(S)$.

Lemma 8.1. *If $R = \mathbf{C}[\![\varpi^n, \varpi^d]\!]$ with n, d coprime, then*

$$c(M) = \delta - \min(\Delta) \quad \textit{for all } \Delta \in I^\delta(S) \textit{ and } M \in \mathbf{A}_\Delta(\mathbf{C}).$$

In particular, $\overline{\mathcal{J}}(c) = \bigcup_{\Delta \mid \min(\Delta) = \delta - c} \mathbf{A}_\Delta$ and $\overline{\mathcal{J}}_{\leq c} = \bigcup_{\Delta \mid \min(\Delta) \geq \delta - c} \mathbf{A}_\Delta$.

Proof. In the notation of Section 2, we have $\varpi^{-\min(\Delta)} M \in \mathcal{D}$ for all $\Delta \in I^\delta(S)$ and $M \in \mathbf{A}_\Delta$. Now observe that

$$c(M) = c(\varpi^{-\min(\Delta)} M) = -\min(\Delta) + \ell(M) = -\min(\Delta) + \delta,$$

where the second equality holds by Lemma 2.5. □

8.4. **The Hikita Filtration.** Next, we (re)turn to Hikita's work in [Hi]. In the notation of Section 7.4, recall that Proposition 7.4 gives us an isomorphism

$$\begin{array}{ccc} \hat{\mathcal{G}}^{\psi(d),0} & \xrightarrow{\sim} & \overline{\mathcal{J}}, \\ g\hat{K} & \mapsto & M_g, \end{array}$$

where $\hat{\mathcal{G}}^{\psi(d),0}$ is the affine Springer fiber over $\psi(d)$ with structure group SL_n. Hikita first defines a filtration of $\hat{\mathcal{G}}^{\psi(d),0}$, then lifts it to $\hat{\mathcal{B}}_\nu^{\psi(d),0}$ along the projection $\hat{\mathcal{B}}_\nu \to \hat{\mathcal{B}}_{(n)} = \hat{\mathcal{G}}$. Thus, as with the gap filtration, we can largely reduce to studying the $\nu = (n)$ case.

Recall that the partition of $\hat{\mathcal{G}}$ into $\hat{I}$-orbits, where $\hat{I} = \hat{K}_{(1^n)}$ acts on $\hat{\mathcal{G}}$ by left multiplication, forms a stratification:

$$\hat{\mathcal{G}} = \coprod_{\mu \in X_\bullet} \hat{\mathcal{G}}_\mu, \quad \text{where } \hat{\mathcal{G}}_\mu := \hat{I} x^\mu \hat{K}/\hat{K}.$$

Above, $X_\bullet$ is the same cocharacter lattice as in Section 7.4, and for any $\mu \in X_\bullet$, we write x^μ to mean the image of x under $\mu : \hat{\mathbf{G}}_m \to \hat{G}$. The strata $\hat{\mathcal{G}}_\mu$ are affine spaces known as *affine Schubert cells*. Henceforth, we fix the identification $X_\bullet = \mathbf{Z}^n$ under which $x^\mu = \mathrm{diag}(x^{\mu_1}, \ldots, x^{\mu_n})$. Then the affine Grassmannian of SL_n is the sub-ind-scheme $\hat{\mathcal{G}}_{\mathrm{SL}_n} \subseteq \hat{\mathcal{G}}$ given by

$$\hat{\mathcal{G}}_{\mathrm{SL}_n} = \coprod_{\mu \in X_\bullet^0} \hat{\mathcal{G}}_\mu, \quad \text{where } X_\bullet^0 := \{\mu \in X_\bullet \mid \mu_1 + \cdots + \mu_n = 0\}.$$

The proof of [Hi, Prop. 4.1] shows that there is a bijection

$$a : X_\bullet^0 \xrightarrow{\sim} \mathbf{Z}_{\geq 0}^{n-1}$$

defined as follows:

(1) $a_i(0, \ldots, 0) = 0$ for all i.
(2) If $\mu \neq (0, \ldots, 0)$, then

$$\begin{aligned}(8.1) \qquad &(a_1, \ldots, a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}) \\ &\quad = (\mu_{k+1} - \mu_k - 1, \ldots, \mu_n - \mu_k - 1, \mu_1 - \mu_k, \ldots, \mu_{k-1} - \mu_k),\end{aligned}$$

where k is the largest index in $\{1, \ldots, n\}$ such that $\mu_k = \min_i \mu_i$. Note that since $\mu_1 + \cdots + \mu_n = 0$, we must have $\mu_k < 0$.

For all $a \in \mathbf{Z}_{\geq 0}^{n-1}$, let $|a| = a_1 + \cdots + a_{n-1}$. For any integer c, let

$$\hat{\mathcal{G}}_{\mathrm{SL}_n, \leq c} = \bigcup_{\substack{\mu \in X_\bullet^0 \\ |a(\mu)| \leq c}} \hat{\mathcal{G}}_\mu.$$

Following [Hi, Cor. 4.7], we define the *Hikita filtration* on $\hat{\mathcal{G}}^{\psi(d),0}$ to be its increasing filtration by the subvarieties

$$\hat{\mathcal{G}}_{\leq c}^{\psi(d),0} = \hat{\mathcal{G}}^{\psi(d),0} \cap \hat{\mathcal{G}}_{\mathrm{SL}_n, \leq c}.$$

For each integer composition ν of n, we define $\hat{\mathcal{B}}_{\nu, \leq c}^{\psi(d),0}$ to be the preimage of $\hat{\mathcal{G}}_{\leq c}^{\psi(d),0}$ along the projection $\hat{\mathcal{B}}_\nu \to \hat{\mathcal{G}}$. We define the *Hikita filtration* on $\hat{\mathcal{B}}_\nu^{\psi(d),0}$ to be its increasing filtration by these subvarieties. This recovers the definition for $\nu = (1^n)$ in the proof of [Hi, Thm. 4.17].

8.5. **The Involution ι.** For any $g \in G$, let g^τ be the "anti-transpose" given by $g^\tau = J g^t J$, where g^t is the usual transpose and $J \in G$ the matrix with 1's along the anti-diagonal and 0's elsewhere. The map $\iota : G \to G$ given by

$$\iota(g) := (g^\tau)^{-1} = (g^{-1})^\tau$$

is an involutory automorphism with differential $\iota : \mathfrak{g} \to \mathfrak{g}$ given by

$$\iota(\gamma) = -\gamma^\tau.$$

We extend these automorphisms to $\hat{G}$ and its Lie algebra by linearity and completion. We see that $\iota(\hat{K}) = \hat{K}$ and $\iota(\hat{K}_{(1^n)}) = \hat{K}_{(1^n)}$, from which we deduce that ι descends to involutions of $\hat{\mathcal{G}}$ and $\hat{\mathcal{B}}_{(1^n)}$.

From the definition (7.3), we also see that $\iota(\psi(d)) = -\psi(d)$. We deduce that the affine Springer fibers $\hat{\mathcal{G}}^{\psi(d)}$ and $\hat{\mathcal{B}}_{(1^n)}^{\psi(d)}$ are stable under ι, as are their SL_n variants $\hat{\mathcal{G}}^{\psi(d),0}$ and $\hat{\mathcal{B}}_{(1^n)}^{\psi(d),0}$.

Lemma 8.2. *The involutions above have the following properties:*

(1) *For all $\mu \in X_\bullet$, we have $\iota(\hat{\mathcal{G}}_\mu) = \hat{\mathcal{G}}_{\iota(\mu)}$, where*

$$\iota(\mu_1, \ldots, \mu_n) = (-\mu_n, \ldots, -\mu_1).$$

(2) *For any integer e, the involution on the Borel–Moore homology of $\hat{\mathcal{B}}_{(1^n)}^{\psi(d),e}$ induced by ι is equivariant with respect to the Springer action of S_n. Moreover, it preserves the homological degree and weight filtration.*

In preparation for the proof of part (2), we set up some notation. Recall that $\hat{\mathcal{G}}^{\psi(d),0} \subseteq \hat{\mathcal{G}}_{\mathrm{SL}_n}$, and hence,

$$\hat{\mathcal{G}}^{\psi(d),0} = \coprod_{\mu \in X_\bullet^0} \mathbf{A}_\mu, \quad \text{where } \mathbf{A}_\mu := \hat{\mathcal{G}}^{\psi(d),0} \cap \hat{\mathcal{G}}_\mu.$$

Let $X_\bullet^{\psi(d),0} \subseteq X_\bullet^0$ be the subset of cocharacters μ for which $\mathbf{A}_\mu$ is nonempty. It is explained in [Hi, §2.3], following [GKM06], that these $\mathbf{A}_\mu$ are affine spaces. Moreover, [Hi, Thm. 2.7] is an explicit combinatorial formula for their dimensions, which shows that

$$\dim(\mathbf{A}_\mu) = \dim(\mathbf{A}_{\iota(\mu)}) \tag{8.2}$$

for all $\mu \in X_\bullet^{\psi(d),0}$.

Proof of Lemma 8.2. Part (1) follows from computing $\iota(x^\mu) = x^{\iota(\mu)}$.

To show part (2): First, recall that the Springer action in question is defined via Proposition 3.1 and Remark 3.2 via the outer rectangle of (7.5). The bottom arrow of this outer rectangle sends $g\hat{K} \mapsto [\mathrm{Ad}(g^{-1})\psi(d) \bmod x]$. So we must show that the residues of $\mathrm{Ad}(g^{-1})\psi(d)$ and $\mathrm{Ad}(\iota(g)^{-1})\psi(d) \bmod x$ have the same Jordan types as nilpotent elements of $\mathfrak{g}$. This follows from computing

$$\mathrm{Ad}(\iota(g)^{-1})\psi(d) = -\mathrm{Ad}(\iota(g)^{-1})\iota(\psi(d)) = -\iota(\mathrm{Ad}(g^{-1})\psi(d)),$$

then observing that ι commutes with reduction mod x and preserves the Jordan types of nilpotent elements.

The fact that the involution on $\mathrm{H}_*^{\mathrm{BM}}(\hat{\mathcal{B}}_{(1^n)}^{\psi(d),e})$ preserves the homological degree and weight filtration follows from $\hat{\mathcal{B}}_{(1^n)}^{\psi(d),e}$ being paved by the affine spaces $\mathbf{A}_\mu$, together with the identity (8.2). $\square$

In what follows, we set $\overline{\mathcal{J}}_\nu = \overline{\mathcal{P}}_\nu^0$ and $\overline{\mathcal{J}}_{\nu,\leq c} = \overline{\mathcal{P}}_{\nu,\leq c}^0$ to make parallels in notation clearer. Together with Lemma 8.2(2), the following result completes a necessary step in proof (B) of case (1) of Theorem 5.

Theorem 8.3. *Suppose that $R = \mathbf{C}[\![\varpi^n, \varpi^d]\!]$ for some $d > 0$ coprime to n. Then the composition of isomorphisms*

$$\hat{\mathcal{B}}^{\psi(d),0}_{(1^n)} \xrightarrow{\iota} \hat{\mathcal{B}}^{\psi(d),0}_{(1^n)} \xrightarrow{\text{Prop 7.4}} \overline{\mathcal{J}}_{(1^n)} \tag{8.3}$$

restricts to an isomorphism $\iota(\hat{\mathcal{B}}^{\psi(d),0}_{(1^n),\leq c}) \xrightarrow{\sim} \overline{\mathcal{J}}_{(1^n),\leq c}$ for all c.

Since $\overline{\mathcal{J}}_{(1^n),\leq c}$ and $\hat{\mathcal{B}}^{\psi(d),0}_{(1^n),\leq c}$ are respectively the preimages of $\overline{\mathcal{J}}_{\leq c}$ and $\hat{\mathcal{G}}^{\psi(d),0}_{\leq c}$, the ι-equivariance of the projection $\hat{\mathcal{B}}^{\psi(d),0}_{(1^n)} \to \hat{\mathcal{G}}^{\psi(d),0}$ and the commutativity of the left square of (7.5) allows us to replace $\nu = (1^n)$ with $\nu = (n)$.

We will match the strata $\mathbf{A}_\mu \subseteq \hat{\mathcal{G}}^{\psi(d),0}$ with the strata $\mathbf{A}_\Delta \subseteq \overline{\mathcal{J}}$. Let $- \cdot_{1/n} -$ denote the $\mathbf{G}_m$-action on $\hat{G}$ defined by

$$t \cdot_{1/n} g(x) := c^{2\rho^\vee} g(c^{2n}\varpi) c^{-2\rho^\vee}$$

for all $t \in \mathbf{G}_m$ and $g \in \hat{G}$. It descends to a $\mathbf{G}_m$-action on $\hat{\mathcal{G}}$ that we again denote by $- \cdot_{1/n} -$. As explained in [Hi, GKM06], we have

$$\begin{aligned} \hat{\mathcal{G}}_\mu &= \{g\hat{K} \in \hat{\mathcal{G}} \mid \lim_{t\to 0}\,(t \cdot_{1/n} g\hat{K}) = x^\mu \hat{K}\} && \text{for all } \mu \in X_\bullet, \\ \mathbf{A}_\mu &= \{g\hat{K} \in \hat{\mathcal{G}}^{\psi(d),0} \mid \lim_{t\to 0}\,(t \cdot_{1/n} g\hat{K}) = x^\mu \hat{K}\} && \text{for all } \mu \in X_\bullet^0. \end{aligned}$$

So we must match $- \cdot_{1/n} -$ with the $\mathbf{G}_m$-action on $\overline{\mathcal{J}}$ in Section 8.3.

Proposition 8.4. *The map (8.3) transports the $\mathbf{G}_m$-action $- \cdot_{1/n} -$ on $\hat{\mathcal{G}}$ onto the $\mathbf{G}_m$-action on $\mathcal{M}_{(n)}$ induced by $t \cdot_2 \varpi := t^2\varpi$.*

Proof. It suffices to work on $\mathbf{C}$-points. First, ι is equivariant under $- \cdot_{1/n} -$ because $\iota(c^{2\rho^\vee}) = c^{2\rho^\vee}$, so we can replace (8.3) with (7.4). Observe that if $g = g(x) \in G(\mathbf{C}(\!(x)\!))$, and $g'(x) = t \cdot_{1/n} g(x)$ for some $t \in \mathbf{C}^\times$, then $g'(x)^{-1} = t \cdot_{1/n} g^{-1}(x)$. Thus the entries of the matrix $g'(x)^{-1}$ are given by

$$(g'(x)^{-1})_{i,j} = t^{2(j-i)}(g(t^{2n}\varpi)^{-1})_{i,j}.$$

We deduce that

$$\begin{aligned} \varpi^\delta \,\mathsf{ab}(\mathsf{v}_i \cdot (t \cdot_{1/n} g(x))^{-1}) &= \varpi^\delta \sum_j (g'(\varpi^n)^{-1})_{i,j}\varpi^j \\ &= t^{-2i}\varpi^\delta \sum_j (g((t^2\varpi)^n)^{-1})_{i,j}(t^2\varpi)^j \\ &= t^{-2\delta-2i}(t \cdot_2 \varpi^\delta \,\mathsf{ab}(\mathsf{v}_i \cdot g(x)^{-1})). \end{aligned}$$

Above, $t^{-2\delta-2i}$ is just a nonzero scalar depending on i. So the calculation shows that the vector subspaces of $\mathbf{C}(\!(\varpi)\!)$ formed by $M_{t\cdot_{1/n} g}$ and $t \cdot_2 M_g$ coincide. □

In the notation of Section 8.3, let $\Delta : X_\bullet^{\psi(d),0} \to I^\delta(S)$ be defined by

$$\mathsf{Gen}_n(\Delta(\mu)) = \{n\mu_i + n - i + \delta \mid 1 \leq i \leq n\}. \tag{8.4}$$

Then the map $x^\mu \mapsto M_{\Delta(\mu)}$ is precisely the effect of (8.3) on the $(- \cdot_{1/n} -)$-fixed points of $\hat{\mathcal{G}}^{\psi(d),0}$, as we can check from the definition of ι and (7.1)–(7.2). Now (8.2) and Proposition 8.4 imply:

Corollary 8.5. *The map* $\Delta : X_\bullet^{\psi(d),0} \to I^\delta(S)$ *is bijective, and for all* $\mu \in X_\bullet^{\psi(d),0}$, (8.3) *restricts to an isomorphism*

$$\mathbf{A}_\mu \xrightarrow{\sim} \mathbf{A}_{\Delta(\mu)}.$$

To finish the proof of Theorem 8.3, it remains to show that for all $\mu \in X_\bullet^{\psi(d),0}$, we have $|a(\mu)| = c(M_{\Delta(\mu)})$. By Lemma 8.1 and (8.4), this is equivalent to:

Lemma 8.6. *For all* $\mu \in X_\bullet^{\psi(d),0}$, *we have*

$$|a(\mu)| = -\min\{n\mu_i + n - i \mid 1 \leq i \leq n\}.$$

Proof. If $\mu = (0, \ldots, 0)$, then both sides equal 0. If $\mu \neq (0, \ldots, 0)$, then (8.1) gives

$$|a(\mu)| = (\mu_1 + \cdots + \mu_n) - (n\mu_k + n - k),$$

where k is the largest index in $\{1, \ldots, n\}$ such that $\mu_k = \min_i \mu_i$. Since $\mu \in X_\bullet^0$, the right-hand side above simplifies to $-(n\mu_k + n - k)$. □

Remark 8.7. It is natural to ask what the involution ι on $\hat{\mathcal{G}}^{\psi(d),0}$ looks like after being transported through (7.4), to an involution on $\overline{\mathcal{J}}$. From (8.4), we can check that it is precisely the duality that Gorsky–Mazin denote by $\Delta \mapsto \widehat{\Delta}$ in [GM14]. Explicitly, for any $\Delta \in I^\delta(S)$, we have $\mathsf{Gen}_n(\widehat{\Delta}) = \{d(n-1) - k \mid k \in \mathsf{Gen}_n(\Delta)\}$.

Remark 8.8. In [GMV16], Gorsky–Mazin–Vazirani assert a simplification of Hikita's work in terms of a generalized Pak–Stanley bijection $\mathcal{PS}$. In particular, their maps $\mathcal{A}$ and $\mathcal{PS}$ respectively encode the area and codinv statistics in Section 4: See Section 3.3 in their paper. It would be interesting to know how their combinatorics is related to our geometry.

Example 8.9. Take $(n, d) = (3, 4)$. We compute

$$\begin{aligned} X_\bullet^{\psi(d),0} &= \{(0,0,0), (-1,0,1), (-1,1,0), (0,-1,1), (1,0,-1)\}, \\ I^\delta(S) &= \{\Delta_{3,4,5}, \Delta_{6,4,2}, \Delta_{3,7,2}, \Delta_{6,1,5}, \Delta_{0,4,8}\}. \end{aligned}$$

Above, we have labeled the elements of $I^\delta(S)$ in the form Δ_{b_1,b_2,b_3}, where $\mathsf{Gen}_n = \{b_1, b_2, b_3\}$ and $b_i \equiv d(i-1) \pmod{n} \equiv i - 1 \pmod{3}$ for all i. We compute these statistics:

μ	$a(\mu)$	$\|a(\mu)\|$	$(n\mu_i + n - i)_i$	$\Delta(\mu)$	$\min(\Delta(\mu))$
$(0,0,0)$	$(0,0)$	0	$(2,1,0)$	$\Delta_{3,4,5}$	3
$(-1,0,1)$	$(0,1)$	1	$(-1,1,3)$	$\Delta_{6,4,2}$	2
$(-1,1,0)$	$(1,0)$	1	$(-1,4,0)$	$\Delta_{3,7,2}$	2
$(0,-1,1)$	$(1,1)$	2	$(2,-2,3)$	$\Delta_{6,1,5}$	1
$(1,0,-1)$	$(2,1)$	3	$(5,1,-3)$	$\Delta_{0,4,8}$	0

Compare to Example 4.16.

8.6. **The Perverse Filtration.** We return to the setup of Section 8.2, where $f(x, y)$ is arbitrary. For simplicity, we ignore the map to the x-axis in what follows. Besides $\mathsf{Q}^{\geq *}$, there is another filtration on the cohomology of $\overline{\mathcal{P}}/\Gamma$, defined as follows by Maulik–Yun:

Fix a complex, integral, projective curve C, whose normalization has genus zero, and which is smooth away from a unique planar singularity given in local coordinates by $f(x, y) = 0$. We emphasize that while C is integral, the germ f can still have multiple branches. Fix an embedding of C into a family of curves $\mathcal{C}$, whose base is irreducible, and which satisfies conditions (A1)–(A4) in [MY, §2.1]. For instance, in any versal deformation of C, we can obtain such a family after base change to a small-enough Zariski neighborhood around C [MY, Prop. 3.5].

Condition (A4) entails a nonsingular basepoint $s \in C(\mathbf{C})$. Let $\overline{\mathcal{J}}(C)$ be the compactified Jacobian of (C, s) [AIK]. In more detail, $\overline{\mathcal{J}}(C)$ is a projective variety whose $\mathbf{C}$-points parametrize torsion-free, coherent sheaves on C of degree 0 and generic rank 1, equipped with a trivialization at s. In this setting, Section 2.14 of *ibid.* defines an increasing perverse filtration $\mathsf{P}_{\leq *}$ on $\mathrm{H}^*(\overline{\mathcal{J}}(C))$, in terms of the perverse truncation of the pushforward of the constant sheaf along the structure map of $\overline{\mathcal{J}}(\mathcal{C})$. Proposition 2.15 of *ibid.* shows that $\mathsf{P}_{\leq *}$ is invariant under base change of the family of curves, so it is canonical. It is strictly compatible with the weight filtration $\mathsf{W}_{\leq *}$.

Finally, the proof of Theorem 3.11 of *ibid.* shows that there is a weight-preserving isomorphism $\mathrm{H}^*(\overline{\mathcal{J}}(C)) \simeq \mathrm{H}^*(\overline{\mathcal{P}}/\Gamma)$, canonical up to the choice of uniformization that defines the Γ-action on $\overline{\mathcal{P}}$. We define the *perverse filtration* $\mathsf{P}_{\leq *}$ on $\mathrm{H}^*(\overline{\mathcal{P}}/\Gamma)$ by transport along this isomorphism. Following Maulik–Yun, we normalize $\mathsf{P}_{\leq *}$ so that it sits in degrees 0 through 2δ.

For any filtration $\mathsf{F}_{\leq *}$ on the cohomology of $\overline{\mathcal{P}}/\Gamma$, strictly compatible with the weight filtration, we may form the *virtual Poincaré polynomial*

$$\mathsf{P}^{vir,\mathsf{F}}(\mathsf{q}, \mathsf{t}) = \sum_{i,j,k} (-1)^i \mathsf{q}^j \mathsf{t}^k \dim \mathrm{gr}^{\mathsf{F}}_j \mathrm{gr}^{\mathsf{W}}_k \mathrm{H}^i(\overline{\mathcal{P}}/\Gamma).$$

Explicitly, Theorem 3.11 of [MY] states that

$$\mathsf{Hilb}(\mathsf{q}, \mathsf{t}) = \frac{1}{(1-\mathsf{q})^b} \mathsf{P}^{vir,\mathsf{P}}(\mathsf{q}, \mathsf{t}).$$

By comparison, Theorem 3 implies that

$$\mathsf{Quot}(\mathsf{q}, \mathsf{t}) = \frac{1}{(1-\mathsf{q})^b} \mathsf{P}^{vir,\mathsf{Q}}(\mathsf{q}, \mathsf{t}).$$

We deduce that:

Corollary 8.10. *Conjecture 1 is equivalent to*

$$\mathsf{P}^{vir,\mathsf{P}}(\mathsf{q}, \mathsf{t}) = \mathsf{P}^{vir,\mathsf{Q}}(\mathsf{q}, \mathsf{q}^{\frac{1}{2}}\mathsf{t}). \tag{8.5}$$

The motivation behind Conjecture 8 is that it would strictly imply (8.5), and hence, Conjecture 1. We emphasize again that while $\mathsf{P}_{\leq *}$ is defined via auxiliary global methods, $\mathsf{Q}^{\geq *}$ is intrinsic and purely local. For this reason, Corollary 8.10 seems remarkable to us.

8.7. **The Rennemo Splitting.** We keep the curve C and the family of curves $\mathcal{C}$ from above. Note that the conditions on $\mathcal{C}$ in [MY, §2.1] are essentially the same as those in [R, §2.1].

For any integer $\ell \geq 0$, let $\mathcal{H}^\ell(C)$ be the Hilbert scheme of ℓ points on C. Let

$$AJ : \coprod_\ell \mathcal{H}^\ell(C) \to \overline{\mathcal{J}}(C)$$

be the Abel–Jacobi map, constructed for general ℓ via the basepoint s from earlier. Theorem 1.3 of [R] shows that $\bigoplus_\ell \mathrm{H}^*(\mathcal{H}^\ell(C))$ admits a quotient W, defined through the action of a certain Weyl algebra by Hecke correspondences, such that the composition

$$\mathrm{H}^*(\overline{\mathcal{J}}(C)) \xrightarrow{AJ^*} \bigoplus_\ell \mathrm{H}^*(\mathcal{H}^\ell(C)) \to W$$

is an isomorphism. Moreover, the bigrading on the middle term descends to W.

Following Rennemo, let D_* be the grading on $\mathrm{H}^*(\overline{\mathcal{J}}(C))$ obtained by pulling back the non-cohomological grading on W. Proposition 7.1 of *ibid.* states that D_* is a splitting of the perverse filtration $\mathsf{P}_{\leq *}$. Again, we can transport D_* to the cohomology of $\overline{\mathcal{P}}/\Gamma$ via the isomorphism $\mathrm{H}^*(\overline{\mathcal{J}}(C)) \simeq \mathrm{H}^*(\overline{\mathcal{P}}/\Gamma)$.

8.8. **The GORS Filtrations.** To conclude, we review the three filtrations $\mathsf{F}^{\mathrm{alg}}$, $\mathsf{F}^{\mathrm{ind}}$, $\mathsf{F}^{\mathrm{geom}}$ proposed in [GORS] and their relationship to our story.

Again, suppose that $R \simeq \mathbf{C}[\![\varpi^n, \varpi^d]\!]$ for some $d > 0$ coprime to n. For clarity below, let $\hat{\mathcal{B}}_{d/n} = \overline{\mathcal{J}}_{(1^n)}$. Let $\mathrm{H}^*_{\mathbf{G}_m}(\hat{\mathcal{B}}_{d/n})$ be the equivariant cohomology of $\hat{\mathcal{B}}_{d/n}$ with respect to the $\mathbf{G}_m$-action induced by scaling ϖ. The perverse filtration $\mathsf{P}_{\leq *}$ from earlier can be lifted to this equivariant, parabolic setting.

Writing $\mathrm{H}^*_{\mathbf{G}_m}(pt) = \mathbf{C}[\epsilon]$, with $\deg(\epsilon) = 2$, we can form the $\mathbf{C}[\epsilon]$-module

$$\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n} := \mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{\mathbf{G}_m}(\hat{\mathcal{B}}_{d/n}).$$

By work of Oblomkov–Yun [OY16, OY17], a symplectic reflection algebra known as the rational Cherednik algebra (RCA) of S_n with central charge $\frac{d}{n}$ acts on $\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n}|_{\epsilon \to 1}$, and under this action, it becomes graded-isomorphic to the simple RCA module usually denoted $L_{d/n}$. More precisely, the perverse grading on the cohomology corresponds to the grading on $L_{d/n}$ induced by the action of the so-called Euler element. As the notation suggests, the RCA action is constructed by viewing $\hat{\mathcal{B}}_{d/n}$ as an affine Springer fiber in $\hat{\mathcal{B}}_{(1^n)}$, though over an element of $\mathfrak{sl}_n(\mathbf{C}[\![x]\!])$ different in general from the element $\psi(d)$ in Section 8.4.

In [GORS], Gorsky–Oblomkov–Rasmussen–Shende construct two filtrations on $L_{d/n}$ called $\mathsf{F}^{\mathrm{alg}}$ and $\mathsf{F}^{\mathrm{ind}}$ by purely algebraic methods. The former uses the construction of $L_{d/n}$ as a quotient of the RCA representation on the polynomial ring $\mathbf{C}[x_1, \ldots, x_{n-1}]$, while the latter uses the shift functors relating $L_{d/n}$ and $L_{(d+n)/n}$. We refer to Section 4 of *ibid.* for the details.

Via transport from $\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n}|_{\epsilon \to 1}$, there is another grading on $L_{d/n}$ of geometric nature. Namely, following [GORS, 2782], let $\mathsf{F}^{\mathrm{geom}}_{\leq *}$ be defined by

$$\mathsf{F}^{\mathrm{geom}}_i L_{d/n} = \bigoplus_{\substack{j,k \\ j-k \leq i-2\delta}} \mathrm{gr}^{\mathsf{P}}_j \mathrm{H}^k_{d/n}(1). \tag{8.6}$$

Now, [GORS, Conj. 1.4] is the three-way equality $\mathsf{F}^{\mathrm{alg}} = \mathsf{F}^{\mathrm{ind}} = \mathsf{F}^{\mathrm{geom}}$.

There is another filtration $\mathsf{C}_{\leq *}$ on $\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n}|_{\epsilon \to 1}$, called the Chern filtration in [OY16], which ultimately arises from the affine paving of $\hat{\mathcal{B}}_{d/n}$ discussed earlier. Proposition 8.1.2 of [OY16] shows that $\mathsf{F}^{\mathrm{alg}}$ can be constructed from $\mathsf{C}_{\leq *}$ by a saturation formula exactly analogous to (8.6). Theorem 8.2.3(1) of *ibid.* shows that $\mathsf{C}_{\leq *} = \mathsf{P}_{\leq *}$. Therefore, the work of Oblomkov–Yun proves $\mathsf{F}^{\mathrm{alg}} = \mathsf{F}^{\mathrm{geom}}$.

Remark 8.11. While our paper was in revision, Xinchun Ma uploaded to arXiv a proof of the equality $\mathsf{F}^{\mathrm{alg}} = \mathsf{F}^{\mathrm{ind}}$ [M].

Note that $\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n}|_{\epsilon \to 0}$ is the bigraded vector space $\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*(\overline{\mathcal{J}}_{(1^n)})$ discussed elsewhere in this section. It is essentially conjectured in [OY17] that

$$\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n}|_{\epsilon \to 0} \simeq \mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n}|_{\epsilon \to 1}.$$

More precisely, Oblomkov–Yun introduce in *ibid.* a certain family of rings in two parameters ϵ and s, which recover the cohomology rings $\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n}(\epsilon)$ at $s = 1$. The isomorphism above would match the rings at $(\epsilon, s) = (0, 1)$ and $(1, 1)$. Conjecture 1.1.7 of *ibid.* would match those at $(\epsilon, s) = (0, 1)$ and $(1, 0)$. Conjecture 5.5.1 of *ibid.* would match the rings at all values of (ϵ, s).

8.9. **The Carlsson–Oblomkov Filtration.** We now specialize to the case where $d = n + 1$. Recall that the *ring of diagonal coinvariants* of S_n is the bigraded $\mathbf{C}S_n$-module formed by

$$\mathrm{DR}_n = \frac{\mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]}{\mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}_+},$$

where $\mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}_+$ is the ideal of S_n-invariant homogeneous polynomials, under the grading where $\deg(x_i) = (1, 1)$ and $\deg(y) = (-1, 1)$ for all i. An argument in [Go, §5] shows that $\mathrm{gr}^{\mathsf{F}^{\mathrm{ind}}}_* L_{(n+1)/n} \simeq \mathrm{sgn} \otimes \mathrm{DR}_n$, where sgn denotes the sign twist.

In [CO], Carlsson–Oblomkov construct a filtration $\mathsf{F}^{\mathrm{desc}}$ on DR_n that is triangular with respect to one of the gradings, using so-called descent monomials. They match $\mathrm{gr}^{\mathsf{F}^{\mathrm{desc}}}_* \mathrm{DR}_n$ with a bigraded space of the form $\mathrm{gr}^{\mathsf{F}^{fp}}_* \mathrm{H}^{\mathrm{BM}}_*(\hat{\mathcal{B}}_{(n+1)/n})$, where F^{fp} is a filtration defined using fixed-point localization and special features of the inclusion $\hat{\mathcal{B}}_{(n+1)/n} \subseteq \hat{\mathcal{B}}_{(1^n)}$. It is claimed on page 4 of *ibid.* that this isomorphism also matches DR_n with $\mathrm{gr}^{\mathsf{H}}_* \mathrm{H}^{\mathrm{BM}}_*(\hat{\mathcal{B}}_{(n+1)/n})$, where $\mathsf{H}_{\leq *}$ denotes the filtration induced by the Hikita filtration in Section 8.4, and we have suppressed the necessary grading shifts and substitutions.

We expect their claim is true with the filtration $\mathsf{Q}_{\leq *}$ from Section 8.2 in place of $\mathsf{H}_{\leq *}$, and up to taking a (bi)graded dual. Indeed, this change would suggest a route to proving Conjecture 8 for $R = \mathbf{C}[\![\varpi^n, \varpi^{n+1}]\!]$. Suppose that:

(1) $\mathsf{F}^{\mathrm{alg}} = \mathsf{F}^{\mathrm{ind}}$. (See Remark 8.11.)
(2) $\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n}|_{\epsilon \to 0} \simeq \mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{d/n}|_{\epsilon \to 1}$.
(3) $\mathrm{DR}_n \simeq \mathrm{gr}^{\mathsf{Q}}_* \mathrm{H}^{\mathrm{BM}}_*(\hat{\mathcal{B}}_{(n+1)/n})^\vee$ up to appropriate shifts and substitutions.

Then we could match:

$$\begin{aligned}\mathrm{gr}_*^{\mathsf{P}}\,\mathrm{H}^*(\hat{\mathcal{B}}_{(n+1)/n}) = \mathrm{gr}_*^{\mathsf{P}}\,\mathrm{H}^*_{d/n}|_{\epsilon\to 0} &\xleftrightarrow{(2)} \mathrm{gr}_*^{\mathsf{P}}\,\mathrm{H}^*_{d/n}|_{\epsilon\to 1}\\ &\xleftrightarrow{(8.6)} \mathrm{gr}_*^{\mathsf{F}^{\mathrm{geom}}} L_{(n+1)/n}\\ &\xleftrightarrow{[OY16]} \mathrm{gr}_*^{\mathsf{F}^{\mathrm{alg}}} L_{(n+1)/n}\\ &\xleftrightarrow{(1)} \mathrm{gr}_*^{\mathsf{F}^{\mathrm{ind}}} L_{(n+1)/n}\\ &\xleftrightarrow{[Go]} \mathrm{DR}_n\\ &\xleftrightarrow{(3)} \mathrm{gr}_*^{\mathsf{Q}}\,\mathrm{H}_*^{\mathrm{BM}}(\hat{\mathcal{B}}_{(n+1)/n})^\vee = \mathrm{gr}^*_{\mathsf{Q}}\,\mathrm{H}^*(\hat{\mathcal{B}}_{(n+1)/n}).\end{aligned}$$

Note that in this case, (8.5) is as strong as Conjecture 8, since the affine paving of $\hat{\mathcal{B}}_{(n+1)/n}$ shows that its cohomology is pure and supported in even degrees.

Appendix A. Gradings on Link Homology

A.1. In this appendix, we specify our grading conventions for Khovanov–Rozansky homology; compare them to those of other published works; and illustrate on the smallest examples (unknot, Hopf link, trefoil, $(3,4)$ torus knot) to aid the reader's sanity. Our exposition closely follows [GH, §1.6], but we correct some mistakes: See Remarks A.1–A.2.

A.2. **Soergel Bimodules.** Let $T = \mathbf{G}_m^n$, and let

$$\mathbf{S} := \mathrm{H}_T^*(pt) = \mathbf{C}[t_1, \ldots, t_n].$$

We regard $\mathbf{S}$ as a graded ring, with $\deg(t_i) = 2$ for all i. Thus the S_n-action on T that permutes coordinates also preserves the grading on $\mathbf{S}$. Let $s_i \in S_n$ be the transposition that swaps t_i and t_{i+1}.

In the category of graded $\mathbf{S}$-bimodules, we write (m) for the grading shift $\mathbf{B}(m)^i = \mathbf{B}^{i+m}$. Let SBim be the full subcategory generated by the identity bimodule $\mathbf{S}$ and the bimodules $\mathbf{S} \otimes_{\mathbf{S}^{s_i}} \mathbf{S}(1)$ for all i under isomorphisms, direct sums, tensor products $\otimes = \otimes_{\mathbf{S}}$, direct summands, and grading shifts. Objects of SBim are called *Soergel bimodules*. We write $\mathsf{K}^b(\mathsf{SBim})$ for the bounded homotopy category of SBim. It is a monoidal additive category whose unit is the complex consisting of the identity bimodule in degree zero.

Let Br_n be the group of braids on n strands up to isotopy. Any braid $\beta \in Br_n$ defines an object $\bar{\mathcal{T}}_\beta \in \mathsf{K}^b(\mathsf{SBim})$ called the Rouquier complex of β. See, *e.g.*, [GH, §2.1] for the precise definition. If $\beta = \beta'\beta''$ in Br_n, then any sequence of braid moves that transforms $\beta'\beta''$ into β defines an isomorphism from $\bar{\mathcal{T}}_{\beta'} \otimes \bar{\mathcal{T}}_{\beta''}$ onto $\bar{\mathcal{T}}_\beta$. Thus, the braid group is categorified by the objects $\bar{\mathcal{T}}_\beta$ under $\otimes$.

Let Vect_2 be the category of $\mathbf{Z}^2$-graded vector spaces that are finite-dimensional in each bidegree, such that the first grading is bounded below and the second is bounded. Let $\overline{\mathsf{HH}} = \overline{\mathsf{HH}}^{*,*} : \mathsf{SBim} \to \mathsf{Vect}_2$ be the *Hochschild cohomology* functor:

$$\overline{\mathsf{HH}}^{i,j}(\mathbf{B}) = \mathrm{Ext}^i_{\mathbf{S}\otimes_{\mathbf{C}}\mathbf{S}^{\mathrm{op}}}(\mathbf{S}, \mathbf{B}(j)).$$

These Ext's can be computed using a Koszul resolution of $\mathbf{S}$ over $\mathbf{S} \otimes_{\mathbf{C}} \mathbf{S}^{\mathrm{op}}$, which shows that the Ext grading sits in degrees 0 through (at most) n.

Let Vect_3 be the category of $\mathbf{Z}^3$-graded vector spaces that are finite-dimensional in each tridegree, such that the first grading is bounded below and the other two gradings are bounded. Let $\overline{\mathsf{HHH}} = \overline{\mathsf{HHH}}^{*,*,*}$ be the composition of functors

$$\mathsf{K}^b(\mathbf{S}\mathsf{Bim}) \xrightarrow{\overline{\mathsf{HH}}} \mathsf{K}^b(\mathsf{Vect}_2) \xrightarrow{\mathrm{H}^*} \mathsf{Vect}_3.$$

Explicitly, the gradings are ordered so that $\overline{\mathsf{HHH}}^{I,J,K} = \mathrm{H}^k(\overline{\mathsf{HH}}_n^{I,J})$.

The story above can be redone with the quotient torus $T_0 := T/T^{S_n}$ in place of T. Note that T_0 is just the image of T along the quotient map $\mathrm{GL}_n \to \mathrm{PGL}_n$. Replacing T with T_0 entails replacing $\mathbf{S}$ with its subring $\mathbf{S}_0 := \mathrm{H}^*_{T_0}(pt)$. We write $\mathcal{T}_\beta$, HH, HHH for the objects that respectively replace $\bar{\mathcal{T}}_\beta$, $\overline{\mathsf{HH}}$, $\overline{\mathsf{HHH}}$.

Let L be the link closure of β. In [Kh], Khovanov proved that $\mathsf{HHH}(\mathcal{T}_\beta)$ matches the *reduced* version of the triply-graded homology of L proposed in [DGR] and constructed in [KhR], up to an affine transformation of the trigrading. One can show that

$$\overline{\mathsf{HHH}}(\bar{\mathcal{T}}_\beta) \simeq \overline{\mathsf{HHH}}(\bar{\mathcal{T}}_{\mathrm{id}}) \otimes \mathsf{HHH}(\mathcal{T}_\beta), \tag{A.1}$$

and that in consequence, $\overline{\mathsf{HHH}}(\bar{\mathcal{T}}_\beta)$ matches the *unreduced* version of the homology constructed in [KhR], up to similar regradings.

A.3. **The Main Dictionary.** For any $\beta \in Br_n$, let

$$\overline{\mathsf{hhh}}_\beta(A, Q, T) = \sum_{I,J,K} A^I Q^J T^K \, \dim \overline{\mathsf{HHH}}^{I,J,K}(\bar{\mathcal{T}}_\beta),$$

$$\mathsf{hhh}_\beta(A, Q, T) = \sum_{I,J,K} A^I Q^J T^K \, \dim \mathsf{HHH}^{I,J,K}(\mathcal{T}_\beta).$$

That is:

(1) $\overline{\mathsf{hhh}}_\beta(A, Q, T)$ is the series denoted $\mathcal{P}_\beta(Q, A, T)$ in [EH, §A] and [GH, §1.6], and hhh_β is the analogue of $\overline{\mathsf{hhh}}$ for reduced homology.

We write:

(2) $\bar{\mathcal{P}}_L^{\mathrm{norm}}(A, Q, T)$ for the series denoted $\mathcal{P}_L^{\mathrm{norm}}(Q, A, T)$ in *loc. cit.*
(3) $\mathcal{P}_{L,\mathrm{ORS}}(a, q, t)$ for the series denoted $\mathcal{P}(L)$ in [ORS]. It is denoted $\mathcal{P}(L^-)$ in [DGR], where L^- is the chiral mirror of L.
(4) $\bar{\mathcal{P}}_{L,\mathrm{ORS}}(a, q, t)$ for the series denoted $\bar{\mathcal{P}}(L)$ in [ORS], which satisfies

$$\bar{\mathcal{P}}_{L,\mathrm{ORS}}(a, q, t) = \bar{\mathcal{P}}_{U,\mathrm{ORS}}(a, q, t)\mathcal{P}_{L,\mathrm{ORS}}(a, q, t). \tag{A.2}$$

Remark A.1. Contrary to statements suggested by [ORS, 651] and [GH, §1.6], the series $\bar{\mathcal{P}}_{L,\mathrm{ORS}}$ does *not* match the series called the unreduced superpolynomial of L^- and denoted $\bar{\mathcal{P}}(L^-)$ in [DGR], even after further regrading. Indeed, the series denoted $\mathcal{P}(L^-)$ and $\bar{\mathcal{P}}(L^-)$ in [DGR] are *not* proportional to each other by any constant factor, as can be checked from Propositions 6.1 and 6.2 of *ibid.*

Let e be the *writhe* of β, meaning its net number of crossings counted with sign, and let b be the number of components of L. After correction, [GH, §1.6] states:

$$\begin{aligned} \bar{\mathcal{P}}_L^{\mathrm{norm}}(A,Q,T) &= (A^{\frac{1}{2}})^{e-n+b}Q^{-e+2n-2b}(T^{\frac{1}{2}})^{-e-n+b}\,\overline{\mathsf{hhh}}_\beta(A,Q,T), \\ \bar{\mathcal{P}}_{L,\mathrm{ORS}}(a,q,t) &= a^{-b}q^b\,\bar{\mathcal{P}}_L^{\mathrm{norm}}(a^2q^2t,q,t^{-1}) \\ &= a^{e-n}q^nt^e\,\overline{\mathsf{hhh}}_\beta(a^2q^2t,q,t^{-1}). \end{aligned} \tag{A.3}$$

By combining the last identity above with (A.1)–(A.2), we get a reduced version:

$$\mathcal{P}_{L,\mathrm{ORS}}(a,q,t) = a^{e-n+1}q^{n-1}t^e\,\mathsf{hhh}_\beta(a^2q^2t,q,t^{-1}).$$

In general, we will not work with $\bar{\mathcal{P}}_L^{\mathrm{norm}}$. Moreover, we will not discuss at all the normalizations used in the series $\mathcal{P}(U), \mathcal{P}(T(2,3))$ in [GH, Rem. 1.27].

Remark A.2. Above, (A.3) fixes a few more typos in [GH, §1.6]:

First, the discussion on [GH, 599] relates their series $\mathcal{P}_L^{\mathrm{norm}}$ to the series we call $\bar{\mathcal{P}}_{L,\mathrm{ORS}}$, not to the superpolynomial in [DGR]. As explained in Remark A.1, the latter two are different. Next, the identity relating $\mathcal{P}_L^{\mathrm{norm}}$ and $\bar{\mathcal{P}}_{L,\mathrm{ORS}}$ in *loc. cit.* has the wrong prefactor. There, the authors express $\bar{\mathcal{P}}_{L,\mathrm{ORS}}$ in terms of variables r, α, Q, T, which correspond to our b, a, q, t^{-1}, respectively. Their prefactor $Q^{2r}\alpha^{-r}$ should be $Q^r\alpha^{-r}$.

By way of comparison: The variables α, Q, T in [EH, §A] also correspond to our a, q, t^{-1}. Hence, their series $\mathcal{P}_L(Q,\alpha,T)$ is our series $\bar{\mathcal{P}}_{L,\mathrm{ORS}}(a,q,t)$. The identity relating $\mathcal{P}_\beta$ and $\mathcal{P}_L$ in *loc. cit.* is correct.

Example A.3. The unknot U is the knot closure of the identity in Br_1, for which $(n,e,b) = (1,0,1)$. The Hochschild cohomology of the identity Soergel bimodule is

$$\overline{\mathsf{HH}}_1^{*,j}(\mathbf{S}) = \begin{cases} \mathbf{S} & j = 0, \\ \mathbf{S}(2) & j = 1, \\ 0 & j \neq 0, 1. \end{cases}$$

Thus $\bar{\mathcal{P}}_U^{\mathrm{norm}}(A,Q,T) = \overline{\mathsf{hhh}}_{\mathrm{id}}(A,Q,T) = \dfrac{1+AQ^{-2}}{1-Q^2}$, from which

$$\bar{\mathcal{P}}_{U,\mathrm{ORS}}(a,q,t) = \frac{a^{-1}+at}{q^{-1}-q}.$$

A.4. **"Our" Series.** For any braid $\beta \in Br_n$ with writhe e whose link closure L has b components, let

$$\begin{aligned} \bar{\mathsf{X}}_\beta(\mathsf{a},\mathsf{q},\mathsf{t}) &:= \mathsf{t}^{\frac{e}{2}}\,\overline{\mathsf{hhh}}_\beta(\mathsf{aq},\mathsf{q}^{\frac{1}{2}},\mathsf{q}^{\frac{1}{2}}\mathsf{t}^{-\frac{1}{2}}), \\ \mathsf{X}_\beta(\mathsf{a},\mathsf{q},\mathsf{t}) &:= \frac{\bar{\mathsf{X}}_\beta(\mathsf{a},\mathsf{q},\mathsf{t})}{\bar{\mathsf{X}}_{\mathrm{id}}(\mathsf{a},\mathsf{q},\mathsf{t})} = \mathsf{t}^{\frac{e}{2}}\,\mathsf{hhh}_\beta(\mathsf{aq},\mathsf{q}^{\frac{1}{2}},\mathsf{q}^{\frac{1}{2}}\mathsf{t}^{-\frac{1}{2}}). \end{aligned}$$

Above, note that $\mathsf{X}_{\mathrm{id}}(\mathsf{a},\mathsf{q},\mathsf{t}) = \dfrac{1+\mathsf{a}}{1-\mathsf{q}}$. We can check that

$$\begin{aligned} \bar{\mathcal{P}}_{L,\mathrm{ORS}}(a,q,t) &= (aq^{-1})^{e-n}\,\bar{\mathsf{X}}_\beta(a^2t,q^2,q^2t^2), \\ \mathcal{P}_{L,\mathrm{ORS}}(a,q,t) &= (aq^{-1})^{e-n+1}\,\mathsf{X}_\beta(a^2t,q^2,q^2t^2). \end{aligned}$$

It turns out that in the rest of this paper, $\bar{\mathsf{X}}_\beta$ and X_β are the most convenient series for us to use.

In particular, suppose that $f(x,y) \in \mathbf{C}[\![x]\!][y]$ such that $f(x,y) = 0$ defines a generically separable, degree-n cover of the x-axis, fully ramified at $(x,y) = (0,0)$. Then the preimage in the cover of a positively-oriented loop around $x = 0$ is a braid $\beta_f \in Br_n$, whose link closure is the link L_f introduced in Section 1.5. We see that $\bar{\mathsf{X}}_{\beta_f}$ is precisely the series $\bar{\mathsf{X}}_f$ introduced in (1.4).

A.5. **Torus Links.** For integers $n, d > 0$, let $T_{n,d}$ be the positive (n,d) torus link, considered negative in [DGR]. Its number of components is $b = \gcd(n,d)$. Taking $f(x,y) = y^n - x^d$ in the construction above shows that $T_{n,d}$ is the link closure of a braid $\beta_{n,d} \in Br_n$ for which $e = (n-1)d$. Let

$$\delta = \frac{1}{2}(e - n + b) = \frac{1}{2}(nd - n - d + \gcd(n,d)).$$

Let $\bar{\mathsf{X}}_{n,d} = \bar{\mathsf{X}}_{\beta_{n,d}}$, as in the rest of this paper, and $\mathsf{X}_{n,d} = \mathsf{X}_{\beta_{n,d}}$.

Example A.4. For the Hopf link $T_{2,2}$, we have

$$\begin{aligned} \mathsf{X}_{2,2}(\mathsf{a},\mathsf{q},\mathsf{t}) &= 1 + \frac{\mathsf{qt}}{1-\mathsf{q}} + \frac{\mathsf{at}}{1-\mathsf{q}}, \\ \mathcal{P}_{T_{2,2},\mathrm{ORS}}(a,q,t) &= aq^{-1} + \frac{aq^3t^2}{1-q^2} + \frac{a^3qt^3}{1-q^2}. \end{aligned}$$

Example A.5. For the trefoil $T_{2,3}$, we have

$$\begin{aligned} \mathsf{X}_{2,3}(\mathsf{a},\mathsf{q},\mathsf{t}) &= 1 + \mathsf{qt} + \mathsf{at}, \\ \mathcal{P}_{T_{2,3},\mathrm{ORS}}(a,q,t) &= a^2(q^{-2} + q^2t^2) + a^4t^3. \end{aligned}$$

The latter series is [DGR, Ex. 3.3].

Example A.6. For the $(3,4)$ torus knot $T_{3,4}$, we have

$$\begin{aligned} \mathsf{X}_{3,4}(\mathsf{a},\mathsf{q},\mathsf{t}) &= 1 + \mathsf{qt} + \mathsf{qt}^2 + \mathsf{q}^2\mathsf{t}^2 + \mathsf{q}^3\mathsf{t}^3 \\ &\quad + \mathsf{a}(\mathsf{t} + \mathsf{t}^2 + \mathsf{qt}^2 + \mathsf{qt}^3 + \mathsf{q}^2\mathsf{t}^3) \\ &\quad + \mathsf{a}^2\mathsf{t}^3, \\ \mathcal{P}_{T_{3,4},\mathrm{ORS}}(a,q,t) &= a^6(q^{-6} + q^{-2}t^2 + t^4 + q^2t^4 + q^6t^6) \\ &\quad + a^8(q^{-4}t^3 + q^{-2}t^5 + t^5 + q^2t^7 + q^4t^7) \\ &\quad + a^{10}t^8. \end{aligned}$$

The latter series is [DGR, Ex. 3.4].

In Section 4, we implicitly need the following identities that match $\bar{\mathsf{X}}_{n,d}, \mathsf{X}_{n,d}$ with other series in the literature.

(1) Let $\tilde{P}_{n,m}(u,q,t)$ be the series in [GN]. For coprime n, d, we have

$$\bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}) = \frac{\mathsf{t}^\delta}{1-\mathsf{q}}\,\tilde{P}_{n,d}(-\mathsf{a},\mathsf{q},\mathsf{t}^{-1}).$$

(2) Let $\mathcal{P}_{m,n} = \mathcal{P}_{m,n}(a,q,t)$ be the series in [M22]. For coprime n, d, we have

$$\bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}) = (-\mathsf{a}^{-1}\mathsf{q}^{\frac{1}{2}}\mathsf{t}^{\frac{1}{2}})^{\delta}\,\mathcal{P}_{n,d}(-\mathsf{a},\mathsf{q},\mathsf{t}^{-1}).$$

Note that the substitution sends $t \mapsto \mathsf{q}$ and $q \mapsto \mathsf{t}^{-1}$, not vice versa.

(3) Let $\hat{P}_{0^M,0^N}(q,t,a), \hat{Q}_{0^M,0^N}(q,t,a), R_{0^M,0^N}(q,t,a)$ be the series in [GMV20]. For any n, d, we have

$$\begin{aligned}\frac{1}{1+\mathsf{a}}\bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}) &= \frac{1}{1-\mathsf{q}}\mathsf{X}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}^{-1}) \\ &= R_{0^n,0^d}(\mathsf{q},\mathsf{t}^{-1},\mathsf{a}\mathsf{q}^{-1}) \\ &= \hat{Q}_{0^n,0^d}(\mathsf{q},\mathsf{t}^{-1},\mathsf{a}\mathsf{q}^{-1}) && \text{by [GMV20, Cor. 5.10]} \\ &= \mathsf{q}^{-d-n}\hat{P}_{0^n,0^d}(\mathsf{q},\mathsf{t},\mathsf{a}\mathsf{q}^{-1}) && \text{by [GMV20, (11)].}\end{aligned}$$

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