LEVEL-RANK DUALITIES FROM Φ-HARISH-CHANDRA SERIES AND AFFINE SPRINGER FIBERS

MINH-TÂM QUANG TRINH AND TING XUE

Abstract. For any generic finite reductive group $\mathbb{G}$, integer $e > 0$, and Φ_e-cuspidal pair $(\mathbb{L}, \lambda)$, Broué–Malle–Michel conjectured that the endomorphism rings of the Deligne–Lusztig representations attached to $\mathbb{G}$, $(\mathbb{L}, \lambda)$ all come from the same generic cyclotomic Hecke algebra. We propose a new conjecture about the Harish-Chandra theory of such pairs, involving two integers e and m: namely, that the intersection of an Φ_e-Harish-Chandra series and a Φ_m-Harish-Chandra series is parametrized by both a union of Φ_m-blocks of the Φ_e-Hecke algebra and a union of Φ_e-blocks of the Φ_m-Hecke algebra, in a way that matches blocks. We also conjecture that when blocks match, there is an equivalence of categories between their highest-weight covers. When $e = 1$, we provide evidence that our bijections are essentially realized by bimodules that Oblomkov–Yun construct from the cohomology of affine Springer fibers. This suggests a strange analogy: Roughly, homogeneous affine Springer fibers are to roots of unity as tensor products of Deligne–Lusztig representations are to prime powers.

We predict the generic Hecke parameters for arbitrary Φ-cuspidal pairs of the groups $\mathbb{GL}_n$ and $\mathbb{GU}_n$, unifying the known cases. We prove that they would imply our conjectural bijections for these groups and coprime e, m. Then we show that the bijections for $\mathbb{GL}_n$ would be related by affine permutations to Uglov's bijections between bases of higher-level Fock spaces. This would reduce our block equivalences for $\mathbb{GL}_n$ to those conjectured by Chuang–Miyachi and proved by several authors under the name of level-rank duality. Finally, for many cases in exceptional types, we verify that the parameters predicted by Broué–Malle are compatible with our conjectures.

Contents

1. Introduction

1.1. *Double affine Hecke algebras*, or *DAHAs*, were introduced by Cherednik in proving Macdonald's conjectures about orthogonal polynomials [Ch]. Their rational degenerations, also known as *rational Cherednik algebras*, were introduced in [D, EG], and have since found independent applications to symplectic geometry and representation theory. In particular, rational DAHAs may be viewed as rings of quantized differential operators, which admit analogues of the Bernstein–Gelfand–Gelfand category O of a semisimple Lie algebra [GGOR]. Via a Riemann–Hilbert-type construction, their categories O form the highest-weight covers of the module categories of *cyclotomic Hecke algebras* at roots of unity [R08].

In [S76], Springer constructed representations of finite Weyl groups using the cohomology of fixed-point varieties in flag manifolds, now known as Springer fibers. In [L96], Lusztig extended this work to affine Weyl groups, using ind-varieties now known as *affine Springer fibers*. In [OY], motivated by prior K-theoretic work in [VV], Oblomkov–Yun developed a double-affine analogue, involving actions of trigonometric and rational DAHAs on the modified cohomology of *homogeneous* affine Springer fibers. In this setting, the Springer actions intertwine with the actions of certain braid groups, arising via monodromy and associated with smaller complex reflection groups than the original Weyl group. This paper grew from our attempts to find formulas for the resulting (DAHA, braid-group) bimodules.

In [T], motivated by a putative Betti analogue of Oblomkov–Yun's setup, the first author conjectured a formula for the virtual graded character of their rational DAHA module in the split case, taking the form

$$\sum_{\chi} \mathrm{Deg}_{\chi}(e^{2\pi i\nu})[\Delta_{\nu}(\chi)]. \tag{1.1}$$

Above, $\nu \in \mathbf{Q}_{>0}$ is the constant central charge of the rational DAHA, in lowest terms. The sum runs over the irreducible characters of the Weyl group W. The expression $[\Delta_{\nu}(\chi)]$ is the graded character of the Verma module of the DAHA indexed by χ, while its coefficient $\mathrm{Deg}_{\chi}(e^{2\pi i\nu}) \in \mathbf{Z}$ is the value at $x = e^{2\pi i\nu}$ of the *generic-degree polynomial* $\mathrm{Deg}_{\chi}(x) \in \mathbf{Q}[x]$. This polynomial can be defined as follows: For any prime power $q > 1$ and split finite reductive group $\mathbb{G}(q)$ over $\mathbf{F}_q$ with Weyl group W, the degree of the unipotent principal series representation of $\mathbb{G}(q)$ indexed by χ is equal to $\mathrm{Deg}_{\chi}(q)$. In this way, (1.1) suggests a connection between the rational DAHA modules in [OY] and representations of finite groups of Lie type.

In [VX], in the context of their ongoing study of character sheaves for graded Lie algebras,[1] Vilonen and the second author constructed a local system over the same base space as the affine Springer fibration in [OY]. They computed that its monodromy, a priori a braid action, factors through a cyclotomic Hecke algebra for the underlying complex reflection group. We expect that the local system of Oblomkov–Yun, after we forget its DAHA action, is essentially the local system of Vilonen–Xue, via a double application of Fourier duality passing through work of Lusztig–Yun [LY17, LY18]. This would mean the braid-group representations in [OY] factor through the same Hecke algebras.

[1] Also known as anti-orbital complexes.

1.2. These starting points have led us to a more general framework, phrased most clearly and symmetrically in terms of Broué–Malle–Michel's formalism of *generic finite reductive groups* [BMM93]. We review it more fully in Sections 2–3.

Roughly, a generic finite reductive group $\mathbb{G}$ consists of a root datum $\Gamma_{\mathbb{G}}$ and a finite-order automorphism of $\Gamma_{\mathbb{G}}$, defined up to composition with the Weyl group. For simplicity, let us exclude the Suzuki and Ree cases. Then, for each prime power $q > 1$, the root datum defines a connected reductive group G over $\bar{\mathbf{F}}_q$, and the automorphism defines a q-Frobenius $F : G \to G$, hence a finite reductive group $\mathbb{G}(q) := G^F$. The work of Broué–Malle–Michel concerns the parts of the representation theory of $\mathbb{G}(q)$ where q can be treated as an indeterminate, or which do not depend on q at all. In particular, the irreducible characters of $\mathbb{G}(q)$ which are *unipotent* in the sense of Deligne–Lusztig theory can be indexed by a set $\mathrm{Uch}(\mathbb{G})$ depending only on $\mathbb{G}$.

As in Harish-Chandra's philosophy of cusp forms, the elements of $\mathrm{Uch}(\mathbb{G})$ can be grouped, according to their behavior under Lusztig restriction, into subsets $\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda)$ indexed by equivalence classes of pairs $(\mathbb{L}, \lambda)$, where:

(1) $\mathbb{L}$ is the generic version of a Levi subgroup whose root system is a parabolic subsystem of that of $\mathbb{G}$.
(2) $\lambda \in \mathrm{Uch}(\mathbb{L})$ is the generic version of a cuspidal unipotent irreducible character of that Levi.

The equivalence relation is conjugacy under the Weyl group of $\mathbb{G}$.

For any integer $e > 0$, we say that $\mathbb{L}$ is *Φ_e-split* iff it is the centralizer of a *Φ_e-torus*, meaning one whose generic order is a power of the eth cyclotomic polynomial Φ_e. We say that λ is *Φ_e-cuspidal* iff it is not induced from any smaller Φ_e-split Levi. The main result of [BMM93] states that for fixed e, the subsets $\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda)$ arising from Φ_e-split $\mathbb{L}$ and Φ_e-cuspidal λ are disjoint and partition $\mathrm{Uch}(\mathbb{G})$. They recover the partition into usual Harish-Chandra series when $e = 1$, and hence, are called *Φ_e-Harish-Chandra series*.

Each Φ_e-cuspidal pair $(\mathbb{L}, \lambda)$ defines a finite complex reflection group $W_{\mathbb{G},\mathbb{L},\lambda}$: the centralizer of λ in the generic relative Weyl group of $(\mathbb{G}, \mathbb{L})$. Broué–Malle–Michel predict the existence of an algebra $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ over a suitable cyclotomic extension of $\mathbf{Z}[x]$, such that:

- The group algebra of $W_{\mathbb{G},\mathbb{L},\lambda}$ is isomorphic to $H_{\mathbb{G},\mathbb{L},\lambda}(\zeta_e)$, where ζ_e is any primitive eth root of unity.
- For all prime powers $q > 1$, the endomorphism algebra of the Deligne–Lusztig representation of $\mathbb{G}(q)$ arising from $(\mathbb{L}, \lambda)$ is $H_{\mathbb{G},\mathbb{L},\lambda}(q)$.

See Conjecture 3.3. Above, the notation $H_{\mathbb{G},\mathbb{L},\lambda}(\alpha)$ connotes a specialization of the form $x^{\frac{1}{n}} \mapsto \alpha^{\frac{1}{n}}$. By the double-centralizer theorem, the existence of $H_{\mathbb{G},\mathbb{L},\lambda}(x)$, together with an appropriate flatness statement, would imply a bijection

$$\chi^{\mathbb{G}}_{\mathbb{L},\lambda} : \mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda) \xrightarrow{\sim} \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda}) \tag{1.2}$$

compatible with induction from smaller Φ_e-split Levis. Even though $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ has been constructed only in certain cases, Broué–Malle–Michel were able to construct the putative bijections $\chi^{\mathbb{G}}_{\mathbb{L},\lambda}$ for all $\mathbb{G}$ and $(\mathbb{L}, \lambda)$, via case-by-case arguments.

1.3. Suppose that $\mathbb{A}$ is a Φ_1-split maximal torus of $\mathbb{G}$ for which $H_{\mathbb{G},\mathbb{A},1}(x)$ exists, and that the automorphisms defining $\mathbb{G}$ and $\mathbb{A}$ are induced by the same Dynkin-diagram automorphism. In this case, the BGG category O of the rational DAHA in [OY], depending on a *slope* $\nu \in \mathbf{Q}_{>0}$, defines a highest-weight cover of the module category of $H_{\mathbb{G},\mathbb{A},1}(e^{2\pi i\nu})$. Note that by the discussion above, the specialization $H_{\mathbb{G},\mathbb{A},1}(1)$ must be the group algebra of the relative Weyl group of $(\mathbb{G},\mathbb{A})$, but for $\nu \notin \mathbf{Z}$, the specializations $H_{\mathbb{G},\mathbb{A},1}(e^{2\pi i\nu})$ need not even be semisimple.

On the other hand, if $\mathbb{T}$ is a Φ_m-split maximal torus, then $W_{\mathbb{G},\mathbb{T},1}$ is the complex reflection group whose braid group appears in [OY], when $\nu = \frac{d}{m}$ for any $d > 0$ coprime to m. Recall that we expect the braid actions in *ibid.* to factor through the Hecke algebras in [VX]. It turns out that the latter are all of the form $H_{\mathbb{G},\mathbb{T},1}(1)$, for various m and $\mathbb{T}$.

These observations suggest that the (DAHA, braid-group) bimodules in [OY] reify some sort of duality between the specializations $H_{\mathbb{G},\mathbb{A},1}(\zeta_m)$ and $H_{\mathbb{G},\mathbb{T},1}(1)$. One is led to guess that beyond this geometric setting, the specific choices $\mathbb{A}$ and $\mathbb{T}$ are artificial, and that dualities should exist between certain blocks of $H_{\mathbb{G},\mathbb{L},\lambda}(\zeta_m)$ and of $H_{\mathbb{G},\mathbb{M},\mu}(\zeta_e)$ for any primitive roots ζ_e and ζ_m, any Φ_e-cuspidal pair $(\mathbb{L},\lambda)$, and any Φ_m-cuspidal pair $(\mathbb{M},\mu)$. We have found evidence that this is so.

To state our conjectures, we use the following notation. Assuming that $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ has been defined, and that b is a block of the module category for $H_{\mathbb{G},\mathbb{L},\lambda}(\zeta_m)$, let $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathsf{b}} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})$ be the subset of irreducible characters that index the standard objects in the highest-weight cover of b. Equivalently, they index the simple $KH_{\mathbb{G},\mathbb{L},\lambda}(x)$-modules in the preimage of b along the Brauer decomposition map [GP, GJ] from virtual $KH_{\mathbb{G},\mathbb{L},\lambda}(x)$-modules to virtual $H_{\mathbb{G},\mathbb{L},\lambda}(\zeta_m)$-modules, for an appropriate field $K \supseteq \mathbf{Q}(x)$.

For $e, m, (\mathbb{L},\lambda), (\mathbb{M},\mu)$ as above, let $\mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu)$ be the intersection of the Harish-Chandra series $\mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda)$ and $\mathrm{Uch}(\mathbb{G},\mathbb{M},\mu)$. Let $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})$ and $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})$ be the images of the maps

$$\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda}) \xleftarrow{\chi^{\mathbb{G}}_{\mathbb{L},\lambda}} \mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu) \xrightarrow{\chi^{\mathbb{G}}_{\mathbb{M},\mu}} \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu}). \tag{1.3}$$

Assuming that $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ and $H_{\mathbb{G},\mathbb{M},\mu}(x)$ have been defined, consider the following assertions about these Hecke algebras:

(I) $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu}$, *resp.* $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda}$, is partitioned by sets taking the form $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathsf{b}}$ for blocks b of $H_{\mathbb{G},\mathbb{L},\lambda}(\zeta_m)$, *resp.* $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathsf{c}}$ for blocks c of $H_{\mathbb{G},\mathbb{M},\mu}(\zeta_e)$.

(II) Above, $\chi^{\mathbb{G}}_{\mathbb{L},\lambda}$ and $\chi^{\mathbb{G}}_{\mathbb{M},\mu}$ induce a bijection between the set of b such that $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathsf{b}} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu}$ and the set of c such that $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathsf{c}} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda}$: hence, a bijection between $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathsf{b}}$ and $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathsf{c}}$ when b and c correspond to each other.

(III) Above, the bijection $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathsf{b}} \xrightarrow{\sim} \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathsf{c}}$ is categorified by an equivalence between the derived categories of the highest-weight covers of b and c.

The statement below comprises Conjectures 4.1 and 4.3 in the body text.

Conjecture 1. *Whenever Broué–Malle–Michel's Conjecture 3.3 holds, the Hecke algebras $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ and $H_{\mathbb{G},\mathbb{M},\mu}(x)$ that it predicts satisfy properties (I)–(III).*

1.4. In Section 5, we review the precise setup of Oblomkov–Yun. Then we present evidence toward a conjectural formula for their (DAHA, braid-group) bimodules, or rather, its class in a certain Grothendieck group: See Proposition 5.4.

For now, we merely sketch their geometry. Suppose that the root datum $\Gamma_{\mathbb{G}}$ is irreducible, that $\mathbb{G}$ arises from a Dynkin-diagram automorphism, and that $\mathbb{A}$ is a Φ_1-split maximal torus of $\mathbb{G}$. Let $\mathfrak{g}$ be the Lie algebra over $\mathbf{C}[\![z]\!]$ corresponding to $\mathbb{G}$, and let $L\mathfrak{g}$ be the formal loop space for which $L\mathfrak{g}(\mathbf{C}) = \mathfrak{g}(\mathbf{C}(\!(z)\!))$ with z an indeterminate. For any $\nu \in \mathbf{Q}$, Oblomkov–Yun construct:

(1) A (finite-dimensional) subvariety $L\mathfrak{g}_\nu^{\mathrm{rs}} \subseteq L\mathfrak{g}$, stable under a $\mathbf{G}_m$-action on $L\mathfrak{g}$ that depends on ν and a choice of simple roots. The superscript rs is meant to connote regular semisimple elements.

We write $Br_{\nu,\gamma}$ for the fundamental group of $L\mathfrak{g}_\nu^{\mathrm{rs}}$ with basepoint γ. We previously mentioned that if m is the denominator of ν in lowest terms, and $\mathbb{T}$ is a Φ_m-split maximal torus of $\mathbb{G}$, then $Br_{\nu,\gamma}$ is the braid group of $W_{\mathbb{G},\mathbb{T},1}$.

Let $D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$ be the rational DAHA of $(\mathbb{G},\mathbb{A})$ with central charge $\vec{\nu}$ depending on ν as in [OY, §4.2]. (Note that some of our variable names do not match theirs.) When $\mathbb{G}$ is split, $\nu > 0$, and m is a *regular elliptic number* for the relative Weyl group $W_{\mathbb{G},\mathbb{A}}$ [VV, §1.1], Oblomkov–Yun construct:

(2) A local system of bigraded vector spaces over $L\mathfrak{g}_\nu^{\mathrm{rs}}$, which we will denote $\mathcal{E}_\nu$. The construction uses the $\mathbf{G}_m$-equivariant cohomology of a certain affine Springer fibration, together with its *perverse filtration*.
(3) A fiberwise $D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$-action on $\mathcal{E}_\nu$, commuting with the action of $Br_{\nu,\gamma}$ on $\mathcal{E}_{\nu,\gamma}$ by monodromy.

The $Br_{\nu,\gamma}$-invariants of $\mathcal{E}_{\nu,\gamma}$ form the simple $D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$-module often denoted $L_{\vec{\nu}}(1)$. We write $[\mathcal{E}_{\nu,\gamma}]$ for the virtual graded bimodule formed from $\mathcal{E}_{\nu,\gamma}$ by taking the alternating sum over cohomological degrees.

To state our conjectural formula, recall from [BMM93] that for any generic character $\rho \in \mathrm{Uch}(\mathbb{G},\mathbb{T},1)$ and primitive mth root of unity ζ, the generic degree $\mathrm{Deg}_\rho(x) \in \mathbf{Q}[x]$ satisfies

$$\mathrm{Deg}_\rho(\zeta) = \varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\rho) \deg \chi^{\mathbb{G}}_{\mathbb{T},1}(\rho) \quad \text{for some sign } \varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\rho) \in \{\pm 1\}. \tag{1.4}$$

We also write χ_1 for the character of $H_{\mathbb{G},\mathbb{T},1}(1)$ corresponding to a character χ of $W_{\mathbb{G},\mathbb{T},1}$ under Brauer decomposition.

Conjecture 2. *Let $\mathbb{G},\mathbb{A},\nu,m,\mathbb{T}$ be as above, with $\nu > 0$ and m a regular elliptic number for $W_{\mathbb{G},\mathbb{A}}$, and let $\gamma \in L\mathfrak{g}_\nu^{\mathrm{rs}}(\mathbf{C})$. Assume that either $\mathbb{G}$ is split or* [OY, Conj. 8.2.5] *holds, and that Conjecture 3.3 holds for $\mathbb{G}, m, (\mathbb{T},1)$. Then:*

(1) *The $Br_{\nu,\gamma}$-action on $\mathcal{E}_{\nu,\gamma}$ factors through $H_{\mathbb{G},\mathbb{T},1}(1)$.*
(2) *In the Grothendieck group $\mathsf{K}_0(\mathbf{C}W_{\mathbb{G},\mathbb{A}} \otimes H_{\mathbb{G},\mathbb{T},1}(1)^{\mathrm{op}})[\![t]\!][t^{-1}]$, we have*

$$[\mathcal{E}_{\nu,\gamma}] = \sum_{\rho \in \mathrm{Uch}(\mathbb{G},\mathbb{A},1,\mathbb{T},1)} \varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\rho)[\Delta_{\vec{\nu}}(\chi^{\mathbb{G}}_{\mathbb{A},1}(\rho)) \otimes \chi^{\mathbb{G}}_{\mathbb{T},1}(\rho)_1],$$

where $\Delta_{\vec{\nu}}(\chi)$ denotes the standard module of $D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$ indexed by χ, and the variable t tracks its $W_{\mathbb{G},\mathbb{A}}$-equivariant Euler grading.

In the split case, Conjecture 2 refines (1.1), via (1.4). The refined formula bears a remarkable analogy with a virtual bimodule that, under the Broué–Malle–Michel conjectures, can be constructed from Deligne–Lusztig representations:

(1.5)

bimodule	algebras	H-parameters
$\mathcal{E}_{\nu,\gamma}$	$(D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu}), H_{\mathbb{G},\mathbb{T},1}(1))$	$(e^{2\pi i\nu}, 1)$
$R^G_L(\lambda_q) \otimes_{\mathbf{C}G^F} R^G_M(\mu_q)$	$(H_{\mathbb{G},\mathbb{L},\lambda}(q), H_{\mathbb{G},\mathbb{M},\mu}(q))$	(q, q)

Above, $q > 1$ is a prime power; $G, L, \lambda_q, M, \mu_q$ are the finite-group data arising from $q, \mathbb{G}, \mathbb{L}, \lambda, \mathbb{M}, \mu$; and $R^G_L(\lambda_q), R^G_M(\mu_q)$ are the compactly-supported cohomologies of appropriate Deligne–Lusztig varieties. The top right entry alludes to how $D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$ provides a highest-weight cover of $H_{\mathbb{G},\mathbb{A},1}(e^{2\pi i\nu})$. Section 5 gives more details.

1.5. Most work on the existence and explicit parameters of the Hecke algebras $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ has focused on the case where $\mathbb{L}$ is a maximal torus of $\mathbb{G}$. Beyond tori, the work of Dudas in [Du] implicitly determines their parameters for the generic general linear groups $\mathbb{G} = \mathbb{GL}_n$ and cuspidal pairs $(\mathbb{L}, \lambda)$ such that $W_{\mathbb{G},\mathbb{L},\lambda}$ is cyclic, confirming a prediction of Broué–Malle [BM93, §2.10].

In Conjecture 6.1, we predict the Hecke parameters for arbitrary Φ-cuspidal pairs of $\mathbb{GL}_n$. Then, in Conjecture 6.4, we use Ennola duality to predict the parameters for the corresponding Φ-cuspidal pairs of $\mathbb{GU}_n$. In Propositions 6.2 and 6.5, we show that our predictions are consistent with all known cases. Note that for $\mathbb{G} = \mathbb{GL}_n, \mathbb{GU}_n$, the groups $W_{\mathbb{G},\mathbb{L},\lambda}$ are wreath products $\mathbf{Z}_e \wr S_a := (\mathbf{Z}/e\mathbf{Z})^a \rtimes S_a$; hence, the Hecke algebras are specializations of Ariki–Koike algebras, whose blocks at roots of unity were described combinatorially by Lyle–Mathas [LM]. At the conclusion of Section 6, we use their work to prove:

Theorem 3. *If $\mathbb{G} = \mathbb{GL}_n$,* resp. *$\mathbb{G} = \mathbb{GU}_n$, and the integers e, m are coprime, then the Hecke algebras $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ and $H_{\mathbb{G},\mathbb{M},\mu}(x)$ defined by* (6.2), resp. (6.3)*, satisfy properties (I) and (II). In fact,* $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu}$ *and* $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda}$ *each correspond to a single block.*

In fact, for the general linear cases, we will relate the bijections in part (2) of Conjecture 1 to bijections that previously appeared in Uglov's work on *higher-level Fock spaces* [U]. Recall that for each tuple $\vec{s} \in \mathbf{Z}^e$, the v-deformed Fock space of level e and charge $\vec{s}$ is the vector space $\Lambda^{\vec{s}}_v$ over $\mathbf{Q}(v)$ spanned by symbols $|\vec{\lambda}, \vec{s}\rangle$, where $\vec{\lambda}$ runs over e-tuples of integer partitions, or *e-partitions*. For each integer $m > 0$, it may be viewed as a module over the quantum affine algebra $\mathrm{U}_v(\widehat{\mathfrak{sl}}_m)$, in which case the residue of $\vec{s}$ modulo m describes the highest weight of the simple submodule generated by $|\vec{0}, \vec{s}\rangle$. Generalizing the work of Leclerc–Thibon on the level-1 case, Uglov constructed a canonical basis for $\Lambda^{\vec{s}}_v$, related to the standard basis by an upper-triangular transition matrix of affine Kazhdan–Lusztig polynomials. To do so, he made use of vector-space isomorphisms

$$\bigoplus_{\substack{\vec{s} \in \mathbf{Z}^e \\ s_1 + \cdots + s_e = s}} \Lambda^{\vec{s}}_v \xleftarrow{\sim} \Lambda^s_v \xrightarrow{\sim} \bigoplus_{\substack{\vec{r} \in \mathbf{Z}^m \\ r_1 + \cdots + r_m = s}} \Lambda^{\vec{r}}_v$$

for each integer s, relating level-e and level-m Fock spaces to the level-1 Fock space of charge s. Crucially, these isomorphisms are not defined symmetrically. Their composition defines a nontrivial bijection from charged e-partitions onto charged m-partitions. In [G], this bijection is described as a *level-rank duality*.

As already observed in [U], the left-hand map is essentially induced by the map sending an integer partition to its e-core and e-quotient. The right-hand map is a twisted version of the analogous map for m. It turns out that for $\mathbb{G} = \mathbb{GL}_n$, where $\mathrm{Uch}(\mathbb{G})$ is indexed by partitions of n, the Φ_e-Harish-Chandra series of a unipotent irreducible character is indexed by the e-core of its partition, while its image under (1.2) is indexed by a shifted version of its e-quotient. This similarity of structure motivates the following result, proved in Section 7:

Theorem 4. *Let Π be the set of all integer partitions. In the setup of Theorem 3 for $\mathbb{G} = \mathbb{GL}_n$, the maps of* (1.3) *fit into a commutative diagram*

$$\begin{tikzcd}
\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda}) \arrow[d] & \mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu) \arrow[l, "\chi^{\mathbb{G}}_{\mathbb{L},\lambda}"'] \arrow[r, "\chi^{\mathbb{G}}_{\mathbb{M},\mu}"] \arrow[d] & \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu}) \arrow[d] \\
\Pi^e \times \mathbf{Z}^e \arrow[d, "\tilde{w}_{e,m,e+\ell_\lambda}"'] & \Pi \arrow[l, "\Upsilon^1_e(|\rho, e+\ell_\lambda\rangle) \leftarrow\!\shortmid \rho"'] \arrow[r, "\rho \mapsto \Upsilon^1_m(|\rho, m+\ell_\mu\rangle)"] & \Pi^m \times \mathbf{Z}^m \arrow[d, "\tilde{w}_{m,e,m+\ell_\mu}"] \\
\Pi^e \times \mathbf{Z}^e \arrow[rr, "\Upsilon^e_m"] & & \Pi^m \times \mathbf{Z}^m
\end{tikzcd}$$

where the integers ℓ_λ, ℓ_μ are the respective lengths of λ, μ; the maps $\tilde{w}_{(-,-,-)}$ are induced by affine permutations; and the map Υ^e_m is Uglov's bijection in [U, §4.1], *up to normalization. All maps in the diagram are injective or bijective.*

Furthermore, the diagram is compatible with Lusztig induction between Φ_e- and Φ_m-split Levis in a precise sense, given by Theorem 2.4(2d).

Chuang–Miyachi conjectured that Uglov's bijections could be categorified by Koszul dualities between blocks of highest-weight covers for Ariki–Koike algebras, *i.e.*, blocks of categories O for cyclotomic rational DAHAs [ChuM]. This categorification of level-rank duality was proved by Shan–Varagnolo–Vasserot in [SVV], through equivalences between such categories O and truncations of the parabolic categories O of the affine Lie algebras $\widehat{\mathfrak{sl}}_e$. The latter equivalences were proved by Losev [L] Rouquier–Shan–Varagnolo–Vasserot [RSVV], and Webster [W] independently. Using these results, we deduce:

Corollary 5. *If $\mathbb{G} = \mathbb{GL}_n$ and the integers e, m are coprime, then the Hecke algebras $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ and $H_{\mathbb{G},\mathbb{M},\mu}(x)$ defined by* (6.2) *satisfy property (III).*

1.6. In Section 8, we give evidence for Conjecture 1 in exceptional types, again excluding the Suzuki and Ree cases. More precisely, for explicit Hecke parameters that were either determined in [L-Cox, L78] or conjectured in [BM93, M], we verify the partitioning of sets in (I), and the agreement of cardinalities implied by (II), though not the bijections: See Proposition 8.1. Note that by Corollary 2.7, it suffices to check pairs of *singular numbers* e, m. For (split) $\mathbb{G}$ of type G_2 or F_4, we give the explicit details for all such pairs.

1.7. **Future Work.** We expect Conjectures 6.1–6.4 to be tractable, as well as certain extensions of Theorems 3–4 and Corollary 5:

- In Theorem 3, we expect to remove the hypothesis that e, m be coprime from the first assertion. In this generality, the second assertion is false: the partitions in (I) need not be singletons.
- We expect precise analogues of Conjectures 6.1–6.4, Theorems 3–4, and Corollary 5 for reductive groups in types B, C, D, possibly after restricting the whole setup to the unipotent principal series.

We will address the items above in a sequel.

The bimodule in Conjecture 2 also seems closely related to work of Boixeda Alvarez–Losev [BL]. Using the same setup as Oblomkov–Yun, but at an *integral* slope d rather than a regular elliptic slope, they construct a $(D^{trig}_{\mathbb{G},\mathbb{A}}(d), D^{trig}_{\mathbb{G},\mathbb{A}}(0))$-bimodule, where $D^{trig}_{\mathbb{G},\mathbb{A}}(\nu)$ denotes the *trigonometric* DAHA rather than the rational DAHA. This seems to add a third row to the analogy in (1.5). In a separate future paper, we will address evidence toward an analogue of Conjecture 2(2) for the bimodule in [BL].

1.8. **Acknowledgments.** We thank Pablo Boixeda-Alvarez, Olivier Dudas, George Lusztig, Andrew Mathas, Kari Vilonen, and Zhiwei Yun for useful comments. During part of the preparation of this work, the first author was supported by an NSF Mathematical Sciences Research Fellowship, Award DMS-2002238.

2. Φ-Harish-Chandra Series

2.1. If H is a finite group, then we write $\mathsf{Rep}(H)$ to denote the category of finite-dimensional representations of H over an appropriate algebraically-closed field of characteristic zero. When we work with algebraic varieties over finite fields and their algebraic closures, it will be convenient to take the field $\bar{\mathbf{Q}}_\ell$, for a fixed prime $\ell > 0$. Elsewhere, we take the field $\mathbf{C}$ and fix isomorphisms $\bar{\mathbf{Q}}_\ell \simeq \mathbf{C}$.

We identify isomorphism classes of representations of H with their characters. We write $\mathrm{Irr}(H)$ for the set of irreducible characters.

2.2. Consider a prime power $q > 1$ and a connected, reductive algebraic group G over $\bar{\mathbf{F}}_q$, equipped with a q-Frobenius map $F : G \to G$ defining an $\mathbf{F}_q$-structure. For any F-stable Levi subgroup $L \subseteq G$, Lusztig introduced induction and restriction functors of the following form [L-Fin]:

$$R_L^G : \mathsf{Rep}(L^F) \rightleftarrows \mathsf{Rep}(G^F) : {}^*R_L^G$$

For any parabolic subgroup $P \subseteq G$ containing L, not necessarily F-stable, the induction functor R_L^G may be defined as $\mathrm{H}_c^*(Y^G_{L\subseteq P}) \otimes_{\mathbf{C}L^F} (-)$, where $Y^G_{L\subseteq P}$ is an algebraic variety over $\mathbf{F}_q$ equipped with commuting actions of G^F and L^F. Here, we have written $\mathrm{H}_c^*(Y)$ to denote the compactly-supported ℓ-adic cohomology of $Y_{\bar{\mathbf{F}}_q}$, where ℓ is invertible in $\mathbf{F}_q$. The restriction functor ${}^*R_L^G$ is defined as the right adjoint of R_L^G. The fact that $R_L^G, {}^*R_L^G$ do not depend on P was shown by Lusztig [L90]. For a more detailed exposition of these functors, we refer to the second edition of the book by Digne–Michel [DM20].

Under its L^F-action, the variety $Y^G_{L\subseteq P}$ forms a torsor over a G^F-variety $X^G_{L\subseteq P}$. An irreducible character of G^F is *unipotent* iff it occurs in $\mathrm{H}^*_c(X^G_{T\subseteq B})$ for some F-stable maximal torus T and Borel B, or equivalently, in $R^G_T(1)$ for some such T. Lusztig observed that while $\mathrm{Irr}(G^F)$ grows in size with q, the subset of unipotent irreducible characters $\mathrm{Uch}(G^F)$ can be indexed in a way depending only on the root datum of G and its Frobenius action, not on q itself [L78, L84].

Example 2.1. Suppose that $A \subseteq G$ is a maximally F-split maximal torus, and that B is an F-stable Borel containing A. Then $Y^G_{A\subseteq B} = G^F/U^F$, where U is the unipotent radical of B, and $X^G_{A\subseteq B} = G^F/B^F$. The representation $R^G_A(1)$ is just $\mathrm{H}^0(G^F/B^F)$, which we can view as the space of functions on G^F/B^F. The irreducible constituents of $R^G_A(1)$ form a subset of $\mathrm{Uch}(G^F)$ called the *unipotent principal series*, parametrized by the irreducible characters of the group $W_{G^F,A^F} := N_{G^F}(A^F)/A^F$. Later, we will discuss how to generalize this parametrization.

2.3. Broué–Malle–Michel introduced generic finite reductive groups in order to study properties of the functors $R^G_L, {}^*R^G_L$ and the sets $\mathrm{Uch}(G^F)$ that only depend on q through specializations of an indeterminate variable [BMM93, §1]. A *generic finite reductive group* $\mathbb{G}$, or *generic group* for short, consists of:

(1) A root datum $\Gamma_{\mathbb{G}} = (X, R, X^\vee, R^\vee)$.
(2) A coset of the form $[f]_{\mathbb{G}} := W_{\Gamma_{\mathbb{G}}} f \subseteq \mathrm{Aut}(\Gamma_{\mathbb{G}})$, where $W_{\Gamma_{\mathbb{G}}}$ is the Weyl group of $\Gamma_{\mathbb{G}}$ and f a finite-order automorphism of $\Gamma_{\mathbb{G}}$ normalizing $W_{\Gamma_{\mathbb{G}}}$. We say that $\mathbb{G}$ is *split* iff $[f]_{\mathbb{G}} = W_{\Gamma_{\mathbb{G}}}$.

An *isomorphism* between generic groups is an isomorphism between the root data in (1), matching the coset data in (2).

We say that a generic group $\mathbb{G}'$ is a *generic subgroup* of $\mathbb{G}$ iff its cocharacter lattice embeds into that of $\mathbb{G}$, its root system embeds into that of $\mathbb{G}$ as a parabolic subsystem, and $[f]_{\mathbb{G}'} \subseteq [f]_{\mathbb{G}}$. In this case, we write $\mathbb{G}' \leq \mathbb{G}$. To indicate that $\mathbb{G}' \neq \mathbb{G}$ as well, we write $\mathbb{G}' < \mathbb{G}$. Henceforth:

Assumption 2.2. We exclude from consideration any generic group $\mathbb{G}$ with a generic subgroup of the *Suzuki* type ${}^2C_2$ or the *Ree* types ${}^2G_2, {}^2F_4$. (The letter and subscript indicate the root datum, while the superscript is the minimal order among elements of the coset datum.)

Under Assumption 2.2, any choice of prime power $q > 1$ and representative $f \in [f]_{\mathbb{G}}$ gives rise to a tuple (G, T, F), where G is a connected, reductive algebraic group over $\bar{\mathbf{F}}_q$ with q-Frobenius map F, as above, and $T \subseteq G$ is an F-stable maximal torus. Conversely, every such (non-Suzuki/Ree) tuple comes from a generic group this way.

In the situation above, we set $\mathbb{G}(q) = G^F$ in a slight abuse of notation. As an abstract group, $\mathbb{G}(q)$ only depends on q and $\mathbb{G}$. The orders of the groups $\mathbb{G}(q)$ are generic, in the sense that we can define a polynomial $|\mathbb{G}|(x) \in \mathbf{Q}[x]$ such that $|\mathbb{G}|(q) = |\mathbb{G}(q)|$ for all q.

A *generic torus* is a generic group $\mathbb{T}$ whose root and coroot lattices are empty. For such $\mathbb{T}$, the Weyl group is trivial, so we can write $[f]_{\mathbb{T}} = \{f_{\mathbb{T}}\}$. The order of $\mathbb{T}$ satisfies $|\mathbb{T}|(x) = \det(x - f_{\mathbb{T}} \mid X^\vee)$, where $X^\vee$ is the cocharacter lattice of $\mathbb{T}$. The orbit of $f_{\mathbb{T}}$ in $[f]_{\mathbb{G}}$ under conjugation by $W_{\Gamma_{\mathbb{G}}}$ is called the *type* of $\mathbb{T}$ in $\mathbb{G}$.

2.4. Henceforth, fix a generic group $\mathbb{G}$.

A *Levi subgroup* of $\mathbb{G}$ is a generic subgroup $\mathbb{L} \leq \mathbb{G}$ whose cocharacter lattice is the same as that of $\mathbb{G}$. For such $\mathbb{L}$, the *relative Weyl group*

$$W_{\mathbb{G},\mathbb{L}} := N_{W_{\Gamma_{\mathbb{G}}}}([f]_{\mathbb{L}})/W_{\Gamma_{\mathbb{L}}} \simeq (N_{W_{\Gamma_{\mathbb{G}}}}(W_{\Gamma_{\mathbb{L}}})/W_{\Gamma_{\mathbb{L}}})^{[f]_{\mathbb{L}}}$$

is a complex reflection group, isomorphic to the relative Weyl group $W_{\mathbb{G}(q),\mathbb{L}(q)} := N_{\mathbb{G}(q)}(\mathbb{L}(q))/\mathbb{L}(q)$ for all q by the Lang–Steinberg theorem. Note that the definition of $W_{\mathbb{G},\mathbb{L}}$ in [B, 75] has a typo.

A *subtorus* of $\mathbb{G}$ is a generic subgroup $\mathbb{T} \leq \mathbb{G}$ given by a generic torus. Every subtorus $\mathbb{T} \leq \mathbb{G}$ defines a Levi subgroup $Z_{\mathbb{G}}(\mathbb{T}) \leq \mathbb{G}$, called its *centralizer*: The root lattice of $Z_{\mathbb{G}}(\mathbb{T})$ is the centralizer of the cocharacter lattice of $\mathbb{T}$ in the root lattice of $\mathbb{G}$, and $[f]_{Z_{\mathbb{G}}(\mathbb{T})} = W_{\Gamma_{Z_{\mathbb{G}}(\mathbb{T})}} f_{\mathbb{T}}$.

2.5. There is a set $\mathrm{Uch}(\mathbb{G})$ that indexes the sets $\mathrm{Uch}(\mathbb{G}(q))$ uniformly in q. Its construction is functorial with respect to isomorphisms between generic groups. The degrees of the unipotent irreducible characters of the groups $\mathbb{G}(q)$ are generic, in the sense that we can define a polynomial $\mathrm{Deg}_\rho(x) \in \mathbf{Q}[x]$ for each $\rho \in \mathrm{Uch}(\mathbb{G})$ such that $\mathrm{Deg}_\rho(q) = \deg \rho_q$ for all q, where ρ_q is the corresponding element of $\mathrm{Uch}(\mathbb{G}(q))$.

Recall that $W_{\mathbb{G}(q),\mathbb{L}(q)}$ acts on $\mathrm{Irr}(\mathbb{L}(q))$, stabilizing $\mathrm{Uch}(\mathbb{L}(q))$. The action on $\mathrm{Uch}(\mathbb{L}(q))$ is generic, in the sense that it lifts to an action of $W_{\mathbb{G},\mathbb{L}}$ on $\mathrm{Uch}(\mathbb{L})$. For all $\lambda \in \mathrm{Uch}(\mathbb{L})$, we define $W_{\mathbb{G},\mathbb{L},\lambda}$ to be the centralizer of λ in $W_{\mathbb{G},\mathbb{L}}$.

The maps induced by $R_L^G, {}^*R_L^G$ on Grothendieck rings have generic versions on the summands spanned by unipotent characters: If G and F arise from $\mathbb{G}$, and L from $\mathbb{L}$ for some Levi subgroup $\mathbb{L} \leq \mathbb{G}$, then we have linear maps

$$R_{\mathbb{L}}^{\mathbb{G}} : \mathbf{Z}\mathrm{Uch}(\mathbb{L}) \rightleftarrows \mathbf{Z}\mathrm{Uch}(\mathbb{G}) : {}^*R_{\mathbb{L}}^{\mathbb{G}}$$

that recover Lusztig's maps on $\mathbf{Z}\mathrm{Uch}(G^F)$ and $\mathbf{Z}\mathrm{Uch}(L^F)$. The maps $R_{\mathbb{L}}^{\mathbb{G}}, {}^*R_{\mathbb{L}}^{\mathbb{G}}$ are moreover compatible with isomorphisms between Levi subgroups induced by conjugation by $W_{\Gamma_{\mathbb{G}}}$.

2.6. Motivated by observations from the ℓ-modular representation theory of G^F for large primes ℓ, Broué–Malle–Michel used generic groups to formalize a generalization of Harish-Chandra theory, depending on an integer $e > 0$ by way of the cyclotomic polynomial $\Phi_e(x) \in \mathbf{Z}[x]$ [BMM93, §3]. (For a fixed prime power $q > 1$, they take e to be the order of ℓ in $\mathbf{F}_q$.)

As preparation: We say that a generic torus $\mathbb{T}$ is a *Φ_e-torus* iff $|\mathbb{T}|(x)$ is a power of $\Phi_e(x)$. We say that a Levi subgroup $\mathbb{L} \leq \mathbb{G}$ is *Φ_e-split* iff $\mathbb{L} = Z_{\mathbb{G}}(\mathbb{T})$ for some Φ_e-subtorus $\mathbb{T} \leq \mathbb{G}$. We say that $\lambda \in \mathrm{Uch}(\mathbb{L})$ is *Φ_e-cuspidal* iff ${}^*R_{\mathbb{M}}^{\mathbb{L}}(\lambda) = 0$ for any smaller Φ_e-split Levi $\mathbb{M} < \mathbb{L}$.

Remark 2.3. Note that a maximal torus of $\mathbb{G}$ that is Φ_e-split as a Levi subgroup of $\mathbb{G}$ need not be a Φ_e-torus. This already happens for the generic general linear group $\mathbb{G} = \mathbb{GL}_2$, discussed in Section 6, and $e = 2$.

A *Φ_e-cuspidal pair* for $\mathbb{G}$ is a pair $(\mathbb{L}, \lambda)$ in which $\mathbb{L} \leq \mathbb{G}$ is a Φ_e-split Levi subgroup and $\lambda \in \mathrm{Uch}(\mathbb{L})$ is Φ_e-cuspidal. The corresponding *Φ_e-Harish-Chandra* *series* of $\mathrm{Uch}(\mathbb{G})$ is the set

$$\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda) = \{\rho \in \mathrm{Uch}(\mathbb{G}) \mid \rho \text{ occurs in } R_{\mathbb{L}}^{\mathbb{G}}(\lambda)\}.$$

The action of $W_{\Gamma_{\mathbb{G}}}$ on the set of Levi subgroups extends to an action on the set of Φ_e-cuspidal pairs, and the Harish-Chandra series of a pair only depends on its $W_{\Gamma_{\mathbb{G}}}$-orbit. Let $\mathsf{HC}_e(\mathbb{G})$ be the set of $W_{\Gamma_{\mathbb{G}}}$-orbits of Φ_e-cuspidal pairs.

Theorem 2.4 (Broué–Malle–Michel)**.** *For any integer $e > 0$:*

(1) *The Φ_e-Harish-Chandra series form a partition of* $\mathrm{Uch}(\mathbb{G})$*: That is,*

$$\mathrm{Uch}(\mathbb{G}) = \coprod_{[\mathbb{L},\lambda] \in \mathsf{HC}_e(\mathbb{G})} \mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda).$$

(2) *For any Φ_e-cuspidal pair $(\mathbb{L}, \lambda)$ of $\mathbb{G}$, there is a map*

$$(\varepsilon_{\mathbb{L},\lambda}^{\mathbb{G}}, \chi_{\mathbb{L},\lambda}^{\mathbb{G}}) : \mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda) \to \{\pm 1\} \times \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda}).$$

These maps are compatible with the action of $W_{\Gamma_{\mathbb{G}}}$ on the set of Φ_e-cuspidal pairs. Moreover:

(a) *$\chi_{\mathbb{L},\lambda}^{\mathbb{G}}$ is bijective.*

(b) *$\chi_{\mathbb{L},\lambda}^{\mathbb{G}}(\lambda)$ is the trivial character $1_{W_{\mathbb{G},\mathbb{L},\lambda}}$.*

(c) *For all $\rho \in \mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda)$, we have*

$$\mathrm{Deg}_\rho(x) \equiv \varepsilon_{\mathbb{L},\lambda}^{\mathbb{G}}(\rho) \deg \chi_{\mathbb{L},\lambda}^{\mathbb{G}}(\rho) \pmod{\Phi_e(x)}.$$

(d) *For any Φ_e-split Levi $\mathbb{M}$ with $\mathbb{L} \leq \mathbb{M} \leq \mathbb{G}$, we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{Z}\mathrm{Uch}(\mathbb{M}, \mathbb{L}, \lambda) \arrow[r, "(\varepsilon_{\mathbb{L},\lambda}^{\mathbb{M}}, \chi_{\mathbb{L},\lambda}^{\mathbb{M}})"] \arrow[d, "R_{\mathbb{M}}^{\mathbb{G}}"'] & \mathbf{Z}\mathrm{Irr}(W_{\mathbb{M},\mathbb{L},\lambda}) \arrow[d, "\mathrm{Ind}_{W_{\mathbb{M},\mathbb{L},\lambda}}^{W_{\mathbb{G},\mathbb{L},\lambda}}"] \\
\mathbf{Z}\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda) \arrow[r, "(\varepsilon_{\mathbb{L},\lambda}^{\mathbb{G}}, \chi_{\mathbb{L},\lambda}^{\mathbb{G}})"] & \mathbf{Z}\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})
\end{tikzcd}$$

in which the horizontal arrows are induced by linearity.

Example 2.5. The generic version of the setup in Example 2.1 consists of a generic group $\mathbb{G}$ and a Φ_1-split maximal torus $\mathbb{A} \leq \mathbb{G}$. If G, F, A arise from $q, \mathbb{G}, \mathbb{A}$, then A is a maximally F-split maximal torus of G, and $\mathrm{Uch}(\mathbb{G}, \mathbb{A}, 1)$ parametrizes the unipotent principal series of G^F. Moreover, the endomorphism algebra $H_{G,A,1}(q) := \mathrm{End}_{\mathbf{C}G^F}(R_A^G(1))$ is isomorphic to $\mathbf{C}W_{\mathbb{G},\mathbb{A}}$.

Let $\rho_q \in \mathrm{Uch}(G^F)$ correspond to $\rho \in \mathrm{Uch}(\mathbb{G}, \mathbb{A}, 1)$. For all $\rho \in \mathrm{Uch}(\mathbb{G}, \mathbb{A}, 1)$, the sign $\varepsilon_{\mathbb{A},1}^{\mathbb{G}}(\rho)$ is positive. The map $\chi_{\mathbb{A},1}^{\mathbb{G}} : \mathrm{Uch}(\mathbb{G}, \mathbb{A}, 1) \to \mathrm{Irr}(W_{\mathbb{G},\mathbb{A}})$ is determined by the existence of a $(G^F, H_{G,A,1}(q))$-bimodule isomorphism

$$R_A^G(1) = \bigoplus_{\rho \in \mathrm{Uch}(\mathbb{G},\mathbb{A},1)} \rho_q \otimes \chi_{\mathbb{A},1}^{\mathbb{G}}(\rho)_q$$

in which $\chi_{\mathbb{A},1}^{\mathbb{G}}(\rho)_q$ is the $H_{G,A,1}(q)$-module corresponding to $\chi_{\mathbb{A},1}^{\mathbb{G}}(\rho)$.

2.7. **Singular Numbers.** Observe that if $\rho \in \mathrm{Uch}(\mathbb{G})$ is itself Φ_e-cuspidal, then there is a Φ_e-Harish-Chandra series $\mathrm{Uch}(\mathbb{G}, \mathbb{G}, \rho)$ consisting of ρ alone.

For any polynomial $\mathsf{f}(x) \in \mathbf{Q}[x]$, let $r_e(\mathsf{f})$ be the largest power to which Φ_e divides f. The following fact is [BMM93, Prop. 2.9]:

Proposition 2.6 (Broué–Malle–Michel)**.** *$\rho \in \mathrm{Uch}(\mathbb{G})$ is Φ_e-cuspidal if and only if $r_e(\mathrm{Deg}_\rho) = r_e(|\mathbb{G}|)$.*

We say that e is a *singular number* for $\mathbb{G}$ iff $r_e(|\mathbb{G}|) > 0$. Observe that there are finitely many such numbers; for groups of exceptional type, we list them in Section 8. It is useful to record that:

Corollary 2.7. *If e is* not *a singular number for $\mathbb{G}$, then every element of* $\mathrm{Uch}(\mathbb{G})$ *is Φ_e-cuspidal. In other words, the Φ_e-Harish-Chandra partition is nontrivial only if e is a singular number.*

Proof. We always have $r_e(\mathrm{Deg}_\rho) \leq r_e(|\mathbb{G}|)$, because for any prime power $q > 1$ and G, F, ρ_q arising from $q, \mathbb{G}, \rho$, we know that $\mathrm{Deg}_\rho(q) = \mathrm{Deg}_{\rho_q}(1)$ divides $|\mathbb{G}|(q) = |G^F|$ in $\mathbf{Z}$ by usual character theory. So when e is not a singular number, we always have $r_e(\mathrm{Deg}_\rho) = r_e(|\mathbb{G}|) = 0$. Now the result follows from Proposition 2.6. □

2.8. **Regular Numbers.** The existence of a Φ_e-split maximal torus constrains the number e even more strongly. To explain how this works, we review some notions due to Springer, following [BMM99, §5B].

Suppose that W is a complex reflection group with reflection representation V over $\mathbf{C}$, and that f is a finite-order automorphism of V normalizing W. For any root of unity $\zeta \in \mathbf{Q}_{cyc}^\times$ and $w \in W$, we say that wf is *ζ-regular*, or just *regular*, iff wf has an eigenvector $v \in V^{\mathrm{reg}}$ with eigenvalue ζ.

Example 2.8. Suppose that there is an f-stable system of simple reflections $S \subseteq W$, or in other words, f is an automorphism of the Coxeter system (W, S). Let $w \in W$ be the product, in any order, of a full set of representatives for the f-orbits on S, and let $c = wf$. Then c is a regular element of Wf. Elements that take this form for some f-stable S and choice of f-orbit representatives are called *twisted Coxeter elements* [S74, §7.3].

Suppose that in addition, $W = W_{\Gamma_\mathbb{G}}$ and $Wf = [f]_\mathbb{G}$ for some generic group $\mathbb{G}$. Then W is crystallographic, so by Galois theory, wf is ζ-regular if and only if it is ζ'-regular for any other root of unity ζ' of the same order. In this case, we say that the order of ζ is a *regular number* for $\mathbb{G}$.

If $\Gamma_\mathbb{G}$ is irreducible and f induced by a Dynkin-diagram automorphism, then we define the *twisted Coxeter number* of $\mathbb{G}$ to be the order of any twisted Coxeter element of $[f]_\mathbb{G}$.

Proposition 2.9 (Broué–Malle–Michel)**.**
(1) *If e is a regular number for $\mathbb{G}$, then $\mathbb{G}$ admits a Φ_e-split maximal torus.*
(2) *Conversely, if $\mathbb{T} \leq \mathbb{G}$ is a Φ_e-split maximal torus, then $\mathbb{T}$ has type $[wf]$ for some Φ_e-regular element $wf \in [f]_\mathbb{G}$.*

Proof. Part (1) follows from [BMM99, Prop. 5.8]. To show part (2): Suppose that $\mathbb{T} = Z_\mathbb{G}(\mathbb{T}')$ for some Φ_e-torus $\mathbb{T}'$. If $\mathbb{T}$ is itself a torus, then $\mathbb{T}'$ must be maximal or *Sylow* in the terminology of [BMM93, B]. Then by [BMM99, Cor. 5.10], e must be a regular number for $\mathbb{G}$, so by part (1), there is a torus of type $[wf]$ for some Φ_e-regular wf. Again by [BMM99, Cor. 5.10], $f_{\mathbb{T}}$ must be conjugate to wf under $W_{\Gamma_{\mathbb{G}}}$. Thus $\mathbb{T}$ has the same type. □

Corollary 2.10. *Any regular number for* $\mathbb{G}$ *is a singular number.*

Proof. If e is a regular number for $\mathbb{G}$, then by Proposition 2.9(1), a Φ_e-maximal torus $\mathbb{T} \leq \mathbb{G}$ exists. As in the proof of Corollary 2.7, we see that $|\mathbb{T}(x)|$ divides $|\mathbb{G}(x)|$ in $\mathbf{Q}[x]$ because $|\mathbb{T}|(q)$ divides $|\mathbb{G}|(q)$ in $\mathbf{Z}$ for any prime power $q > 1$. Therefore, $r_e(|\mathbb{G}|) \geq r_e(|\mathbb{T}|) > 0$. □

3. Cyclotomic Hecke Algebras

3.1. In [L-Cox], Lusztig determined the G^F-equivariant endomorphism algebra of $R_T^G(1)$ in the case where $T \subseteq G$ is a Coxeter torus, *i.e.*, a maximal torus of type $[c]$ for some twisted Coxeter element c. He observed that this algebra could be viewed as a generalized Hecke algebra for the cyclic group W_{G^F,T^F}, with parameters given by the eigenvalues of F on $\mathrm{H}_c^*(X_{T\subseteq B}^G)$. Soon afterward, in [L77], Lusztig determined the algebras $\mathrm{End}_{\mathbf{C}G^F}(R_L^G(\lambda_q))$ for various F-split Levi subgroups L of classical groups G and cuspidal unipotent λ_q. He observed that again, these could be viewed as Hecke algebras for the relative Weyl groups W_{G^F,L^F}. These calculations were later extended to general F-split Levis in [L78], and to non-unipotent characters by Howlett–Lehrer in [HL].

Let C be a complex reflection group of the form $W_{\mathbb{G},\mathbb{L},\lambda}$ for some generic group $\mathbb{G}$, integer $e > 0$, and Φ_e-cuspidal pair $(\mathbb{L}, \lambda)$, as above. In [BM93] and [BMR], Broué–Malle–Rouquier study a ring $H_C(\vec{u})$ that only depends on the structure of C as a complex reflection group. It may be viewed as a multi-parameter version of the Hecke algebras studied in the works above, in which the parameters are also allowed to be as generic as possible.

In [BM93], motivated both by [L-Cox] and by Broué's Abelian Defect Group Conjecture, Broué–Malle conjectured the existence of a certain specialization of $H_C(\vec{u})$, depending on $\mathbb{G}, (\mathbb{L}, \lambda)$ and sending its parameters to various powers of a single variable x, such that the resulting algebra $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ would itself specialize to the algebras $\mathrm{End}_{\mathbf{C}L^F}(R_L^G(\lambda_q))$ arising from $\mathbb{G}, \mathbb{L}, \lambda$ by varying q. We review their conjectures below, mainly following [BM93] and [B].

3.2. First, we review $H_C(\vec{u})$, following [BMR] and [B, Ch. II–III].

Let C be an arbitrary finite complex reflection group and $V = V(C)$ its reflection representation over $\mathbf{C}$. Let $\mathcal{A} = \mathcal{A}(C)$ be the set of hyperplanes in V fixed by pseudo-reflections in C. Let $V^{\mathrm{reg}} = V - \bigcup_{H\in\mathcal{A}} H$, the open locus where C acts freely, and let $Br_C = \pi_1(V^{\mathrm{reg}}/C)$, the braid group of C.

For each orbit $\mathcal{C} \in \mathcal{A}/C$ and hyperplane $H \in \mathcal{C}$, let $\sigma_H \in Br_C$ be a choice of a generator of the monodromy around H that is a distinguished braid reflection in the sense of [B, 21–22]. Let the pseudo-reflection s_H be the image of $\sigma_{\mathcal{C}}$ under the quotient map $Br_C \to C$, and let $e_{\mathcal{C}}$ be the order of s_H, which only depends on $\mathcal{C}$.

Let $\mathbf{Z}[\vec{u}^{\pm 1}] = \mathbf{Z}[u_{\mathcal{C},j}^{\pm 1} \mid \mathcal{C} \in \mathcal{A}/C,\, 0 \leq j < e_{\mathcal{C}}]$, and let

$$H_C(\vec{u}) = \frac{\mathbf{Z}[\vec{u}^{\pm 1}]Br_C}{\langle (\sigma_H - u_{\mathcal{C},0}) \cdots (\sigma_H - u_{\mathcal{C},e_{\mathcal{C}}-1}) \mid \mathcal{C} \in \mathcal{A}/C,\, H \in \mathcal{C}\rangle}.$$

It was conjectured in [BMR] that $H_C(\vec{u})$ is always free over $\mathbf{Z}[\vec{u}^{\pm 1}]$ of rank $|C|$. This statement has been verified in all of the infinite families of irreducible complex reflection groups, and some of the exceptional cases. For our purposes, the following result from [E] suffices:

Theorem 3.1 (Etingof–Rains, Losev, Marin–Pfeiffer)**.** *For any field K of characteristic zero, $K \otimes_{\mathbf{Z}} H_C(\vec{u})$ is a free module over $K \otimes_{\mathbf{Z}} \mathbf{Z}[\vec{u}^{\pm 1}]$ of rank $|C|$. In particular, if we fix a homomorphism $\mathbf{Z}[\vec{u}^{\pm 1}] \to K$, then the corresponding base change $K \otimes_{\mathbf{Z}[\vec{u}^{\pm 1}]} H_C(\vec{u})$ is a K-algebra of dimension $|C|$.*

3.3. Next, we review $H_{\mathbb{G},\mathbb{L},\lambda}(x)$, following [BM93] and [B].

Let $\mathbf{Q}_{cyc}$ be the maximal cyclotomic field extension of $\mathbf{Q}$, and let $\mathbf{Z}_{cyc}$ be the ring of integers of $\mathbf{Q}_{cyc}$. For each integer $n > 0$, fix a primitive nth root of unity $\zeta_n \in \mathbf{Z}_{cyc}$. A *$\Phi_e$-specialization* of $\mathbf{Z}[\vec{u}^{\pm 1}]$ is a homomorphism $\mathcal{S} : \mathbf{Z}[\vec{u}^{\pm 1}] \to \mathbf{Z}_{cyc}[x^{\pm \frac{1}{\infty}}]$ of the form

$$\mathcal{S}(u_{\mathcal{C},j}) = \zeta_{e_{\mathcal{C}}}^j (\zeta_e^{-1} x)^{m_{\mathcal{C},j}} \quad \text{for some } m_{\mathcal{C},j} = m_{\mathcal{C},j}(\mathcal{S}) \in \mathbf{Q}.$$

Example 3.2. The $\mathbf{Z}_C$-linear homomorphism $\mathbf{Z}[\vec{u}^{\pm 1}] \to \mathbf{Z}_{cyc}[x^{\pm \frac{1}{\infty}}]$ that sends $u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j$ is a Φ_e-specialization for every value of e. The base change of $H_C(\vec{u})$ along this map is the group algebra of C over $\mathbf{Z}_{cyc}[x^{\pm \frac{1}{\infty}}]$ [B, 45–46].

Suppose that G, F, L arise from $q, \mathbb{G}, \mathbb{L}$, where we exclude the Suzuki and Ree cases as usual (Assumption 2.2). Recall the G^F-variety $Y_{L \subseteq P}^G$ over $\mathbf{F}_q$. For any $\lambda_q \in \mathrm{Uch}(L^F)$, we write $\mathrm{H}_c^*(Y_{L\subseteq P}^G)[\lambda_q]$ to denote the λ_q-isotypic component of $\mathrm{H}_c^*(Y_{L\subseteq P}^G)$, viewed as a graded representation of G^F of finite dimension over $\bar{\mathbf{Q}}_\ell$. What follows is essentially conjecture (HC) of [B, 84].

Conjecture 3.3 (Broué–Malle–Michel)**.** *For any integer $e > 0$ and Φ_e-cuspidal pair $(\mathbb{L}, \lambda)$ for $\mathbb{G}$, there is a Φ_e-specialization $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda} : \mathbf{Z}_{W_{\mathbb{G},\mathbb{L},\lambda}}[\vec{u}^{\pm 1}] \to \mathbf{Z}_{cyc}[x^{\pm \frac{1}{\infty}}]$ such that the base change*

$$H_{\mathbb{G},\mathbb{L},\lambda}(x) := \mathbf{Z}_{cyc}[x^{\pm \frac{1}{\infty}}] \otimes_{\mathbf{Z}_{W_{\mathbb{G},\mathbb{L},\lambda}}[\vec{u}^{\pm 1}]} H_{W_{\mathbb{G},\mathbb{L},\lambda}}(\vec{u})$$

of $H_{W_{\mathbb{G},\mathbb{L},\lambda}}(\vec{u})$ along $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ has the following property: For any prime power $q > 1$ and prime ℓ invertible in $\mathbf{F}_q$, there is an isomorphism of algebras

$$\bar{\mathbf{Q}}_\ell \otimes_{\mathbf{Z}_{cyc}[x^{\pm \frac{1}{\infty}}]} H_{\mathbb{G},\mathbb{L},\lambda}(x) \simeq \mathrm{End}_{\bar{\mathbf{Q}}_\ell G^F}(\mathrm{H}_c^*(Y_{L\subseteq P}^G)[\lambda_q]), \tag{3.1}$$

where on the left, the base change sends $x^{\frac{1}{n}}$ to an nth root of q in $\bar{\mathbf{Q}}_\ell$ for each integer $n > 0$, and on the right, G, F, L, λ_q arise from $q, \mathbb{G}, \mathbb{L}, \lambda$.

Strictly speaking, $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ is an abuse of notation, as this algebra may depend on higher roots of x. Note that the conclusion of the conjecture also implies that $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ is uniquely determined up to Galois automorphisms of $\mathbf{Z}_{cyc}$.

Remark 3.4. It is explained in [B, 81–83] that the Φ_e-specialization $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ should obey several further properties. In particular, (U5.3.4) of *loc. cit.* states that the specializations $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ should be compatible with inclusions of Levi subgroups, in the manner of Theorem 2.4(2d).

Example 3.5. In Example 2.5, write $W = W_{\mathbb{G},\mathbb{A}}$. Then $H_W(\vec{u})$ takes the form

$$H_W(\vec{u}) = \mathbf{Z}_W[\vec{u}^{\pm 1}]Br_W/\langle(\sigma_H - u_{\mathcal{C},0})(\sigma_H - u_{\mathcal{C},1}) \mid \mathcal{C} \in \mathcal{A}/W,\ H \in \mathcal{C}\rangle.$$

Meanwhile, $H_{G,A,1}(q)$ is generated by Hecke operators T_s for each simple reflection $s \in W$, modulo braid relations and quadratic relations.

Suppose that $\mathbb{G}$ is split, so that $W = W_{\Gamma_{\mathbb{G}}}$. Here, the quadratic relations take the form $(T_s + 1)(T_s - q) = 0$, by [I]. Following the conventions of [BM93], let $\mathcal{S}_{\mathbb{G},\mathbb{A},1}$ be the Φ_1-specialization that sends $(u_{\mathcal{C},0}, u_{\mathcal{C},1}) \mapsto (1, -x)$ for all $\mathcal{C}$. Then (3.1) sends $\sigma_H \mapsto -T_{s_H}$.

3.4. In the case where $(\mathbb{L}, \lambda) = (\mathbb{T}, 1)$ for some maximal torus $\mathbb{T}$, Broué–Michel constructed a $Br_{W_{\mathbb{G},\mathbb{T}}}$-action on the associated Deligne–Lusztig cohomologies, and conjectured that the $H_{\mathbb{G},\mathbb{T},1}(x)$-action of Conjecture 3.3 would arise from this braid action. See [BM97] and [B, 84–88] for details. This conjecture has since been established in many cases by Digne, Michel, and Rouquier [DMR, DM06], building on [L-Cox]. To summarize the state of the art, we follow [DM06].

Theorem 3.6 (Lusztig, Digne–Michel–Rouquier)**.** *Let $\mathbb{T} \leq \mathbb{G}$ be a maximal torus. Then Conjecture 3.3 holds for $(\mathbb{L}, \lambda) = (\mathbb{T}, 1)$, with an explicit $\mathcal{S}_{\mathbb{G},\mathbb{T},1}$, in the following cases. Throughout, f is a Dynkin-diagram automorphism representing $[f]_{\mathbb{G}}$.*

(1) *$\mathbb{T}$ is Φ_1-split.*
(2) *$\mathbb{T}$ is Coxeter in the sense of [L-Cox, (1.14)]. That is, $\mathbb{T}$ is of type $[c]$ for some twisted Coxeter element $c \in [f]_{\mathbb{G}}$, as defined in Example 2.8.*
(3) *$\mathbb{T}$ is of type $[w_0 f]$, where $w_0 \in W_{\Gamma_{\mathbb{G}}}$ is the longest element.*
(4) *$\mathbb{G}$ is split of type A.*
(5) *$\mathbb{G}$ is of type B and $\mathbb{T}$ is Φ_e-split for some even $e > 0$.*
(6) *$\mathbb{G}$ is split of type D_4 and $\mathbb{T}$ is Φ_4-split.*

Proof. (1) If $\mathbb{G}$ is split, then this is due to Iwahori [I]. The specialization was described above, in Example 3.5. If $\mathbb{G}$ is not split, then it is due to Lusztig [L78], and the specialization is given by Table II of *ibid.*

(2) This is due to Lusztig [L-Cox]. The specialization $\mathcal{S}_{\mathbb{G},\mathbb{T},1}$ sends the variables $u_{\mathcal{C},j}$ to the powers of x that specialize to the eigenvalues of F on $\mathrm{H}_c^*(X_{T\subseteq B}^G)$, when G, F, T arise from $q, \mathbb{G}, \mathbb{T}$. When G is almost-simple, the eigenvalues are listed in (7.3) of *ibid.*

(3) This is [DMR, Th. 5.4.1], reviewed in [DM06, Prop. 7.2]. The specialization is described there.

(4) This is [DM06, Thm. 10.1]. Note that in type A, the Weyl group contains two conjugacy classes of regular elements, corresponding to the two cases of the theorem. The specialization is described there. The requirement that $\mathbb{G}$ be split is due to Section 9 of *ibid.*

(5) This is [DM06, Thm. 11.1]. The specialization is described there.

(6) This is [DM06, Prop. 12.2]. The specialization is described there. □

3.5. Beyond tori, we can also establish Conjecture 3.3 with an explicit $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ in the cases where:

(1) $\mathbb{L}$ is an arbitrary Φ_1-split Levi. This case is again due to Lusztig, with the specialization given by [L78, Table II].
(2) $\mathbb{G}$ is a generic general linear group and $W_{\mathbb{G},\mathbb{L},\lambda}$ is cyclic. This case follows from work of Dudas [Du]: See Proposition 6.2(2) below.

Case (2) above, along with case (4) of Theorem 3.6, motivates a prediction for arbitrary Φ-cuspidal pairs of generic general linear groups, and consequently, of generic general unitary groups via Ennola duality. We defer the precise statements to Section 6.

4. Conjectures about Blocks

4.1. Throughout this section, Assumption 2.2 remains in force.

When K is a field and R a finite-dimensional K-algebra, we write $\mathsf{Rep}_K(R)$ for the category of free R-modules that are finite-dimensional over K. When K can be inferred, we write $\mathsf{K}_0(R)$ for the Grothendieck ring of $\mathsf{Rep}_K(R)$.

4.2. To state the first of our conjectures, we will draw freely upon the theory of blocks and Brauer decomposition [GP, Ch. 7] [GJ, Ch. 3].

Let C be any finite complex reflection group. We keep the notation $\mathbf{Z}[\vec{u}^{\pm 1}]$ of Section 3. Let $\mathcal{S} : \mathbf{Z}[\vec{u}^{\pm 1}] \to \mathbf{Z}_{cyc}[x^{\pm\frac{1}{\infty}}]$ be an arbitrary specialization, and let $H_{C,\mathcal{S}}(x)$ be the base change of $H_C(\vec{u})$ along $\mathcal{S}$.

For any root of unity $\zeta = \zeta_m \in \mathbf{Z}_{cyc}^\times$, let $H_{C,\mathcal{S}}(\zeta)$ be the base change of $H_{C,\mathcal{S}}(x)$ along the morphism $\mathbf{Z}_{cyc}[x^{\pm\frac{1}{\infty}}] \to \mathbf{Q}_{cyc}$ that sends $x^{\frac{1}{n}} \mapsto \zeta_{mn}$ for all n. Then the category $\mathsf{Rep}_{\mathbf{Q}_{cyc}}(H_{C,\mathcal{S}}(\zeta))$ is partitioned into blocks, describing the failure of the ring $H_{C,\mathcal{S}}(\zeta)$ to be semisimple. This partition defines a corresponding direct-sum decomposition of $\mathsf{K}_0(H_{C,\mathcal{S}}(\zeta))$ into block ideals. For any block b, let I_b denote the block ideal generated by its objects.

Let $K \supseteq \mathbf{Q}_{cyc}(u_{\mathcal{C},j} \mid \mathcal{C}, j)$ be a splitting field for $H_C(\vec{u})$, and let

$$KH_C(\vec{u}) = K \otimes_{\mathbf{Z}[\vec{u}^{\pm 1}]} H_C(\vec{u}).$$

Then we can form the Brauer decomposition map

$$d_\zeta : \mathsf{K}_0(KH_C(\vec{u})) \to \mathsf{K}_0(H_{C,\mathcal{S}}(\zeta))$$

with respect to the composition $H_C(\vec{u}) \xrightarrow{\mathcal{S}} H_{C,\mathcal{S}}(x) \to H_{C,\mathcal{S}}(\zeta)$.

Let $\mathrm{Irr}(KH_C(\vec{u}))$ be the set of simple $KH_C(\vec{u})$-modules up to isomorphism. By combining Theorem 3.1, Example 3.2, and Tits deformation [GP, Thm. 7.4.6], we obtain a bijection $\mathrm{Irr}(C) \xrightarrow{\sim} \mathrm{Irr}(KH_C(\vec{u}))$. Let $\mathrm{Irr}(C)_\mathsf{b}$ be the preimage of I_b along the composition of maps

$$\mathrm{Irr}(C) \xrightarrow{\sim} \mathrm{Irr}(KH_C(\vec{u})) \subset \mathsf{K}_0(KH_C(\vec{u})) \xrightarrow{d_\zeta} \mathsf{K}_0(H_{C,\mathcal{S}}(\zeta)). \tag{4.1}$$

The composition does not depend on K. Abusing language, we refer to the sets $\mathrm{Irr}(C)_\mathsf{b}$ as the *$(\Phi_m, \mathcal{S})$-blocks* of $\mathrm{Irr}(C)$.

4.3. Now let $\mathbb{G}$ be a generic group. For the first time, we fix two separate integers $e, m > 0$. Let $(\mathbb{L}, \lambda)$, *resp.* $(\mathbb{M}, \mu)$, be a Φ_e-cuspidal pair, *resp.* a Φ_m-cuspidal pair, for $\mathbb{G}$. As in the introduction, let

$$\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda, \mathbb{M}, \mu) = \mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda) \cap \mathrm{Uch}(\mathbb{G}, \mathbb{M}, \mu),$$

and let $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})$ and $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})$ be the images of $\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda, \mathbb{M}, \mu)$ along the maps

$$\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda}) \xleftarrow{\chi^{\mathbb{G}}_{\mathbb{L},\lambda}} \mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda, \mathbb{M}, \mu) \xrightarrow{\chi^{\mathbb{G}}_{\mathbb{M},\mu}} \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu}).$$

Let $\mathcal{S}_e$ be a Φ_e-specialization of $\mathbf{Z}_{W_{\mathbb{G},\mathbb{L},\lambda}}[\vec{u}^{\pm 1}]$, and let $\mathcal{S}_m$ be a Φ_m-specialization of $\mathbf{Z}_{W_{\mathbb{G},\mathbb{M},\mu}}[\vec{u}^{\pm 1}]$. We define *properties (I) and (II) for $(\mathcal{S}_e, \mathcal{S}_m)$* to be:

(I) $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu}$, *resp.* $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda}$, is a union of $(\Phi_m, \mathcal{S}_e)$-blocks, *resp.* $(\Phi_e, \mathcal{S}_m)$-blocks.

(II) The bijection between $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu}$ and $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda}$ induced by $\chi^{\mathbb{G}}_{\mathbb{L},\lambda}$ and $\chi^{\mathbb{G}}_{\mathbb{M},\mu}$ descends to a bijection

$$\chi^{\mathbb{L},\lambda}_{\mathbb{M},\mu} : \{\mathsf{b} \mid \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathsf{b}} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu}\} \xrightarrow{\sim} \{\mathsf{c} \mid \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathsf{c}} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda}\}.$$

In particular, we obtain a bijection

$$\chi^{\mathsf{b}}_{\mathsf{c}} : \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathsf{b}} \xrightarrow{\sim} \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathsf{c}}$$

whenever $\chi^{\mathbb{L},\lambda}_{\mathbb{M},\mu}(\mathsf{b}) = \mathsf{c}$.

Conjecture 4.1. *For any $\mathbb{G}, e, \mathbb{L}, \lambda, m, \mathbb{M}, \mu$ such that Conjecture 3.3 holds for $\mathbb{G}, e, (\mathbb{L}, \lambda)$ and analogously for $\mathbb{G}, m, (\mathbb{M}, \mu)$, properties (I) and (II) hold for the pair of specializations $(\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}, \mathcal{S}_{\mathbb{G},\mathbb{M},\mu})$ that it predicts.*

4.4. **The Rational DAHA.** Before stating the next conjecture about blocks, we review background about rational double affine Hecke algebras (DAHAs).

Again, let C be any finite complex reflection group. We keep the notations $V, \mathcal{A}, \mathcal{C}, H$ from Section 3.2. For each $H \in \mathcal{A}$, let $C_H \subseteq C$ be the centralizer of H. Fix $\alpha_H \in V^\vee$ so that $H = \ker(\alpha_H)$, and fix $\alpha_H^\vee \in V$ so that $\mathbf{C}\alpha_H^\vee$ is a C_H-stable complement to H.

Let $\mathsf{Refl} = \mathsf{Refl}(C)$ be the set of pseudo-reflections of C. Fix a vector

$$\vec{\nu} = (\nu_t)_t \in \mathbf{C}^{\mathsf{Refl}}$$

invariant under conjugation by C. We define the *rational Cherednik algebra* or *rational DAHA* of C with *central charge* $\vec{\nu}$ to be the $\mathbf{C}$-algebra

$$D_C^{rat}(\vec{\nu}) = (\mathbf{C}C \ltimes (\mathrm{Sym}(V) \otimes \mathrm{Sym}(V^\vee)))/I(\vec{\nu}),$$

where $I(\vec{\nu})$ is the two-sided ideal

$$I(\vec{\nu}) = \left\langle YX - XY - \langle X, Y\rangle + \sum_{H \in \mathcal{A}} \sum_{\substack{t \in C_H \\ t \neq 1}} \nu_t t \, \frac{\langle X, \alpha_H^\vee\rangle\langle \alpha_H, Y\rangle}{\langle \alpha_H, \alpha_H^\vee\rangle} \;\middle|\; \begin{array}{l} X \in V, \\ Y \in V^\vee \end{array} \right\rangle.$$

This is the definition in [GGOR, §3.1] when their base ring is $\mathbf{C}$.

We write $\mathsf{O}_C^{rat}(\vec{\nu})$ for the BGG category O of $D_C^{rat}(\vec{\nu})$ defined in [GGOR, §3.2]. Its *Verma* or *standard* objects are indexed by $\mathrm{Irr}(C)$. For all $\chi \in \mathrm{Irr}(C)$, we write $\Delta_{\vec{\nu}}(\chi)$ for the corresponding Verma, and $L_{\vec{\nu}}(\chi)$ for its simple quotient.

Let $\mathsf{eu} \in D_C^{rat}(\vec{\nu})$ be the Euler element reviewed at the end of [GGOR, §3.1]. It commutes with C and its action on any object of $\mathsf{O}_C^{rat}(\vec{\nu})$ is locally finite. For any object M of $\mathsf{O}_C^{rat}(\vec{\nu})$, we define the *graded character* of M to be

$$[M] \in \sum_\alpha t^\alpha [M_\alpha] \in \mathsf{K}_0(\mathbf{C}C)[t^{\mathbf{C}}],$$

where M_α is the eigenspace of M with eigenvalue α. In particular,

$$[\Delta_{\vec{\nu}}(\chi)] = t^{\mathsf{h}(\chi)}[\mathrm{Sym}(V)] \cdot \chi,$$

where above, $[\mathrm{Sym}(V)] = \sum_i t^i [\mathrm{Sym}^i(V)]$ and

$$\mathsf{h}(\chi) = \frac{1}{2}\dim(V) - \sum_{t \in \mathsf{Refl}} \frac{2\nu_t}{1 - \det_V(t)} \frac{\chi(t)}{\chi(1)}.$$

4.5. **The KZ Functor.** Let $\{\kappa_{\mathcal{C},j} \in \mathbf{C} \mid \mathcal{C} \in \mathcal{A}/C,\, 0 \leq j < e_{\mathcal{C}}\}$ be defined by

$$\kappa_{\mathcal{C},j} = \frac{2}{e_{\mathcal{C}}} \sum_{1 \leq k < e_{\mathcal{C}}} \nu_{s_H^k} \left(\frac{1 - \det_V(s_H)^{jk}}{1 - \det_V(s_H)^{-k}}\right) \quad \text{for any } H \in \mathcal{C}. \tag{4.2}$$

Note that $\kappa_{\mathcal{C},0} = 0$, and if $e_{\mathcal{C}} = 2$, then $\kappa_{\mathcal{C},1} = \nu_{s_H}$. Let

$$\zeta_{\mathcal{C},j} = \det_V(s_H)^j e^{2\pi i \kappa_{\mathcal{C},j}}. \tag{4.3}$$

In the notation of Section 3, let $\mathcal{T}_{\vec{\zeta}} : \mathbf{Z}[\vec{u}^{\pm 1}] \to \mathbf{C}$ be the ring homomorphism defined by $\mathcal{T}_{\vec{\nu}}(\vec{u}_{\mathcal{C},j}) = \zeta_{\mathcal{C},j}$ for all $\mathcal{C}, j$, and let

$$H_C(\vec{\zeta}) = \mathbf{C} \otimes_{\mathbf{Z}[\vec{u}^{\pm 1}]} H_C(\vec{u})$$

be the base change of $H_C(\vec{u})$ along $\mathcal{T}_{\vec{\zeta}}$. The authors of [GGOR] introduced an exact monoidal functor

$$\mathsf{KZ} : \mathsf{O}_C^{rat}(\vec{\nu}) \to \mathsf{Rep}_{\mathbf{C}}(H_W(\vec{\zeta})),$$

called the *Knizhnik–Zamolodchikov (KZ) functor*. It can be defined geometrically by localizing each object of $\mathsf{O}_C^{rat}(\vec{\nu})$ to a local system over V^{reg}/C, then invoking a monodromy calculation from [BMR]. We refer to [GGOR, §5.3] for details.

The functor KZ is representable by a projective object P_{KZ}, which induces a morphism of $\mathbf{C}$-algebras $H_C(\vec{\zeta}) \to \mathrm{End}(P_{\mathsf{KZ}})^{\mathrm{op}}$. It turns out that this is an isomorphism. In particular, the restriction of KZ to projective objects of $\mathsf{O}_C^{rat}(\vec{\nu})$ is fully faithful. Since $\mathsf{O}_C^{rat}(\vec{\nu})$ also forms a highest weight category, we deduce that $\mathsf{O}_C^{rat}(\vec{\nu})$ is a *highest weight cover* of $\mathsf{Rep}_{\mathbf{C}}(H_W(\vec{\zeta}))$ in the sense of [R08]. Moreover, KZ induces a bijection between the blocks of $\mathsf{O}_C^{rat}(\vec{\nu})$ and the blocks of $\mathsf{Rep}_{\mathbf{C}}(H_W(\vec{\zeta}))$.

In some cases, the behavior of KZ on standard objects is known. To explain, observe that as in Section 4, we can use Brauer decomposition to form a map

$$\mathrm{Irr}(C) \to \mathsf{K}_0(H_C(\vec{\zeta})).$$

We write $\chi_{\vec{\zeta}} \in \mathsf{K}_0(H_C(\vec{\zeta}))$ to denote the image of $\chi \in \mathrm{Irr}(C)$. If there exists $\zeta \in \mathbf{C}$ such that $\zeta_{\mathcal{C},j} = \zeta$ for all $\mathcal{C}, j$, then we write χ_ζ in place of $\chi_{\vec{\zeta}}$. Theorem 6.8 of [GGOR] then states:

Theorem 4.2 (Ginzburg–Guay–Opdam–Rouquier)**.** *If C is a Weyl group and $\vec{\nu} = \nu$ is constant, then $\mathsf{KZ}(\Delta_{\vec{\nu}}(\chi)) = \chi_\zeta$, where $\zeta = e^{2\pi i\nu}$.*

The argument of *loc. cit.* can be extended to other situations where $H_C(\vec{\zeta})$ is cellular in the sense of Graham–Lehrer, *cf.* Section 6.6.

4.6. Henceforth, we fix the embedding $\mathbf{Q}_{cyc} \hookrightarrow \mathbf{C}$ under which $\zeta_n \mapsto e^{2\pi i/n}$. We also assume that $\det_V(s_H) = e^{2\pi i/e_{\mathcal{C}}}$ for all $\mathcal{C} \in \mathcal{A}/C$ and $H \in \mathcal{C}$ [B, 10–11].

The algebra $H_{C,\mathcal{S}}(\zeta)$ in Section 4 now takes the form $H_C(\vec{\zeta})$ whenever $\mathcal{S}$ is a Φ_e-specialization for some e. Indeed, if $\zeta = \zeta_m$ for some integer $m > 0$, then the composition $H_C(\vec{u}) \to H_{C,\mathcal{S}}(x) \to H_{C,\mathcal{S}}(\zeta)$ sends

$$u_{\mathcal{C},j} \mapsto e^{2\pi i(j/e_{\mathcal{C}} + m_{\mathcal{C},j}(\mathcal{S})(1/m - 1/e))} \quad \text{for some } m_{\mathcal{C},j}(\mathcal{S}) \in \mathbf{Q}.$$

We see from (4.2)–(4.3) that it suffices to choose $\vec{\nu} = (\nu_t)_t$ so that

$$\frac{2}{e_{\mathcal{C}}} \sum_{1 \le k < e_{\mathcal{C}}} \nu_{s_H^k} \left(\frac{1 - e^{2\pi ijk/e_{\mathcal{C}}}}{1 - e^{-2\pi ik/e_{\mathcal{C}}}} \right) \in \left(\frac{1}{m} - \frac{1}{e} \right) m_{\mathcal{C},j}(\mathcal{S}) + \mathbf{Z}.$$

Let us say that $\vec{\nu}$ is an *$\mathcal{S}$-charge* iff it satisfies the condition above. In this case, for any block b of $\mathsf{Rep}_{\mathbf{C}}(H_{C,\mathcal{S}}(\zeta))$, we write $\mathsf{O}_C^{rat}(\vec{\nu})_{\mathsf{b}}$ to denote the corresponding block of $\mathsf{O}_C^{rat}(\vec{\nu})$.

Now we return to the setup of Section 4.3, where $\mathcal{S}_e$ is a Φ_e-specialization of $\mathbf{Z}_{W_{\mathbb{G},\mathbb{L},\lambda}}[\vec{u}^{\pm 1}]$ and $\mathcal{S}_m$ is a Φ_m-specialization of $\mathbf{Z}_{W_{\mathbb{G},\mathbb{M},\mu}}[\vec{u}^{\pm 1}]$. Assuming property (II) for $(\mathcal{S}_e, \mathcal{S}_m)$, we define *property (III) for $(\mathcal{S}_e, \mathcal{S}_m)$* to be:

(III) Whenever $\chi_{\mathbb{M},\mu}^{\mathbb{L},\lambda}(\mathsf{b}) = \mathsf{c}$, and $\vec{\nu}_e$, *resp.* $\vec{\nu}_m$, is an $\mathcal{S}_e$-charge, *resp.* $\mathcal{S}_m$-charge, the bijection $\chi_{\mathsf{c}}^{\mathsf{b}}$ categorifies to a (bounded) derived equivalence

$$\mathsf{D}^b(\mathsf{O}_{W_{\mathbb{G},\mathbb{L},\lambda}}^{rat}(\vec{\nu}_e)_{\mathsf{b}}) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{O}_{W_{\mathbb{G},\mathbb{M},\mu}}^{rat}(\vec{\nu}_m)_{\mathsf{c}})$$

that takes standard objects to standard objects.

Conjecture 4.3. *If Conjecture 4.1 holds for some $\mathbb{G}, e, \mathbb{L}, \lambda, m, \mathbb{M}, \mu$, then property (III) holds for the pair $(\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}, \mathcal{S}_{\mathbb{G},\mathbb{M},\mu})$ in the conjecture as well.*

5. Conjectures about Affine Springer Fibers

5.1. In this section, we give the precise definition of the local system of (DAHA, braid-group) bimodules $\mathcal{E}_\nu$, and the evidence for Conjecture 2 arising from twisted $\mathbb{G}$ of rank 2.

Henceforth, we assume that the underlying root datum $\Gamma_{\mathbb{G}}$ is irreducible, and that θ is an automorphism of its Dynkin diagram that represents $[f]_{\mathbb{G}}$. Let δ be the order of θ, and let $\mathbb{A} \le \mathbb{G}$ be the maximal torus defined by $[f]_{\mathbb{A}} = W_{\Gamma_{\mathbb{G}}}\theta$.

Fix a rational number $\nu > 0$. We will use the rational DAHA of $W_{\mathbb{G},\mathbb{A}}$ with central charge $\vec{\nu} = (\nu_t)_t$ defined by

$$\nu_t = \begin{cases} \nu & t \text{ is associated with one of the longest roots,} \\ \delta\nu & \text{else.} \end{cases}$$

We also choose the vectors α_H and $\alpha_H^\vee$ of Section 4.4 to be the roots and coroots of the relative root system for $(\mathbb{G}, \mathbb{A})$. Doing so gives $\langle \alpha_H, \alpha_H^\vee \rangle = 2$ for all hyperplanes H. Altogether, our definition of the rational DAHA recovers the definition in [OY, §4.2.2].

To abbreviate, we will write $D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$ and $\mathsf{O}^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$ in place of $D^{rat}_{W_{\mathbb{G},\mathbb{A}}}(\vec{\nu})$ and $\mathsf{O}^{rat}_{W_{\mathbb{G},\mathbb{A}}}(\vec{\nu})$. If $\nu_t = \nu$ for all t, then we write ν in place of $\vec{\nu}$. Note that this situation occurs when $\mathbb{G}$ is split.

5.2. We now shift our geometric setting from finite fields to the field of complex numbers.

Let G be a simply-connected, connected quasi-split algebraic group over $\mathbf{C}(\!(z)\!)$ defined by the same root datum and the same Dynkin-diagram automorphism as $\mathbb{G}$. Let $A \subseteq G$ be the maximal torus corresponding to $\mathbb{A}$. Explicitly,

$$G = \mathrm{Res}_{\mathbf{C}(\!(z)\!)}^{\mathbf{C}(\!(z^{\frac{1}{e}})\!)}(\bar{G} \otimes \mathbf{C}(\!(z^{\frac{1}{e}})\!))^\theta \quad \text{and} \quad A = \mathrm{Res}_{\mathbf{C}(\!(z)\!)}^{\mathbf{C}(\!(z^{\frac{1}{e}})\!)}(\bar{A} \otimes \mathbf{C}(\!(z^{\frac{1}{e}})\!))^\theta,$$

where Res is Weil restriction, and $\bar{G}$ and $\bar{A}$ are the complex algebraic groups defined by the (split) root data.

Remark 5.1. Note that in [OY, §2.2.2], the variable G is the same as ours, but their T is our A, and other notations do not match. For instance, their e is not our e.

As explained in [OY, §2.2.3], there is a smooth $\mathbf{C}[\![z]\!]$-group scheme $\mathbf{K}$ with generic fiber G and connected special fiber: that is, an integral model of G over $\mathbf{C}[\![z]\!]$. We define $\mathfrak{g}$ to be the Lie algebra of $\mathbf{K}$. Similarly, there is an integral model of A over $\mathbf{C}[\![z]\!]$ with connected special fiber, which we denote by $\mathbf{A}$.

If we fix a θ-stable system of simple roots in $\Gamma_{\mathbb{G}}$, then its orbits under θ are in biijection with the simple roots of G with respect to A. In particular, $(X^\vee)_\theta \simeq A(\mathbf{C}(\!(z)\!))/\mathbf{A}(\mathbf{C}[\![z]\!])$, where $(-)_\theta$ means the θ-coinvariant quotient. The $\mathbf{R}$-span of $(X^\vee)_{\theta,\mathrm{free}} := (X^\vee)_\theta/(X^\vee)_{\theta,\mathrm{tors}}$ can be identified with the apartment for A in the building of G, once we identify the origin with the point corresponding to $\mathbf{K}$. The facets of the apartment cut out by the affine roots of G with respect to A are in bijection with the parahoric subgroups of G containing $\mathbf{A}$.

Let LG be the loop group defined by $LG(R) = G(R[\![z]\!][z^{-1}])$ for any $\mathbf{C}$-algebra R. Let $\mathbf{I} \subseteq \mathbf{K}$ be the Iwahori subgroup defined by the simple roots of (G, A) above, and let $I \subseteq LG$ be the corresponding sub-ind-group, so that $I(R) = \mathbf{I}(R[\![z]\!])$. Recall that the affine flag variety of G is the fpqc quotient $\mathcal{F}l = LG/I$.

Write $\nu = \frac{d}{m}$ in lowest terms, with $d, m > 0$. Let $\rho^\vee$ be the half-sum of the positive coroots of (G, A). Then there is a $\mathbf{G}_m$-action on LG defined by

$$t \cdot g(z) = \mathrm{Ad}(t^{-2d\rho^\vee})g(t^{2m}z).$$

It descends to an action on $\mathcal{F}l$, and also, the loop Lie algebra $L\mathfrak{g} := \mathrm{Lie}(LG)$. If $\gamma \in L\mathfrak{g}$ is an eigenvector for this action, then the *affine Springer fiber*

$$\mathcal{F}l^\gamma = \{[g] \in LG/I \mid \mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}(I)\}$$

is $\mathbf{G}_m$-stable. We write $L\mathfrak{g}_\nu$ for the weight-$2d$ eigenspace of $L\mathfrak{g}$ under the $\mathbf{G}_m$-action, and $L\mathfrak{g}_\nu^{\mathrm{rs}} \subseteq L\mathfrak{g}_\nu$ for its open locus of generically regular semisimple elements.

Recall that if γ is generically regular semisimple, then $\mathcal{F}l^\gamma$ is finite-dimensional by [KL, Prop. 1].

Let $Br_{\nu,\gamma}$ be the topological fundamental group of $L\mathfrak{g}_\nu^{\mathrm{rs}}(\mathbf{C})$ with basepoint γ. If m is a regular number for $\mathbb{G}$ in the sense of Section 2.8, then [OY, §3.3.6] shows that $Br_{\nu,\gamma}$ is the braid group of $W_{\mathbb{G},\mathbb{T}}$ for any Φ_m-split maximal torus $\mathbb{T} \leq \mathbb{G}$. Henceforth, we fix such a generic torus. In *loc. cit.*, $W_{\mathbb{G},\mathbb{T}}$ is called the *little Weyl group* for $\bar{\nu} := \nu \pmod 1$.

5.3. Henceforth, we fix $\gamma \in L\mathfrak{g}_\nu^{\mathrm{rs}}(\mathbf{C})$. Let $G_0 \subseteq LG$ be the connected reductive group whose Lie algebra is the weight-0 eigenspace of the $\mathbf{G}_m$-action on $L\mathfrak{g}$. Let $G_{0,\gamma}$ be the centralizer of γ in G_0. Then the commuting actions of $\mathbf{G}_m$ and G_0 on LG descend to commuting actions of $\mathbf{G}_m$ and $G_{0,\gamma}$ on $\mathcal{F}l^\gamma$. Note that $G_{0,\gamma}$ corresponds to S_a in [OY].

We say that $w\theta \in W_{\Gamma_\mathbb{G}}\theta$ is *elliptic* iff its only fixed point on $X^\vee \otimes \mathbf{Q}$ is zero. We say that it is *regular elliptic* iff it is both regular and elliptic; in this case, its order is called a *regular elliptic number* for $\mathbb{G}$. This definition then agrees with the definitions in [OY, VV]: *e.g.*, by the discussion in [OY, §3.2]. If m is a regular elliptic number for $\mathbb{G}$, then $\mathcal{F}l^\gamma$ is a projective scheme by [KL, Cor. 2] [OY, Lem. 5.2.3], and $G_{0,\gamma}$ is finite by [OY, Lem. 3.3.5(3)] and the definition of ellipticity.

5.4. Henceforth, we assume that m is a regular elliptic number for $\mathbb{G}$. We write $\mathrm{H}^*_{\mathbf{G}_m}(-)$ to denote $\mathbf{G}_m$-equivariant singular cohomology with $\mathbf{C}$-coefficients. In particular, we write $\mathrm{H}^*_{\mathbf{G}_m}(pt) = \mathbf{C}[\epsilon]$, so that ϵ sits in degree 2.

As we vary γ, the groups $G_{0,\gamma}$ form a finite group scheme over $L\mathfrak{g}_\nu^{\mathrm{rs}}$, and the vector spaces $\mathrm{H}^*_{\mathbf{G}_m}(\mathcal{F}l^\gamma)$ form a local system on which this group scheme acts. In particular, $Br_{\nu,\gamma}$ acts on the invariant subspace $\mathrm{H}^*_{\mathbf{G}_m}(\mathcal{F}l^\gamma)^{G_{0,\gamma}}$.

When $\mathbb{G}$ is split, meaning $\theta = 1$, Oblomkov–Yun endow $\mathrm{H}^*_{\mathbf{G}_m}(\mathcal{F}l^\gamma)^{G_{0,\gamma}}$ with an increasing, $Br_{\nu,\gamma}$-stable *perverse filtration* $\mathsf{P}_{\leq *}$. The precise construction uses an equivariant Ngô-type comparison between $\mathcal{F}l^\gamma$ and a fiber of a twisted, parabolic Hitchin system over a stacky projective line; in the Hitchin setting, the filtration is defined using perverse truncation of the Hitchin sheaf complex and the Ngô support theorem. We refer to §8.3.6 of *ibid.* for details. Moreover, Oblomkov–Yun construct a D_ν^{rat}-action on the bigraded vector space

$$\mathcal{E}_{\nu,\gamma} := \mathrm{gr}_*^{\mathsf{P}} \mathrm{H}^*_{\mathbf{G}_m}(\mathcal{F}l^\gamma)^{G_{0,\gamma}}/(\epsilon - 1)$$

that commutes with the $Br_{\nu,\gamma}$-action [OY, Thm. 8.2.3(1)]. What follows are Theorem 8.2.3(2) and Conjecture 8.2.5 of *ibid.*

Theorem 5.2 (Oblomkov–Yun)**.** *In the setup above, where $\nu \in \mathbf{Q}_{>0}$ such that its lowest denominator is a regular elliptic number for $\mathbb{G}$, the $Br_{\nu,\gamma}$-invariants of $\mathcal{E}_{\nu,\gamma}$ form the simple D_ν^{rat}-module $L_\nu(1)$.*

Conjecture 5.3 (Oblomkov–Yun)**.** *The constructions above, and Theorem 5.2, extend from split $\mathbb{G}$ to quasi-split $\mathbb{G}$, while still satisfying the further properties of $\mathsf{P}_{\leq *}$ in* [OY, Thm. 8.2.3(1)]. *(We may need to write $L_{\bar{\nu}}(1)$ in place of $L_\nu(1)$.)*

5.5. **Evidence for Conjecture 2.** In Section 9 of their paper, Oblomkov–Yun verify Conjecture 5.3 for various non-split cases in which $\Gamma_{\mathbb{G}}^{\theta}$ has rank ≤ 2 and $\nu = \frac{1}{m}$. As they remark, the other possibilities for ν can be reduced to this one by [OY, Prop. 5.5.8]. They arrive at the following results and conjectures:

(OY1) The $Br_{\nu,\gamma}$-action on $\mathcal{E}_{\nu,\gamma}$ is trivial for:

(a) The cases where m is the twisted Coxeter number [OY, Ex. 8.2.6].

(b) The cases where

$$(\mathsf{type}, m) = ({}^2A_2, 2), (C_2, 2), (G_2, 3), (G_2, 2).$$

Here, it is the $D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$-module $L_\nu(1)$ by Theorem 5.2.

(OY2) The $Br_{\nu,\gamma}$-action on $\mathcal{E}_{\nu,\gamma}$ can be computed for

$$(\mathsf{type}, m) = ({}^2A_3, 2), ({}^2A_4, 2), ({}^3D_4, 6), ({}^3D_4, 3).$$

Here, Oblomkov–Yun state a conjecture for the $(D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu}), Br_{\nu,\gamma})$-bimodule structure in terms of a direct sum of simple bimodules, but the existence of the perverse filtration on the whole cohomology remains open.

Note that Conjecture 3.3 holds for $\mathbb{G}, m, (\mathbb{T}, 1)$ in (OY1a), by Theorem 3.6(2), and in the cases where $\mathsf{type} = {}^2A_2, {}^2A_3, {}^2A_4$, by Proposition 6.5 below. So in these cases, the Hecke algebra $H_{\mathbb{G},\mathbb{T},1}(1)$ in Conjecture 2 is well-defined.

To explain what we can verify: As in the introduction, let

$$[\mathcal{E}_{\nu,\gamma}] = \sum_{i,j} (-1)^i t^j \operatorname{gr}^{\mathsf{P}}_j \mathrm{H}^i_{\mathbf{G}_m}(\mathcal{F}l^{\gamma})^{G_{0,\gamma}}/(\epsilon - 1) \in \mathsf{K}_0(\mathbf{C}W_{\mathbb{G},\mathbb{A}})[t^{\pm 1}].$$

Let $\chi_1 \in \mathsf{K}_0(H^{\mathbb{G}}_{\mathbb{G},\mathbb{T},1}(1))$ be the image of $\chi \in \mathrm{Irr}(W_{\mathbb{G},\mathbb{T}})$ under Brauer decomposition, or more precisely, under (4.1). Recall that Conjecture 2(2) predicts that

$$[\mathcal{E}_{\nu,\gamma}] \stackrel{?}{=} \sum_{\rho \in \mathrm{Uch}(\mathbb{G},\mathbb{A},1,\mathbb{T},1)} \varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\rho)[\Delta_{\vec{\nu}}(\chi^{\mathbb{G}}_{\mathbb{A},1}(\rho)) \otimes \chi^{\mathbb{G}}_{\mathbb{T},1}(\rho)_1] \tag{5.1}$$

in $\mathsf{K}_0(\mathbf{C}W_{\mathbb{G},\mathbb{A}} \otimes H_{\mathbb{G},\mathbb{T},1}(1)^{\mathrm{op}})[\![t]\!][t^{-1}]$, and via Theorem 2.4(2c), implies that

$$[\mathcal{E}_{\nu,\gamma}] \stackrel{?}{=} \sum_{\rho \in \mathrm{Uch}(\mathbb{G},\mathbb{A},1,\mathbb{T},1)} \mathrm{Deg}_\rho(e^{2\pi i \nu})[\Delta_{\vec{\nu}}(\chi^{\mathbb{G}}_{\mathbb{A},1}(\rho))] \tag{5.2}$$

in $\mathsf{K}_0(\mathbf{C}W_{\mathbb{G},\mathbb{A}})[\![t]\!][t^{-1}]$. We will prove:

Proposition 5.4. (1) *In the cases above, excluding those of type ${}^2C_2$ or ${}^2G_2$, the formulas of* [OY] *imply or would imply* (5.2).

(2) *In case (OY1a), and in the cases where ${}^2A_2, {}^2A_3, {}^2A_4$ and $m = 2$, the $Br_{\nu,\gamma}$-action factors through $H_{\mathbb{G},\mathbb{T},1}(1)$, and the formulas of* [OY] *imply or would imply* (5.1).

Throughout the proof, it will be convenient to write $\mathbf{C}$ for the trivial $\mathbf{C}Br_{\nu,\gamma}$-module. In all cases except (OY1a), the group $W_{\mathbb{G},\mathbb{A}}$ will be dihedral; we will adopt Chmutova's notation for its irreducible characters [Chm]. To prove (1), we use her $D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$-module character formulas in *ibid.*. To prove (2), we use the following observation, together with geometric results of Chen–Vilonen–Xue in [CVX1, CVX2, CVX3]:

Lemma 5.5. *Suppose that $H_{\mathbb{G},\mathbb{T},1}(x)$ is defined, and that there exist* $\mathrm{Irr}(W_{\mathbb{G},\mathbb{A}})_\nu \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{A}})$ *and* $(\varepsilon_\nu, D_\nu) : \mathrm{Irr}(W_{\mathbb{G},\mathbb{A}})_\nu \to \{\pm 1\} \times \mathrm{Irr}(H_{\mathbb{G},\mathbb{T},1}(1)) \sqcup \{0\}$ *such that:*

(1) *In* $\mathsf{K}_0(\mathbf{C}W_{\mathbb{G},\mathbb{A}} \otimes H_{\mathbb{G},\mathbb{T},1}(1)^{\mathrm{op}})[\![t]\!][t^{-1}]$, *we have*

$$[\mathcal{E}_{\nu,\gamma}] = \sum_{\psi \in \mathrm{Irr}(W_{\mathbb{G},\mathbb{A}})_\nu} \varepsilon_\nu(\psi)[L_{\vec{\nu}}(\psi) \otimes D_{\vec{\nu}}(\psi)].$$

(2) *For all* $\rho \in \mathrm{Uch}(\mathbb{G}, \mathbb{A}, 1, \mathbb{T}, 1)$ *and* $\psi \in \mathrm{Irr}(W_{\mathbb{G},\mathbb{A}})_\nu$, *we have*

$$(L_{\vec{\nu}}(\psi) : \Delta_{\vec{\nu}}(\chi^{\mathbb{G}}_{\mathbb{A},1}(\rho))) = \varepsilon_\nu(\psi)\varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\rho)[\chi^{\mathbb{G}}_{\mathbb{T},1}(\rho)_1 : D_\nu(\psi)], \tag{5.3}$$

where the left-hand side is the virtual multiplicity of $\Delta_{\vec{\nu}}(\chi^{\mathbb{G}}_{\mathbb{A},1}(\rho))$ *in* $L_\nu(\psi)$ *given by the inverse decomposition matrix for* $D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu})$.

Then (5.1) *holds.*

Proof. Substitute $[L_{\vec{\nu}}(\psi)] = \sum_\chi (L_{\vec{\nu}}(\psi) : \Delta_{\vec{\nu}}(\chi))[\Delta_{\vec{\nu}}(\chi)]$ into (1). Apply (2). ☐

In practice, the sign $\varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\rho)$ is most easily calculated as the sign of the integer $\mathrm{Deg}_\rho(e^{2\pi i/m})$, via Theorem 2.4(2c). It will be convenient to write Deg_χ and $\varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\chi)$ in place of Deg_ρ and $\varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\rho)$ whenever $\chi = \chi^{\mathbb{G}}_{\mathbb{A},1}(\rho)$.

5.5.1. *The Twisted Coxeter Case.* The $Br_{\nu,\gamma}$-action factors through $H_{\mathbb{G},\mathbb{T},1}(1)$ because it is trivial. Hypothesis (1) in Lemma 5.5 holds with $\mathrm{Irr}(W_{\mathbb{G},\mathbb{A}})_\nu = \{1\}$ and $(\varepsilon_\nu, D_\nu)(1) = (1, 1_1)$. Writing V for the reflection representation of $W_{\mathbb{G},\mathbb{A}}$, we have

$$\begin{aligned}\mathrm{Deg}_\chi(e^{2\pi i/m}) &= \begin{cases} (-1)^k & \chi = \Lambda^k(V), \\ 0 & \text{else} \end{cases} \\ &= (L_{\vec{\nu}}(1) : \Delta_{\vec{\nu}}(\chi)),\end{aligned} \tag{5.4}$$

where the first equality follows from [BGK, Thm. 6.6, Rem. 6.9] and the second from [R08, Thm. 5.15]. So $(L_{\vec{\nu}}(1) : \Delta_{\vec{\nu}}(\chi)) = \varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\chi)$. To show hypothesis (2) in Lemma 5.5, it remains to show that

$$[\chi^{\mathbb{G}}_{\mathbb{T},1}(\rho)_1 : 1_1] = 1 \quad \text{for all } \rho \in \mathrm{Uch}(\mathbb{G}, \mathbb{A}, 1, \mathbb{T}, 1). \tag{5.5}$$

Indeed, this follows from observing that $W_{\mathbb{G},\mathbb{T},1}$ is cyclic of order m [S74, Cor. 4.4, Thm. 7.6(v)], and hence, $H_{\mathbb{G},\mathbb{T},1}(1) \simeq \mathbf{Q}_{cyc}[\sigma]/(\sigma - 1)^m$.

5.5.2. *The Case* $({}^2A_2, 2)$. Here, $W_{\mathbb{G},\mathbb{A}} \simeq W_{A_1} \simeq S_2$, whereas $W_{\mathbb{G},\mathbb{T}} \simeq W_{A_3} \simeq S_3$ by Ennola duality (Section 6.5). Again, the $Br_{\nu,\gamma}$-action factors through $H_{\mathbb{G},\mathbb{T},1}(1)$ because it is trivial, and hypothesis (1) in Lemma 5.5 holds with $\mathrm{Irr}(W_{\mathbb{G},\mathbb{A}})_\nu = \{1\}$ and $(\varepsilon_\nu, D_\nu)(1) = (1, 1_1)$.

Via the case $(\mathsf{type}, m) = (A_1, 2)$, we compute $[L_\nu(1)] = [\Delta_\nu(1)] - [\Delta_\nu(\mathsf{sgn})]$. At the same time, we see from [C, §13.8] or Ennola duality that the generic-degree polynomials are $\mathrm{Deg}_1(x) = 1$ and $\mathrm{Deg}_{\mathsf{sgn}}(x) = x^3$, whence $\mathrm{Deg}_1(-1) = 1$ and $\mathrm{Deg}_{\mathsf{sgn}}(-1) = -1$. So to show hypothesis (2) in Lemma 5.5, we again reduce to showing (5.5). Here, it follows from observing that $H_{\mathbb{G},\mathbb{T},1}(1) \simeq H_{W_{S_3}}(-1)$, where $H_{W_{S_3}}(x)$ is the usual Hecke algebra for S_3 as a Weyl group, *cf.* Example 3.5. Alternately, one can use [GJ, Table 7.2], or more classically, [J].

5.5.3. *The Case* $(C_2, 2)$. Here, $W_{\mathbb{G},\mathbb{A}} \simeq W_{\mathbb{G},\mathbb{T}} \simeq W_{BC_2}$. The argument is analogous to that in Section 5.5.2, except that we compute the left-hand side of (5.3) from [Chm, §3.2]. We again reduce to showing (5.5). It follows from $H_{\mathbb{G},\mathbb{T},1}(1) \simeq \mathbf{C}W_{BC_2}$.

5.5.4. *The Case* $(G_2, 3)$. Here, $W_{\mathbb{G},\mathbb{T}} \simeq \mathbf{Z}_6$. The argument is analogous to the Coxeter case, except that we compute the left-hand side of (5.3) from [Chm, §3.2]. It shows an analogue of (5.4), except with V replaced by the representation where $w \in W_{\mathbb{G},\mathbb{A}}$ acts on V by w^2. Again, (5.5) follows from $H_{\mathbb{G},\mathbb{T},1}(1) \simeq \mathbf{Q}_{cyc}[\sigma]/(\sigma - 1)^6$.

5.5.5. *The Case* $(G_2, 2)$. Here, $W_{\mathbb{G},\mathbb{A}} \simeq W_{\mathbb{G},\mathbb{T}} \simeq W_{G_2}$. The argument is analogous to that in Section 5.5.3.

5.5.6. *The Case* $({}^2A_3, 2)$. Here, $W_{\mathbb{G},\mathbb{A}} \simeq W_{BC_2}$ and $W_{\mathbb{G},\mathbb{T}} \simeq W_{A_3} \simeq S_4$.

Section 9.4 of [OY] predicts an isomorphism of $(D^{rat}_{\mathbb{G},\mathbb{A}}(\vec{\nu}), \mathbf{C}Br_{\nu,\gamma})$-bimodules

$$\mathcal{E}_{\nu,\gamma} \overset{?}{\simeq} L_\nu(1) \otimes \mathbf{C} \oplus L_\nu(\varepsilon_1) \otimes \mathrm{H}^1(C_\gamma),$$

where $\mathrm{H}^1(C_\gamma)$ is the simple $\mathbf{C}Br_{\nu,\gamma}$-module in cohomological degree 1 formed by the monodromy of a certain family of genus-1 curves.

We can compute the left-hand side of (5.3) from [Chm, §3.2]. We can again determine the generic-degree polynomials, and hence the signs $\varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\chi)$, from [C, §13.8] or Ennola.

The Hecke algebra $H_{\mathbb{G},\mathbb{T},1}(1)$ has two simple modules, both in the principal block. One is trivial, and the other arises from the $Br_{\nu,\gamma}$-action on $\mathrm{H}^1(C_\gamma)$ above, by [CVX1, §2.3] combined with [CVX2]. Thus the $Br_{\nu,\gamma}$-action factors through $H_{\mathbb{G},\mathbb{T},1}(1)$. Hypothesis (1) of Lemma 5.5 holds with $\mathrm{Irr}(W_{\mathbb{G},\mathbb{A}})_\nu = \{1, \varepsilon_1\}$ and $(\varepsilon_\nu, D_\nu)(1) = (1, 1_1)$ and $(\varepsilon_\nu, D_\nu)(\varepsilon_1) = (-1, \mathrm{H}^1(C_\gamma))$. Finally, the multiplicities on the right-hand side of (5.3) can be computed from the table labeled $D_{(2)}$ in [GJ, 184], or from [J]. In this way, we can verify hypothesis (2) in Lemma 5.5.

5.5.7. *The Case* $({}^2A_4, 2)$. Here, $W_{\mathbb{G},\mathbb{A}} \simeq W_{BC_2}$ and $W_{\mathbb{G},\mathbb{T}} \simeq W_{A_4} \simeq S_5$.

Section 9.5 of [OY] predicts an isomorphism of bimodules

$$\mathcal{E}_{\nu,\gamma} \overset{?}{\simeq} L_\nu(1) \otimes \mathbf{C} \oplus L_\nu(\varepsilon_2) \otimes \Lambda^2\, \mathrm{H}^1(C_\gamma)_{\mathrm{prim}},$$

where $\Lambda^2\, \mathrm{H}^1(C_\gamma)_{\mathrm{prim}}$ is the simple $\mathbf{C}Br_{\nu,\gamma}$-module in cohomological degree 2 formed by a certain summand of the monodromy of a family of genus-2 curves.

Again, we compute the left-hand side of (5.3) from [Chm, §3.2] and the signs $\varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\chi)$ from [C, §13.8] or Ennola.

The Hecke algebra $H_{\mathbb{G},\mathbb{T},1}(1)$ again has two simple modules, both principal. One is trivial, and the other arises from the $Br_{\nu,\gamma}$-action on $\Lambda^2\, \mathrm{H}^1(C_\gamma)_{\mathrm{prim}}$ by [CVX3, (4.18)] combined with [CVX2]. Thus the $Br_{\nu,\gamma}$-action factors through $H_{\mathbb{G},\mathbb{T},1}(1)$. The rest of the verification of (5.1) is analogous to that in Section 5.5.6, except that $\varepsilon_\nu(\Lambda^2\, \mathrm{H}^1(C_\gamma)_{\mathrm{prim}}) = 1$.

5.5.8. *The Case* $({}^3D_4, 6)$. Here, $W_{\mathbb{G},\mathbb{A}} \simeq W_{G_2}$, and $W_{\mathbb{G},\mathbb{T}} \simeq G_4$ in Shephard–Todd notation for complex reflection groups. Explicitly,

$$G_4 = \langle s, t \mid s^3 = t^3 = (st)^3 = 1 \rangle,$$

so G_4 has the same braid group as $W_{A_2} \simeq S_3$.

This is the first case where the central charge of the rational DAHA is non-constant. Section 9.8 of [OY] predicts an isomorphism of bimodules

$$\mathcal{E}_{\nu,\gamma} \overset{?}{\simeq} L_{\vec{\nu}}(1) \otimes \mathbf{C} \oplus L_{\vec{\nu}}(\varepsilon_1) \otimes M_\gamma,$$

where M_γ is the simple $\mathbf{C}Br_{\nu,\gamma}$-module in cohomological degree 0 formed by the 2-dimensional irreducible representation of its quotient S_3.

We can use [Chm, §3.2] and [C, §13.8] to verify (5.2). Our central charge $\vec{\nu}$ corresponds to the parameters $(\frac{1}{2}, \frac{1}{6})$ in Chmutova's notation. Explicitly,

$$[L_{\vec{\nu}}(1)] + 2[L_{\vec{\nu}}(\varepsilon_1)] = [\Delta_{\vec{\nu}}(1) + 2[\Delta_{\vec{\nu}}(\varepsilon_1)] + 2[\Delta_{\vec{\nu}}(\varepsilon_2)] + [\Delta_{\vec{\nu}}(\mathsf{sgn})] - 3[\Delta_{\vec{\nu}}(\tau_1)].$$

5.5.9. *The Case* $({}^3D_4, 3)$. Here, $W_{\mathbb{G},\mathbb{A}} \simeq W_{G_2}$ and $W_{\mathbb{G},\mathbb{T}} \simeq G_4$ once again. Section 9.9 of [OY] predicts an isomorphism of bimodules

$$\mathcal{E}_{\nu,\gamma} \overset{?}{\simeq} L_{\vec{\nu}}(1) \otimes \mathbf{C} \oplus L_{\vec{\nu}}(\varepsilon_1) \otimes \mathrm{H}^1(C'_\gamma),$$

where $\mathrm{H}^1(C'_\gamma)$ is the simple $\mathbf{C}Br_{\nu,\gamma}$-module in cohomological degree 1 formed by the monodromy of a family of genus-1 curves. The rest of the verification of (5.2) is analogous to that in Section 5.5.8, but now $\vec{\nu}$ corresponds to $(1, \frac{1}{3})$, and $\varepsilon_\nu(\mathrm{H}^1(C'_\gamma)) = -1$. Explicitly,

$$\begin{aligned}[L_{\vec{\nu}}(1)] - 2[L_{\vec{\nu}}(\varepsilon_1)] = [\Delta_{\vec{\nu}}(1) - 2[\Delta_{\vec{\nu}}(\varepsilon_1)] - 2[\Delta_{\vec{\nu}}(\varepsilon_2)] + [\Delta_{\vec{\nu}}(\mathsf{sgn})] - [\Delta_{\vec{\nu}}(\tau_1)]\\ + 2[\Delta_{\vec{\nu}}(\tau_2)].\end{aligned}$$

5.6. **Comparison to Deligne–Lusztig Bimodules.** To conclude this section, we explain the table of analogies (1.5) from the introduction.

Henceforth, we return to using G to denote a reductive algebraic group over $\bar{\mathbf{F}}_q$ arising from $\mathbb{G}$. Thus the group G of our new notation will be most analogous to the group $\bar{G} \otimes \mathbf{C}(\!(z)\!)$, not the group G, of our old notation.

Fix an integer $e > 0$ and a Φ_e-cuspidal pair $(\mathbb{L}, \lambda)$ for $\mathbb{G}$. For any prime power $q > 1$, let $H_{\mathbb{G},\mathbb{L},\lambda}(q)$ be the base change of $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ on the left-hand side of (3.1). There is a stronger form of Conjecture 3.3, roughly stating that whenever G, F, L, λ_q arise from $q, \mathbb{G}, \mathbb{L}, \lambda$, the maps $(\varepsilon^{\mathbb{G}}_{\mathbb{L},\lambda}, \chi^{\mathbb{G}}_{\mathbb{L},\lambda})$ of Theorem 2.4 are induced by the $(G^F, H_{\mathbb{G},\mathbb{L},\lambda}(q))$-bimodule stucture of $R^G_L(\lambda_q)$ under Tits deformation.

Conjecture 5.6. *Assume that Conjecture 3.3 holds for q, G, F, L, λ_q. Then in the Grothendieck group $\mathsf{K}_0(\mathbf{C}G^F \otimes H_{\mathbb{G},\mathbb{L},\lambda}(q)^{\mathrm{op}})$, we have*

$$\sum_i (-1)^i \, \mathrm{H}^i_c(Y^G_{L\subseteq P}[\lambda_q]) = \sum_{\rho \in \mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda)} \varepsilon^{\mathbb{G}}_{\mathbb{L},\lambda}(\rho)[\rho_q \otimes \chi^{\mathbb{G}}_{\mathbb{L},\lambda}(\rho)_q],$$

where $\rho_q \in \mathrm{Uch}(G^F)$ corresponds to $\rho \in \mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda)$, and the $H_{\mathbb{G},\mathbb{L},\lambda}(q)$-module χ_q corresponds to $\chi \in \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})$.

Note that this conjecture recovers Example 2.5, because if $\mathbb{A} \leq \mathbb{G}$ is a Φ_1-split maximal torus, then $\varepsilon^{\mathbb{G}}_{\mathbb{A},1} = 1$ uniformly.

Conjecture 3.3 would imply that for any further integer $m > 0$ and Φ_m-cuspidal pair $(\mathbb{M}, \mu)$ giving rise to M, μ_q, we would have

$$\begin{aligned}(5.6)\qquad &\sum_{i,j} (-1)^{i+j}\, \mathrm{H}^i_c(Y^G_{L\subseteq P}[\lambda_q]) \otimes_{\mathbf{C}G^F} \mathrm{H}^j_c(Y^G_{M\subseteq Q}[\mu_q]) \\ &\quad = \sum_{\rho\in \mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu)} \varepsilon^{\mathbb{G}}_{\mathbb{L},\lambda}(\rho)\varepsilon^{\mathbb{G}}_{\mathbb{M},\mu}(\rho)[\chi^{\mathbb{G}}_{\mathbb{L},\lambda}(\rho)_q \otimes \chi^{\mathbb{G}}_{\mathbb{L},\lambda}(\rho)_q]\end{aligned}$$

in $\mathsf{K}_0(H_{\mathbb{G},\mathbb{L},\lambda}(q) \otimes H_{\mathbb{G},\mathbb{M},\mu}(q)^{\mathrm{op}})$. (Above, $Q \subseteq G$ is a parabolic subgroup, like P.) This is the bimodule that produces the second row of (1.5).

Returning to the notation we used earlier in this section, let $\zeta = e^{2\pi i\nu}$, and let $\bar{\mathcal{E}}_{\nu,\gamma}$ be the $(H_{\mathbb{G},\mathbb{A},1}(\zeta), H_{\mathbb{G},\mathbb{T},1}(1))$-bimodule

$$(5.7)\qquad \bar{\mathcal{E}}_{\nu,\gamma} = \sum_{\rho\in \mathrm{Uch}(\mathbb{G},\mathbb{A},1,\mathbb{T},1)} \varepsilon^{\mathbb{G}}_{\mathbb{T},1}(\rho)[\chi^{\mathbb{G}}_{\mathbb{A},1}(\rho)_\zeta \otimes \chi^{\mathbb{G}}_{\mathbb{M},1}(\rho)_1].$$

By Theorem 3.6(1) and Theorem 4.2, we can construct $\bar{\mathcal{E}}_{\nu,\gamma}$ by applying the KZ functor term by term to the left factors in the conjectural decomposition of $\mathcal{E}_{\nu,\gamma}$ in Conjecture 2(2). Since $\varepsilon^{\mathbb{G}}_{\mathbb{A},1}(\rho) = 1$ for all ρ, the bimodules (5.6) and (5.7) only differ in the following ways:

- (5.6) works for any $(\mathbb{L}, \lambda)$ and $(\mathbb{M}, \mu)$, whereas (5.7) requires us to take $\mathbb{L}, , \mathbb{M}$ to be the tori $\mathbb{A}, \mathbb{T}$.
- (5.6) uses the specializations of $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ and $H_{\mathbb{G},\mathbb{M},\mu}(x)$ at $x = q$, whereas (5.7) uses the specializations of $H_{\mathbb{G},\mathbb{A},1}(x)$ and $H_{\mathbb{G},\mathbb{T},1}(x)$ at $x = \zeta$ and $x = 1$, respectively.

This is the content of the analogy in (1.5).

Remark 5.7. Using an equivariant Künneth formula, it is possible to rewrite the left-hand side of (5.6) in terms of the G-equivariant cohomology of a single derived scheme. Here, equivariant cohomology is interpreted as the hypercohomology of the equivariant constant ℓ-adic sheaf, following [BDR].

In the case where $\mathbb{G}$ is split and $\mathbb{L}, \mathbb{M}$ are maximal tori respectively of types $[w], [v]$, the derived scheme takes the form $\mathcal{Y}(w) \times^{\mathrm{L}}_{GF} \mathcal{Y}(v)$ where $\mathcal{Y}(w), \mathcal{Y}(v)$ are defined as follows. Let $\mathcal{B}$ be the flag variety of G, parametrizing its Borel subgroups. For any $w \in W_{\Gamma_{\mathbb{G}}}$, write $B \xrightarrow{x} B'$ to indicate that a pair of Borels (B, B') has relative position w, and let

$$\mathcal{Y}(w) = \{(gF, B) \in GF \times \mathcal{B} \mid B \xrightarrow{w} g(FB)g^{-1}\}.$$

Let G act on $\mathcal{Y}(w)$ according to $x \cdot (gF, B) = (xgF, xBx^{-1})$. The arguments of [BDR, §2] show that the G-equivariant cohomology of $\mathcal{Y}(w)$ recovers $R^G_L(1)$, and hence, that of $\mathcal{Y}(w) \times^{\mathrm{L}}_{GF} \mathcal{Y}(v)$ recovers $R^G_L(1) \otimes_{\mathbf{C}G^F} R^G_M(1)$. Note that $\mathcal{Y}(w), \mathcal{Y}(v)$ are analogues, with GF in place of G, of the varieties Y_w appearing in Lusztig's work on character sheaves.

6. The General Linear and Unitary Groups

6.1. Fix an integer $n \geq 2$. We write $\mathbb{GL}_n$ for the *generic general linear group* of rank n and $\mathbb{GU}_n$ for the *generic general unitary group* of rank n, corresponding to the finite reductive groups $\mathrm{GL}_n(\mathbf{F}_q)$ and $\mathrm{GU}_n(\mathbf{F}_q)$. These generic groups have the same root datum, but $\mathbb{GL}_n$ is split with Weyl group S_n, whereas $[f]_{\mathbb{GU}_n} = S_n f = -S_n$, where $f \in \mathrm{Aut}(\Gamma_{\mathbb{GU}_n})$ is induced by the nontrivial involution of the Dynkin diagram of type A_{n-1}.

In this section, we first describe the Φ-cuspidal pairs of $\mathbb{GL}_n$ and $\mathbb{GU}_n$ in terms of the combinatorics of partitions. We then predict the explicit Φ-specializations arising from Conjecture 3.3 in these cases, and prove Theorem 3 for the resulting Hecke algebras.

6.2. **Partitions and Abaci.** Let Π be the set of integer partitions of arbitrary size. We view an element $\pi \in \Pi$ as a weakly-descreasing sequence of nonnegative integers $\pi_1 \geq \pi_2 \geq \ldots$ such that $\pi_i = 0$ whenever i is large enough. Its *size* $|\pi|$ is the sum of its entries, and its *length* ℓ_π is its number of nonzero entries. We refer to tuples $\vec{\pi} = (\pi^{(1)}, \ldots, \pi^{(e)}) \in \Pi^e$ as *e-partitions*.

We draw partitions as Young diagrams in French notation, composed of unit squares or *boxes* in the upper-right quadrant of the x, y-plane, flush against the positive x- and y-axes. For any partition $\pi \in \Pi$ and box $\square \in \pi$, the *hook length* of $\square$ equals 1 plus the number of boxes to its right plus the number of boxes above it. By definition, π is an *e-core* iff it contains no boxes whose hook length is divisible by e. In the language of [JK, §2.3, 2.7], this means we cannot obtain any smaller Young diagram by removing rim e-hooks from π.

We write $\Pi_{e\text{-}cor} \subseteq \Pi$ for the subset of e-cores. For each integer $e > 0$, there is a bijection $\Pi \xrightarrow{\sim} \Pi_{e\text{-}cor} \times \Pi^e$, called the corresponding *core-quotient bijection*. For our purposes, it is most convenient to express it in terms of the combinatorics of objects called abaci.

In more detail, let

$$\mathsf{B} = \{\beta \subseteq \mathbf{Z} \mid \mathbf{Z}_{<x} \subseteq \beta \subseteq \mathbf{Z}_{<y} \text{ for some integers } x, y\}.$$

Tuples $\vec{\beta} = (\beta^{(0)}, \ldots, \beta^{(e-1)}) \in \mathsf{B}^e$ are usually called *e-abacus configurations*, or more simply, *e-abaci*. The elements of the sets $\beta^{(i)}$ are sometimes called *beads*. There is a bijection from 1-abaci to e-abaci

$$\upsilon_e : \mathsf{B} \xrightarrow{\sim} \mathsf{B}^e$$

as follows. First, let the bijection $(q_e, r_e) : \mathbf{Z} \xrightarrow{\sim} \mathbf{Z} \times \{0, 1, \ldots, e-1\}$ be defined by

$$x = eq_e(x) + r_e(x).$$

Next, set $\upsilon_e(\beta) = (\upsilon_e^{(0)}(\beta), \ldots, \upsilon_e^{(e-1)}(\beta))$, where

$$\upsilon_e^{(i)}(\beta) = \{q_e(x) \mid x \in \beta \text{ such that } r_e(x) = i\}.$$

We now explain how υ_e produces an e-core and an e-quotient.

Recall that a *charged partition* is a pair $(\pi, s) \in \Pi \times \mathbf{Z}$. Henceforth, to follow convention, we will use the notation $|\pi, s\rangle$ rather than (π, s). There is a bijection from charged partitions to 1-abaci

$$\beta : \Pi \times \mathbf{Z} \xrightarrow{\sim} \mathsf{B},$$

which we abbreviate by writing $\beta_{\pi,s} = \beta(|\pi, s\rangle)$ and define by

$$\beta_{\pi,s} = \{\pi_i - i + s \mid i = 1, 2, 3, \ldots\}.$$

More generally, a *charged e-partition* is a pair $|\vec{\pi}, \vec{s}\rangle \in \Pi^e \times \mathbf{Z}^e$. We again write β to denote the bijection from charged e-partitions to e-abaci: $\beta : \Pi^e \times \mathbf{Z}^e \xrightarrow{\sim} \mathsf{B}^e$. We then get a bijection from charged partitions to charged e-partitions:

$$\Upsilon_e : \Pi \times \mathbf{Z} \xrightarrow{\beta} \mathsf{B} \xrightarrow{\upsilon_e} \mathsf{B}^e \xrightarrow{\beta^{-1}} \Pi^e \times \mathbf{Z}^e.$$

With this notation, a partition $\pi \in \Pi$ is an e-core if and only if, for some (*equiv.*, any) $s \in \mathbf{Z}$, the charged e-partition $\Upsilon_e(|\pi, s\rangle)$ takes the form $(\emptyset^e, \vec{t})$ for some $\vec{t} \in \mathbf{Z}^e$, where $\emptyset^e = (\emptyset, \ldots, \emptyset)$. That is, no bead in the output e-abacus can be pushed to a more negative position.

In general, if $\Upsilon_e(|\pi, s\rangle) = |\vec{\varpi}, \vec{r}\rangle$, then the charged version of the core-quotient bijection sends $|\pi, s\rangle$ to the pair consisting of:

(1) The charged e-core $(\Upsilon_e)^{-1}(|\emptyset^e, \vec{r}\rangle)$. The underlying e-core depends only on π, so we call it the *e-core of π*. As in [JK, §2.7], it is the partition that remains after removing as many rim e-hooks from π as possible.
(2) The e-partition $\vec{\varpi}$, which we call the *e-quotient of* $|\pi, s\rangle$.

Thus the charged core-quotient bijection is a map

$$\Pi \times \mathbf{Z} \xrightarrow{\sim} (\Pi_{e\text{-}cor} \times \mathbf{Z}) \times \Pi^e.$$

In practice, we will use its precomposition with maps $\Pi \to \Pi \times \mathbf{Z}$ of the form $\pi \mapsto |\pi, e + \ell_\pi\rangle$.

6.3. In what follows, we fix a Φ_1-split maximal torus $\mathbb{A} \leq \mathbb{GL}_n$. A special feature of $\mathbb{GL}_n$ is that $\mathrm{Uch}(\mathbb{GL}_n) = \mathrm{Uch}(\mathbb{GL}_n, \mathbb{A}, 1)$: Every generic unipotent irreducible character belongs to the principal series. Since $W_{\mathbb{GL}_n, \mathbb{A}} = S_n$, the symmetric group on n letters, we may now regard $\chi^{\mathbb{GL}_n}_{\mathbb{A},1}$ as a bijection:

$$\chi : \mathrm{Uch}(\mathbb{GL}_n) \xrightarrow{\sim} \{\text{partitions of } n\}.$$

Henceforth, we conflate each element $\rho \in \mathrm{Uch}(\mathbb{G})$ with its partition $\chi(\rho)$.

Following [BMM93, 45–48], we explain how the Φ-Harish-Chandra series for $\mathbb{GL}_n$ are described by cores and quotients. First, a full set of representatives for the elements of $\mathsf{HC}_e(\mathbb{GL}_n)$ are the Φ_e-cuspidal pairs $(\mathbb{L}, \lambda)$ in which:

(1) The Levi subgroup $\mathbb{L}$ takes the form

$$\mathbb{L} = \mathbb{T} \times \mathbb{GL}_{n-ae} \quad \text{for some integer } a > 0,$$

where $\mathbb{T} \leq \mathbb{GL}_{ae}$ is a subtorus such that $|\mathbb{T}|(x) = (x^e - 1)^a$. Note that in this case, $\mathbb{L} = Z_{\mathbb{G}}(\mathbb{T}')$ for a Φ_e-subtorus $\mathbb{T}' \leq \mathbb{T}$.
(2) The character $\lambda \in \mathrm{Uch}(\mathbb{L})$ corresponds to an e-core partition of $n - ae$ under the identification $\mathrm{Uch}(\mathbb{L}) = \mathrm{Uch}(\mathbb{GL}_{n-ae})$ induced by (1).

In general, not every Φ_e-split Levi subgroup of $\mathbb{GL}_n$ is S_n-conjugate to a Levi of the form in (1). But for $\mathbb{L}$ of that form, the Φ_e-cuspidal elements of $\mathrm{Uch}(\mathbb{L})$ are precisely those in (2).

Suppose that $(\mathbb{L}, \lambda)$ is a Φ_e-cuspidal pair of the form in (1)–(2). It turns out that $[f]_{\mathbb{L}} = S_{n-ae}v$ for some $v \in S_{ae}$ of cycle type e^a, where we embed $S_{ae} \times S_{n-ae}$ into S_n as a parabolic subgroup. Moreover,

$$W_{\mathbb{G},\mathbb{L},\lambda} \simeq Z_{S_{ae}}(v) \simeq \mathbf{Z}_e \wr S_a,$$

where $\mathbf{Z}_e := \mathbf{Z}/e\mathbf{Z}$. In Shephard–Todd notation, this wreath product is denoted $G(e, 1, a)$. Explicitly, if $c_1, \ldots, c_a$ are the individual e-cycles that comprise v, then c_i generates the ith copy of $\mathbf{Z}_e$ in the wreath product. Note that the isomorphisms above depend only on $\mathbb{L}$, not on λ.

By Clifford theory, the irreducible characters of $\mathbf{Z}_e \wr S_a$ are indexed by the e-partitions $\vec{\pi} = (\pi_1, \ldots, \pi_e)$ with $\sum_i |\pi_i| = a$. Henceforth, we conflate each element of $\mathrm{Irr}(W_{\mathbb{GL}_n,\mathbb{L},\lambda})$ with the corresponding e-partition.

The Harish-Chandra series indexed by $(\mathbb{L}, \lambda)$ is

$$\mathrm{Uch}(\mathbb{GL}_n, \mathbb{L}, \lambda) = \{\rho \in \mathrm{Uch}(\mathbb{GL}_n) \mid \chi(\rho) \text{ has } e\text{-core } \lambda\}.$$

In this sense, the map sending $\rho \in \mathrm{Uch}(\mathbb{GL}_n)$ to the Φ_e-Harish–Chandra series containing ρ is essentially the map sending a partition of n to its e-core. At the same time, the map

$$\chi^{\mathbb{GL}_n}_{\mathbb{L},\lambda} : \mathrm{Uch}(\mathbb{GL}_n, \mathbb{L}, \lambda) \to \mathrm{Irr}(W_{\mathbb{GL}_n,\mathbb{L},\lambda})$$

is essentially the map sending a partition of n to its e-quotient. To make this more precise: For any $s \in \mathbf{Z}$, let $\vec{\varrho}_{e,s}$ be the composition

$$\vec{\varrho}_{e,s} : \Pi \xrightarrow{|-,e+s\rangle} \Pi \times \mathbf{Z} \xrightarrow{\Upsilon_e} \Pi^e \times \mathbf{Z}^e \twoheadrightarrow \Pi^e.$$

Then the map $\chi^{\mathbb{GL}_n}_{\mathbb{L},\lambda}$ of [BMM93, 46–47] is determined by the commutative diagram below, in which the vertical arrows are injective:

$$\begin{tikzcd} \mathrm{Uch}(\mathbb{GL}_n, \mathbb{L}, \lambda) \arrow[r, "\chi^{\mathbb{GL}_n}_{\mathbb{L},\lambda}"] \arrow[d] & \mathrm{Irr}(W_{\mathbb{GL}_n,\mathbb{L},\lambda}) \arrow[d] \\ \Pi \arrow[r, "\vec{\varrho}_{e,\ell_\lambda}"] & \Pi^e \end{tikzcd} \tag{6.1}$$

In *loc. cit.*, Broué–Malle–Michel verify that under this definition, the maps $\chi^{\mathbb{GL}_n}_{\mathbb{L},\lambda}$ satisfy the commutativity constraint in Theorem 2.4(2d).

6.4. In what follows, we write $C = \mathbf{Z}_e \wr S_a$. For such groups, the Hecke algebra $H_C(\vec{u})$ is more commonly known as an *Ariki–Koike algebra*. To describe it, recall the Coxeter presentation:

$$C = \left\langle t, s_1, \ldots, s_{a-1} \;\middle|\; \begin{array}{l} t^e = s_1^2 = \cdots = s_{a-1}^2 = 1, \\ (ts_1)^4 = (s_1s_2)^3 = \cdots = (s_{a-2}s_{a-1})^3 = 1 \end{array} \right\rangle.$$

In the notation of Section 3.2, the C-orbits on the set of hyperplanes $\mathcal{A}$ correspond to the sets of pseudo-reflections $\{t\}$ and $\{s_1, \ldots, s_{a-1}\}$. Fix distinguished braid reflections $\tau, \sigma_1, \ldots, \sigma_{a-1}$ that respectively lift $t, s_1, \ldots, s_{a-1}$ to generators of the monodromy around these hyperplanes, such that the braid relations in [LM, §2.1] hold with our a, τ, σ_i in place of their n, T_0, T_i. We will write τ, σ_i in place of σ_H, and similarly, $u_{\tau,j}, u_{\sigma,j}$ in place of $u_{\mathcal{C},j}$. With this notation,

$$H_C(\vec{u}) = \frac{\mathbf{Z}[\vec{u}^{\pm 1}][Br_C]}{\langle (\tau - u_{\tau,0}) \cdots (\tau - u_{\tau,e-1}), (\sigma_i - u_{\sigma,0})(\sigma_i - u_{\sigma,1}) \mid 1 \leq i < a\rangle}.$$

Extending Theorem 3.6(4), we will predict the specializations of these Ariki–Koike algebras that arise from Conjecture 3.3 when $\mathbb{G} = \mathbb{GL}_n$ and $(\mathbb{L}, \lambda)$ is arbitrary. As preparation, we define a map

$$\vec{a}_e : \Pi_{e\text{-}cor} \to \mathbf{Z}^e$$

as follows. First, for any $s \in \mathbf{Z}$, let $\vec{b}_{e,s} = (b_{e,s}^{(0)}, \ldots, b_{e,s}^{(e-1)})$ be the composition

$$\vec{b}_{e,s} : \Pi \xrightarrow{|-, e+s\rangle} \Pi \times \mathbf{Z} \xrightarrow{\Upsilon_e} \Pi^e \times \mathbf{Z}^e \twoheadrightarrow \mathbf{Z}^e.$$

Equivalently $\vec{b}_{e,s}$ is given by $|\vec{\varrho}_{e,s}(\pi), \vec{b}_{e,s}(\pi)\rangle = \Upsilon_e(|\pi, e+s\rangle)$. Next, set $\vec{a}_e(\lambda) = (a^{(0)}(\lambda), \ldots, a^{(e-1)}(\lambda))$, where

$$a_e^{(i)}(\lambda) = e b_{e,\ell_\lambda}^{(i)}(\lambda) + i.$$

Conjecture 6.1. *Let $(\mathbb{L}, \lambda)$ be a Φ_e-cuspidal pair for $\mathbb{GL}_n$ taking the form in Section 6.3. Then Conjecture 3.3 holds for $\mathbb{G} = \mathbb{GL}_n$ and $(\mathbb{L}, \lambda)$, with the explicit specialization*

$$\begin{cases} \mathcal{S}_{\mathbb{GL}_n, \mathbb{L}, \lambda}(u_{\tau,j}) = x^{a_e^{(j)}(\lambda)} & \textit{for all } j, \\ \mathcal{S}_{\mathbb{GL}_n, \mathbb{L}, \lambda}(u_{\sigma,0}) = 1, \\ \mathcal{S}_{\mathbb{GL}_n, \mathbb{L}, \lambda}(u_{\sigma,1}) = -x^e. \end{cases} \tag{6.2}$$

(The definition of $\vec{a}_e$ ensures that this is, in fact, a Φ_e-specialization.)

Proposition 6.2. *Conjecture 6.1 holds in the cases where:*

(1) *$\mathbb{L}$ is a maximal torus of $\mathbb{GL}_n$.*
(2) *$\mathbb{L} = \mathbb{T} \times \mathbb{GL}_{n-e}$ for some Coxeter maximal torus $\mathbb{T} \leq \mathbb{GL}_e$. Equivalently, $W_{\mathbb{GL}_n, \mathbb{L}, \lambda}$ is cyclic.*

In the notation of Section 6.3, (1) corresponds to $n - ae = 0, 1$, and (2) to $a = 1$.

Proof. Case (1) is precisely case (4) of Theorem 3.6. Here, λ is the trivial character, which corresponds to the empty partition, so $\vec{a}_e(\lambda) = (0, 1, \ldots, e-1)$. Therefore, (6.2) recovers the specialization in [DM06, Thm. 10.1].

In case (2), the group Br_C is freely generated by τ. So it suffices to construct, uniformly for any prime power $q > 1$ and prime ℓ invertible in $\mathbf{F}_q$, a $\mathrm{GL}_n(\mathbf{F}_q)$-equivariant action of τ on $\mathrm{H}_c^*(Y_{L \subseteq P}^G)[\lambda_q]$ with eigenvalues $q^{a_e^{(0)}(\lambda)}, \ldots, q^{a_e^{(e-1)}(\lambda)}$, where L, λ_q arise from q, L, λ. As it turns out, we can map τ to the Frobenius F. Indeed, the work of Dudas in [Du] shows that for the cuspidal pairs we are consdering, the multiset of eigenvalues of F is precisely $\{q^{a_e^{(i)}(\lambda)}\}_i$. To translate his notation into ours, observe that his d is our e, his μ is our λ, and his multiset $\{\gamma_d(X, x)\}_{x \in X'}$ is our multiset $\{q^{a_e^{(i)}(\lambda)}\}_i$. □

Remark 6.3. The strategy of the proof of [DM06, Thm. 10.1] is to reduce from the general case to that of Coxeter maximal tori, via a subtle interplay between the positive braids and Frobenius maps that define the relevant (braid-theoretic) Deligne–Lusztig varieties. It seems plausible that this strategy could be generalized to the *parabolic Deligne–Lusztig varieties* of [DM14]. One could then try to reduce the general case of Conjecture 6.1 to case (2) of Proposition 6.2.

6.5. **Ennola Duality.** For any generic group $\mathbb{G}$, write $\mathbb{G}^-$ to denote the generic group defined by $\Gamma_{\mathbb{G}^-} = \Gamma_{\mathbb{G}}$ and $[f]_{\mathbb{G}^-} = [-f]_{\mathbb{G}}$. Note that the map $\mathbb{L} \mapsto \mathbb{L}^-$ sends each Φ_e-split Levi subgroup of $\mathbb{G}$ to a Φ_{e^-}-split Levi subgroup of $\mathbb{G}^-$, where

$$\Phi_{e^-}(x) := \Phi_e(-x), \quad \text{meaning } e^- = \begin{cases} 2e & e \text{ odd}, \\ e & e \equiv 0 \pmod 4, \\ \frac{1}{2}e & e \equiv 2 \pmod 4. \end{cases}$$

As explained in [BMM93, 44–45] and [B, 57–58, 72], *Ennola duality* for $\mathbb{G}$ amounts to the existence of a similar map

$$\rho \mapsto (\varepsilon(\rho), \rho^-) : \mathrm{Uch}(\mathbb{G}) \to \{\pm 1\} \times \mathrm{Uch}(\mathbb{G}^-),$$

together with analogous maps for all Φ-split Levi subgroups $\mathbb{L} \leq \mathbb{G}$, such that:

(1) $\rho \mapsto \rho^-$ is bijective.
(2) For any integer $e > 0$ and Φ_e-split Levi $\mathbb{L} \leq \mathbb{G}$, we have a commutative diagram:

$$\begin{tikzcd} \mathbf{Z}\mathrm{Uch}(\mathbb{L}) \arrow[r, "\rho \mapsto \varepsilon(\rho)\rho^-"] \arrow[d, "R_{\mathbb{L}}^{\mathbb{G}}"'] & \mathbf{Z}\mathrm{Uch}(\mathbb{L}^-) \arrow[d, "R_{\mathbb{L}^-}^{\mathbb{G}^-}"] \\ \mathbf{Z}\mathrm{Uch}(\mathbb{G}) \arrow[r, "\lambda \mapsto \varepsilon(\lambda)\lambda^-"] & \mathbf{Z}\mathrm{Uch}(\mathbb{G}^-) \end{tikzcd}$$

(3) For all $\rho \in \mathrm{Uch}(\mathbb{G})$, we have $\mathrm{Deg}_\rho(x) = \varepsilon(\rho)\mathrm{Deg}_{\rho^-}(-x)$. See [BM93, 137] or [B, 72].

Classical Ennola duality is the case where $\mathbb{G} = \mathbb{GL}_n$ and $\mathbb{G}^- = \mathbb{GU}_n$ for some n, using the fact that $[f]_{\mathbb{GU}_n} = [-1]_{\mathbb{GL}_n}$. It turns out that:

(4) For all $\rho \in \mathrm{Uch}(\mathbb{GL}_n)$, we have $\chi_{\mathbb{A},1}^{\mathbb{GL}_n}(\rho) = \chi_{\mathbb{A}^-,1}^{\mathbb{GU}_n}(\rho^-)$ under the canonical isomorphisms $W_{\mathbb{GL}_n,\mathbb{A}} \simeq S_n \simeq W_{\mathbb{GU}_n,\mathbb{A}^-}$. For details, see [BMM93, 45] or [DM20, §11.7].

From items (3)–(4), we are led to expect that $\mathcal{S}_{\mathbb{GU}_n,\mathbb{L}^-,\lambda^-}$ and $H_{\mathbb{GU}_n,\mathbb{L}^-,\lambda^-}(x)$ are related to $\mathcal{S}_{\mathbb{GL}_n,\mathbb{L},\lambda}$ and $H_{\mathbb{GL}_n,\mathbb{L},\lambda}(x)$ by the substitution $x \mapsto -x$.

Conjecture 6.4. *Let $(\mathbb{L}, \lambda)$ be a Φ_e-cuspidal pair for $\mathbb{GL}_n$ taking the form in Section 6.3. Then Conjecture 3.3 holds for $\mathbb{G} = \mathbb{GU}_n$ and $(\mathbb{L}^-, \lambda^-)$, with the explicit specialization*

$$\begin{cases} \mathcal{S}_{\mathbb{GU}_n,\mathbb{L}^-,\lambda^-}(u_{\tau,j}) = (-x)^{a_e^{(j)}(\lambda)} & \textit{for all } j, \\ \mathcal{S}_{\mathbb{GU}_n,\mathbb{L}^-,\lambda^-}(u_{\sigma,0}) = 1, \\ \mathcal{S}_{\mathbb{GU}_n,\mathbb{L}^-,\lambda^-}(u_{\sigma,1}) = -(-x)^e. \end{cases} \tag{6.3}$$

Proposition 6.5. *Conjecture 6.4 holds in the cases where $\mathbb{L}$ is a maximal torus of $\mathbb{GL}_n$ and $e \in \{1, 2, n_{\mathrm{odd}}\}$, where:*

$$n_{\mathrm{odd}} = \begin{cases} n-1 & n \textit{ even}, \\ n & n \textit{ odd}. \end{cases}$$

Proof. The values $1, 2, n_{\mathrm{odd}}$ for e respectively correspond to the values $2, 1, 2n_{\mathrm{odd}}$ for e^-. If $e^- = 1$, then $\mathbb{L}^- \leq \mathbb{GU}_n$ belongs to case (1) of Theorem 3.6. It remains to handle $e^- = 2$ and $e = 2n_{\mathrm{odd}}$.

If $n = 2$, then $2n_{\mathrm{odd}} = 2$. If instead $n > 2$, then $2n_{\mathrm{odd}}$ is the twisted Coxeter number of $\mathbb{GL}_n$ by [S74, §7]. Similarly, 2 is the order of $w_0 f = -1$, where $w_0 \in S_n$ is the longest element and f is the Dynkin-diagram automorphism defining $\mathbb{GU}_n$. So by [BMM99, Cor. 5.10], $e^- = 2$, *resp.* $e^- = 2n_{\mathrm{odd}}$ implies that $\mathbb{L}^-$ belongs to case (2), *resp.* case (3), of Theorem 3.6. □

6.6. **Ariki–Koike Blocks.** To prove Theorem 3, we need the work of Lyle–Mathas describing the blocks of specialized Ariki–Koike algebras.

Recall from Section 6.2 that we draw partitions as Young diagrams in the xy-plane, flush against the positive axes. For any partition π and box $\square \in \pi$, let $(x(\square), y(\square))$ be the coordinates of the top right corner of $\square$. What follows is a version of [LM, Thm. 2.11].

Theorem 6.6 (Lyle–Mathas)**.** *Let $C = \mathbf{Z}_e \wr S_a$. Fix a field $K \supseteq \mathbf{Q}$ and units $\alpha_0, \alpha_1, \ldots, \alpha_{e-1}, \omega \in K^\times$, where $\omega \neq 1$. Let $\mathcal{T} : \mathbf{Z}[\vec{u}^{\pm 1}] \to K$ be the specialization*

$$\begin{cases} \mathcal{T}(u_{\tau,j}) = \alpha_j & \textit{for all } j, \\ \mathcal{T}(u_{\sigma,0}) = 1, \\ \mathcal{T}(u_{\sigma,1}) = -\omega, \end{cases}$$

and let $KH_{C,\mathcal{T}} = K \otimes_{\mathbf{Z}[\vec{u}^{\pm 1}]} H_C(\vec{u})$ be the base change of $H_C(\vec{u})$ along $\mathcal{T}$. For any e-partition $\vec{\varpi}$ with $\sum_i |\varpi_i| = a$, let $\mathsf{c}^{\mathcal{T}}_{\vec{\varpi}} : K^\times \to \mathbf{Z}_{\geq 0}$ be defined by

$$\mathsf{c}^{\mathcal{T}}_{\vec{\varpi}}(u) = \left| \left\{ (j, \square) \;\middle|\; \begin{array}{l} 0 \leq j \leq e-1, \\ \square \in \varpi^{(j)}, \\ u = \omega^{y(\square) - x(\square)} \alpha_j \end{array} \right\} \right|.$$

Then two elements of $\mathrm{Irr}(C)$ *map into the same block ideal of* $\mathsf{K}_0(KH_{C,\mathcal{T}})$ *if and only if they correspond to e-partitions $\vec{\varpi}, \vec{\varrho}$ such that $\mathsf{c}^{\mathcal{T}}_{\vec{\varpi}} = \mathsf{c}^{\mathcal{T}}_{\vec{\varrho}}$.*

Above, the map $\mathrm{Irr}(C) \to \mathsf{K}_0(KH_{C,\mathcal{T}})$ is defined by the following replacement for the construction in Section 4. Note that by the work of Dipper–James–Mathas, $KH_{C,\mathcal{T}}$ is a cellular algebra in the sense of Graham–Lehrer [LM, 857]. Thus there is a map $\mathrm{Irr}(C) \to \mathsf{K}_0(KH_{C,\mathcal{T}})$ that sends each irreducible character of C to a corresponding cell module of $KH_{C,\mathcal{T}}$, called its *Specht module*. When $K = \mathbf{Q}_{cyc}$, and $\mathcal{T}$ is the specialization we consider in the corollary to follow, this construction will agree with that of Section 4.

For any charged e-partition $|\vec{\pi}, \vec{s}\rangle$, written out as $\vec{\pi} = (\pi^{(0)}, \ldots, \pi^{(e-1)})$ and $\vec{s} = (s^{(0)}, \ldots, s^{(e-1)})$, let $\mathsf{c}_{|\vec{\pi}, \vec{s}\rangle} : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ be defined by

$$\mathsf{c}_{|\vec{\pi}, \vec{s}\rangle}(k) = \left| \left\{ (j, \square) \;\middle|\; \begin{array}{l} 0 \leq j \leq e-1, \\ \square \in \pi^{(j)}, \\ k = e(y(\square) - x(\square) + s^{(j)}) + j \end{array} \right\} \right|.$$

That is, $\mathsf{c}_{|\vec{\pi},\vec{s}\rangle}$ describes the multiset of values $e(y(\Box) - x(\Box) + s^{(j)}) + j$ as we run over indices j and boxes $\Box \in \pi^{(j)}$. Let $\mathsf{c}^{\Phi_m}_{|\vec{\pi},\vec{s}\rangle} : \mathbf{Z}_m \to \mathbf{Z}_{\geq 0}$ be defined by

$$\mathsf{c}^{\Phi_m}_{|\vec{\pi},\vec{s}\rangle}(\bar{k}) = \sum_{\bar{k}=k+m\mathbf{Z}} \mathsf{c}_{|\vec{\pi},\vec{s}\rangle}(k). \tag{6.4}$$

Note that $\mathsf{c}_{|\vec{\pi},\vec{s}\rangle}(k) = 0$ for all k positive enough or negative enough, so $\mathsf{c}^{\Phi_m}_{|\vec{\pi},\vec{s}\rangle}$ is well-defined.

Corollary 6.7. *Let $\mathcal{S}_{\mathbb{GL}_n,\mathbb{L},\lambda}$ be the specialization defined by (6.2). Then in the notation of Section 4, two elements of $W_{\mathbb{GL}_n,\mathbb{L},\lambda}$ belong to the same $(\Phi_m, \mathcal{S}_{\mathbb{GL}_n,\mathbb{L},\lambda})$-block of $\mathrm{Irr}(W_{\mathbb{GL}_n,\mathbb{L},\lambda})$ if and only if they correspond to elements π, ρ such that*

$$\mathsf{c}^{\Phi_m}_{\Upsilon_e(|\pi, e+\ell_\lambda\rangle)} = \mathsf{c}^{\Phi_m}_{\Upsilon_e(|\rho, e+\ell_\lambda\rangle)}.$$

Proof. Apply Theorem 6.6, then use the commutativity of (6.1) and the definition of $\vec{a}_e$. $\square$

Corollary 6.8. *An analogue of Corollary 6.7 holds with $\mathbb{GU}_n$ and (6.3) in place of $\mathbb{GL}_n$ and (6.2).*

Proof. The minus signs by which (6.3) differs from (6.2) do not affect the deduction of Corollary 6.7 from Theorem 6.6. $\square$

6.7. **Proof of Theorem 3.** The proof will amount to manipulating generating functions. As preparation: For any function $\mathsf{f} : \mathbf{Z} \to \mathbf{Z}$ such that $\mathsf{f}(k) = 0$ for k positive enough, let $\mathsf{Z}(t, \mathsf{f}) \in \mathbf{Z}[\![t^{-1}]\!][t]$ be defined by

$$\mathsf{Z}(t, \mathsf{f}) = \sum_{k \in \mathbf{Z}} f(k) t^k.$$

For any bounded-above subset $\beta \subseteq \mathbf{Z}$, let $\mathsf{Z}(t, \beta) \in \mathbf{Z}[\![t^{-1}]\!][t]$ be defined by

$$\mathsf{Z}(t, \beta) = \mathsf{Z}(t, 1_\beta),$$

where 1_β is the indicator function on β.

Lemma 6.9. *For any charged partition $|\pi, s\rangle$, we have*

$$(1 - t^{-1})\, \mathsf{Z}(t, \mathsf{c}_{|\pi,s\rangle}) = \mathsf{Z}(t, \beta_{\pi,s}) - \mathsf{Z}(t, \mathbf{Z}_{<s}).$$

More generally, if $e > 0$ is an integer and λ is the e-core of π, then

$$(1 - t^{-e})\, \mathsf{Z}(t, \mathsf{c}_{\Upsilon_e(|\pi,s\rangle)}) = \mathsf{Z}(t, \beta_{\pi,s}) - \mathsf{Z}(t, \beta_{\lambda,s}).$$

Proof. By explicit calculation. $\square$

Proposition 6.10. *Fix integers $n, e > 0$ and partitions π, ρ of size n that have the same e-core. Then the following statements are equivalent for any integer $m > 0$ coprime to e:*

(1) *π, ρ have the same m-core.*
(2) *$\mathsf{c}^{\Phi_m}_{\Upsilon_e(|\pi,s\rangle)} = \mathsf{c}^{\Phi_m}_{\Upsilon_e(|\rho,s\rangle)}$ for any $s \in \mathbf{Z}$.*

Proof. From the abacus definition of the m-core of a partition, we see that

$$(1) \iff 1 - t^{-m} \text{ divides } \mathsf{Z}(t, \beta_{\pi,s}) - \mathsf{Z}(t, \beta_{\rho,s}).$$

Similarly, from (6.4),

$$(2) \iff 1 - t^{-m} \text{ divides } \mathsf{Z}(t, \mathsf{c}_{|\pi,s\rangle}) - \mathsf{Z}(t, \mathsf{c}_{|\rho,s\rangle}).$$

By Lemma 6.9, we can match the right-hand sides when m is coprime to e. □

By combining Proposition 6.10 with Corollary 6.7, *resp.* Corollary 6.8, we get Theorem 3 for the generic general linear groups, *resp.* unitary groups:

Corollary 6.11. *Let* $(\mathbb{L}, \lambda)$, resp. $(\mathbb{M}, \mu)$, *be a* Φ_e*-cuspidal pair,* resp. Φ_m*-cuspidal pair, for* $\mathbb{G} = \mathbb{GL}_n$*. Let* $\mathcal{S}_{\mathbb{GL}_n,\mathbb{L},\lambda}$ *and* $\mathcal{S}_{\mathbb{G},\mathbb{M},\mu}$ *be the specializations defined by* (6.2)*. If* e *and* m *are coprime, then* $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu}$ *is a single* $(\Phi_m, \mathcal{S}_{\mathbb{G},\mathbb{L},\lambda})$*-block and* $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda}$ *is a single* $(\Phi_e, \mathcal{S}_{\mathbb{G},\mathbb{M},\mu})$*-block. Thus Conjecture 4.1 holds for the pair* $(\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}, \mathcal{S}_{\mathbb{G},\mathbb{M},\mu})$*.*

Moreover, analogues of these statements hold with $\mathbb{GU}_n$ *and* (6.3) *in place of* $\mathbb{GL}_n$ *and* (6.2)*.*

7. Uglov's Bijections

7.1. The goal of this section is to prove Theorem 4, relating the bijections in Theorem 3 to those introduced by Uglov in [U]. We first introduce the maps Υ_m^e and $\tilde{w}_{e,m,s}$ needed for the statement. It will be convenient to write

$$\mathbf{Z}_{[0,e)} = \{0, 1, \ldots, e-1\}$$

throughout what follows.

7.2. The maps Υ_m^e will recover the maps Υ_m when $e = 1$. Just as we defined Υ_m in terms of $\upsilon_m : \mathsf{B} \xrightarrow{\sim} \mathsf{B}^m$, so we will define Υ_m^e in terms of

$$\upsilon_m^e : \mathsf{B}^e \xrightarrow{\sim} \mathsf{B}^m.$$

Recalling the bijection $(q_m, r_m) : \mathbf{Z} \xrightarrow{\sim} \mathbf{Z} \times \mathbf{Z}_{[0,m)}$ from Section 6.2, let

$$(q_m^e, r_m^e) : \mathbf{Z} \times \mathbf{Z}_{[0,e)} \xrightarrow{\sim} \mathbf{Z} \times \mathbf{Z}_{[0,m)}$$

be the bijection defined by

$$q_m^e(x, y) = \ell q_m(x) + y,$$
$$r_m^e(x, y) = r_m(x).$$

In particular, $(q_m^1, r_m^1) = (q_m, r_m)$.

Remark 7.1. For any e, m, we have $(q_e^m, r_e^m) \circ (q_m^e, r_m^e) = \mathrm{id}$. Indeed, this follows from observing that for all $(x, y) \in \mathbf{Z} \times \mathbf{Z}_{[0,e)}$, we have $q_e(eq_m(x) + y) = q_m(x)$ and $r_e(eq_m(x) + y) = y$.

Next, we set $\upsilon_m^e(\beta) = (\upsilon_m^{e,(0)}(\beta), \ldots, \upsilon_m^{e,(m-1)}(\beta))$, where

$$\upsilon_m^{e,(i)}(\beta) = \{q_m^e(x,y) \mid x \in \beta \text{ such that } r_m^e(x,y) = i\}.$$

Finally, we define Υ_m^e to be the composition

$$\Upsilon_m^e : \Pi^e \times \mathbf{Z}^e \xrightarrow{\beta} \mathsf{B}^e \xrightarrow{\upsilon_m^e} \mathsf{B}^m \xrightarrow{\beta^{-1}} \Pi^m \times \mathbf{Z}^m.$$

In particular, Υ_m^e is also bijective for any e and m.

Once we set $(e,m) = (l,n)$, the map Υ_m^e is essentially the map $(\lambda_l, s_l) \mapsto (\lambda_n, s_n)$ constructed by Uglov on pages 273–274 in [U, §4.1]. The main source of differences is that Uglov's version of β differs from ours by an overall shift by 1.

For another exposition of Uglov's bijection, see Appendix A in [G], where our map Υ_m^e is essentially the map that Gerber would denote by $\dot{\tau} \circ \tau^{-1}$.

7.3. To motivate the maps $\tilde{w}_{e,m,s}$, we first study the maps (q_m^e, r_m^e) more deeply.

Lemma 7.2. *Let $(a,b) \in \mathbf{Z} \times \mathbf{Z}_{[0,e)}$ and $(c,d) \in \mathbf{Z} \times \mathbf{Z}_{[0,m)}$.*

(1) *If $q_m^e(a,b) = c$ and $r_m^e(a,b) = d$, then $ea + mb = mc + ed$.*
(2) *If e and m are coprime, then the converse of (1) holds.*

Proof. If $q_m^e(a,b) = c$ and $r_m^e(a,b) = d$, then

$$mc + ed = m(eq_m(a) + b) + er_m(a) = e(mq_m(a) + r_m(a)) + mb = ea + mb.$$

Conversely, if $ea + mb = mc + ed$, then $ea \equiv ed \pmod m$. If e and m are coprime, then $a \equiv d \pmod m$, whence $r_m(a) = d$ and

$$q_m^e(a,b) = eq_m(a) + b = e(\tfrac{1}{m}(a-d)) + b = c,$$

as claimed. □

Henceforth, suppose that e and m are coprime. We will rewrite Lemma 7.2(2) in stages. First, it is equivalent to the statement that, for all $b \in \mathbf{Z}_{[0,e)}$ and $d \in \mathbf{Z}_{[0,m)}$ and $x \in \mathbf{Z}$ such that

$$\begin{cases} x \equiv mb \pmod e, \\ x \equiv ed \pmod m, \end{cases} \tag{7.1}$$

the map (q_m^e, r_m^e) sends

$$(\tfrac{1}{e}(x - mb), b) \mapsto (\tfrac{1}{m}(x - ed), d).$$

Fix $s, t \in \mathbf{Z}$. We observe that

$$\begin{aligned} \tfrac{1}{e}(x - mb) &= q_e(x+s) - q_e(mb+s), \\ \tfrac{1}{m}(x - ed) &= q_m(x+t) - q_m(ed+t), \\ r_e(mb+s) &= r_e(x+s), \\ r_m(ed+t) &= r_m(x+t). \end{aligned}$$

Let $w_{e,m,s} : \mathbf{Z}_{[0,e)} \xrightarrow{\sim} \mathbf{Z}_{[0,e)}$ be the permutation such that

$$w_{e,m,s}(r_e(mb+s)) = b \quad \text{for all } b \in \mathbf{Z}_{[0,e)}.$$

Let $\xi_{e,m,s} : \mathbf{Z} \times \mathbf{Z}_{[0,e)} \xrightarrow{\sim} \mathbf{Z} \times \mathbf{Z}_{[0,e)}$ be defined by

$$\xi_{e,m,s}(a, b) = (a - q_e(mb + s), b).$$

Then for all x satisfying (7.1), we have

$$\begin{aligned}\xi_{e,m,s}(q_e(x + s), w_{e,m,s}(r_e(x + s)) &= (\tfrac{1}{e}(x - mb), b),\\ \xi_{m,e,t}(q_m(x + t), w_{m,e,t}(r_m(x + t)) &= (\tfrac{1}{m}(x - ed), d).\end{aligned}$$

Finally, let $\tilde{w}_{e,m,s}$ be the composition

$$\tilde{w}_{e,m,s} : \mathbf{Z} \times \mathbf{Z}_{[0,e)} \xrightarrow{\mathrm{id}\times w_{e,m,s}} \mathbf{Z} \times \mathbf{Z}_{[0,e)} \xrightarrow{\xi_{e,m,s}} \mathbf{Z} \times \mathbf{Z}_{[0,e)}. \tag{7.2}$$

It may be regarded as an element of $\mathbf{Z}^e \rtimes S_e$: that is, an affine permutation of the cocharacter lattice of $\mathbb{GL}_e$. Our work has shown:

Proposition 7.3. *For any coprime integers $e, m > 0$ and arbitrary integers s, t, the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{Z} \times \mathbf{Z}_{[0,e)} \arrow[d, "\tilde{w}_{e,m,s}"'] & \mathbf{Z} \arrow[l, "(q_e(x+s), r_e(x+s)) \mapsfrom x"'] \arrow[r, "x \mapsto (q_m(x+t), r_m(x+t))"] & \mathbf{Z} \times \mathbf{Z}_{[0,m)} \arrow[d, "\tilde{w}_{m,e,t}"] \\
\mathbf{Z} \times \mathbf{Z}_{[0,e)} \arrow[rr, "(q_m^e, r_m^e)"] & & \mathbf{Z} \times \mathbf{Z}_{[0,m)}
\end{tikzcd}$$

We again write $\tilde{w}_{e,m,s}$ and $\tilde{w}_{m,e,t}$ for the self-maps of B^e and B^m that these affine permutations respectively induce. Explicitly, if $\vec{\beta} = (\beta^{(0}, \ldots, \beta^{(e-1)}) \in \mathsf{B}^e$, then $\tilde{w}_{e,m,s}(\vec{\beta}) = (\tilde{w}^{(0)}_{e,m,s}(\vec{\beta}), \ldots, \tilde{w}^{(e-1)}_{e,m,s}(\vec{\beta}))$, where

$$\tilde{w}^{(j)}_{e,m,s}(\vec{\beta}) = \{y \in \mathbf{Z} \mid (y, j) = \tilde{w}_{e,m,s}(x, i) \text{ for some } x \in \beta^{(i)}\}.$$

We do the same for the corresponding self-maps of $\Pi^e \times \mathbf{Z}^e$ and $\Pi^m \times \mathbf{Z}^m$. With this notation, we arrive at:

Corollary 7.4. *For any coprime integers $e, m > 0$ and arbitrary integers s, t, the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{B}^e \arrow[d, "\tilde{w}_{e,m,s}"'] & \mathsf{B} \arrow[l, "\upsilon_e(\beta+s) \mapsfrom \beta"'] \arrow[r, "\beta \mapsto \upsilon_m(\beta+t)"] & \mathsf{B}^m \arrow[d, "\tilde{w}_{m,e,t}"] \\
\mathsf{B}^e \arrow[rr, "\upsilon_m^e"] & & \mathsf{B}^m
\end{tikzcd}$$

Hence the following diagram commutes:

$$\begin{tikzcd}
\Pi^e \times \mathbf{Z}^e \arrow[d, "\tilde{w}_{e,m,s}"'] & \Pi \arrow[l, "\Upsilon_e(|\pi, s\rangle) \mapsfrom \pi"'] \arrow[r, "\pi \mapsto \Upsilon_m(|\pi, t\rangle)"] & \Pi^m \times \mathbf{Z}^m \arrow[d, "\tilde{w}_{m,e,t}"] \\
\Pi^e \times \mathbf{Z}^e \arrow[rr, "\Upsilon_m^e"] & & \Pi^m \times \mathbf{Z}^m
\end{tikzcd}$$

7.4. **Proof of Theorem 4.** The theorem asserts that if $\mathbb{G} = \mathbb{GL}_n$ and e, m are coprime, then we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda}) \arrow[d] & \mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu) \arrow[l, "\chi^{\mathbb{G}}_{\mathbb{L},\lambda}"'] \arrow[r, "\chi^{\mathbb{G}}_{\mathbb{M},\mu}"] \arrow[d] & \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu}) \arrow[d] \\
\Pi^e \times \mathbf{Z}^e \arrow[d, "\tilde{w}_{e,m,e+\ell_\lambda}"'] & \Pi \arrow[l, "\Upsilon^1_e(|\rho, e+\ell_\lambda\rangle) \mapsfrom \rho"'] \arrow[r, "\rho \mapsto \Upsilon^1_m(|\rho, m+\ell_\mu\rangle)"] & \Pi^m \times \mathbf{Z}^m \arrow[d, "\tilde{w}_{m,e,m+\ell_\mu}"] \\
\Pi^e \times \mathbf{Z}^e \arrow[rr, "\Upsilon^e_m"] & & \Pi^m \times \mathbf{Z}^m
\end{tikzcd}$$

That the bottom rectangle commutes is Corollary 7.4. The top left and right squares commute, and are compatible with the commutative squares of Theorem 2.4(2d), due to their definition in (6.1) and the comment that follows it.

7.5. **Proof of Corollary 5.** Let $\mathcal{S}_{\mathbb{GL}_n,\mathbb{L},\lambda}$ and $\mathcal{S}_{\mathbb{G},\mathbb{M},\mu}$ be defined by (6.2). Let b be the unique block of $\mathsf{Rep}_{\mathbf{Q}_{cyc}}(H_{\mathbb{G},\mathbb{L},\lambda}(\zeta_m))$ determined by $(\mathbb{M}, \mu)$, and let c be the unique block of $\mathsf{Rep}_{\mathbf{Q}_{cyc}}(H_{\mathbb{G},\mathbb{M},\mu}(\zeta_e))$ determined by $(\mathbb{L}, \lambda)$. As in Conjecture 4.3, suppose that $\vec{\nu}_e$, *resp.* $\vec{\nu}_m$, is an $\mathcal{S}_{\mathbb{GL}_n,\mathbb{L},\lambda}$-charge, *resp.* $\mathcal{S}_{\mathbb{G},\mathbb{M},\mu}$-charge. We want to show that $\chi^{\mathsf{b}}_{\mathsf{c}}$ categorifies to an equivalence

$$\mathsf{D}^b(\mathsf{O}^{rat}_{W_{\mathbb{G},\mathbb{L},\lambda}}(\vec{\nu}_e)_{\mathsf{b}}) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{O}^{rat}_{W_{\mathbb{G},\mathbb{M},\mu}}(\vec{\nu}_m)_{\mathsf{c}}).$$

We first generalize some constructions from Section 6 in order to state the Chuang–Miyachi equivalence.

In the notation of Section 4, take $C = \mathbf{Z}_e \wr S_a$, and for any $\vec{s} = (s^{(0)}, \dots, s^{(e-1)}) \in \mathbf{Z}^e$, let $\mathcal{T}_{e,m,\vec{s}} : \mathbf{Z}[\vec{u}^{\pm 1}] \to \mathbf{C}$ be the ring homomorphism given by

$$\begin{cases} \mathcal{T}_{e,m,\vec{s}}(u_{\tau,j}) = e^{2\pi i (e s^{(j)} + j)/m} & \text{for all } j, \\ \mathcal{T}_{e,m,\vec{s}}(u_{\sigma,0}) = 1, \\ \mathcal{T}_{e,m,\vec{s}}(u_{\sigma,1}) = -e^{2\pi i e/m}. \end{cases}$$

We say that $\vec{\nu} = (\nu_t, \dots, \nu_{t^{e-1}}, \nu_{s_1}) \in \mathbf{C}^{e-1} \times \mathbf{C}$ is a *$\mathcal{T}_{e,m,\vec{s}}$-charge* iff

$$\frac{2}{e} \sum_{1 \le k < e} \nu_{t^k} \left(\frac{1 - e^{2\pi i jk/e}}{1 - e^{-2\pi i k/e}} \right) \in \frac{e s^{(j)}}{m} + \mathbf{Z},$$

$$\nu_{s_1} \in \frac{e}{m} + \mathbf{Z}.$$

Such a vector $\vec{\nu}$ defines, for all $a > 0$, a vector $\vec{\nu}(a) \in \mathbf{C}^{\mathsf{Refl}(\mathbf{Z}_e \wr S_a)}$ once we impose conjugation invariance under $\mathbf{Z}_e \wr S_a$. For any $\mathcal{T}_{e,m,\vec{s}}$-charge $\vec{\nu}$, we set

$$\mathsf{O}^{rat}_{\vec{s},\vec{\nu}} = \bigoplus_{a \ge 0} \mathsf{O}^{rat}_{\mathbf{Z}_e \wr S_a}(\vec{\nu}(a)).$$

By construction, $\mathsf{O}^{rat}_{\mathbf{Z}_e \wr S_a}(\vec{\nu}(a))$ is a highest-weight cover of $\mathsf{Rep}_{\mathbf{C}}(H_{\mathbf{Z}_e \wr S_a, m, \vec{s}})$, where $H_{\mathbf{Z}_e \wr S_a, m, \vec{s}}$ is the base change of the Hecke algebra $H_{\mathbf{Z}_e \wr S_a}(\vec{u})$ along $\mathcal{T}_{e,m,\vec{s}}$.

Recall that the standard objects of the category $\mathsf{O}^{rat}_{\mathbf{Z}_e \wr S_a}(\vec{\nu}(a))$ are indexed by e-partitions $\vec{\pi}$ such that $\sum_i |\pi_i| = a$. Thus, the Grothendieck group of $\mathsf{O}^{rat}_{\vec{s},\vec{\nu}}$ admits a basis of standard objects indexed by e-partitions of arbitrary size.

The following result is a special case of [SVV, Thm. 3.12], stated in a form closest to that of [ChuM, Conj. 6, Rem. 7]. The key ingredient in [SVV] is the theorem proved independently as [L, Thm. 7.7], [RSVV, Thm. 7.4], and [W, Thm. B(5)].

Theorem 7.5 (Categorical Level-Rank Duality). *Fix coprime integers $e, m > 0$. Let $\vec{s} \in \mathbf{Z}^e$ and $\vec{r} \in \mathbf{Z}^m$ such that $\sum_i s_i = \sum_j r_j$, and let*

$$\Pi^e_{\vec{s}} = \{|\vec{\varpi}, \vec{s}\rangle \mid \vec{\varpi} \in \Pi^e\} \quad \textit{and} \quad \Pi^m_{\vec{r}} = \{|\vec{\varrho}, \vec{r}\rangle \mid \vec{\varrho} \in \Pi^m\}.$$

Let $\vec{\nu}_e$, resp. *$\vec{\nu}_m$, be an $\mathcal{T}_{e,m,\vec{s}}$-charge,* resp. *$\mathcal{T}_{m,e,\vec{r}}$-charge. Then:*

(1) *$\Pi^e_{\vec{s}} \cap \Upsilon^m_e(\Pi^m_{\vec{r}})$ indexes a set of the form $\coprod_a \mathrm{Irr}(\mathbf{Z}_e \wr S_a)_{\mathsf{b}_a}$, where b_a is a block of $\mathsf{Rep}_{\mathbf{C}}(H_{\mathbf{Z}_e \wr S_a, m, \vec{s}})$.*

(2) *$\Pi^m_{\vec{r}} \cap \Upsilon^e_m(\Pi^e_{\vec{s}})$ indexes a set of the form $\coprod_d \mathrm{Irr}(\mathbf{Z}_m \wr S_d)_{\mathsf{c}_d}$, where c_d is a block of $\mathsf{Rep}_{\mathbf{C}}(H_{\mathbf{Z}_m \wr S_d, e, \vec{r}})$.*

(3) *The bijection $\Pi^e_{\vec{s}} \cap \Upsilon^m_e(\Pi^m_{\vec{r}}) \xrightarrow{\sim} \Pi^m_{\vec{r}} \cap \Upsilon^e_m(\Pi^e_{\vec{s}})$ induced by Υ^e_m preserves blocks. If it restricts to a bijection*

$$\mathrm{Irr}(\mathbf{Z}_e \wr S_a)_{\mathsf{b}_a} \xrightarrow{\sim} \mathrm{Irr}(\mathbf{Z}_m \wr S_d)_{\mathsf{c}_d}$$

for some $\mathsf{b}_a, \mathsf{c}_d$, then there is a Koszul duality equivalence

$$\mathsf{U}^e_m : \mathsf{D}^b(\mathsf{O}^{rat,gr}_{\mathbf{Z}_e \wr S_a}(\vec{\nu}_e(a))_{\mathsf{b}_a}) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{O}^{rat,gr}_{\mathbf{Z}_m \wr S_d}(\vec{\nu}_m(d))_{\mathsf{c}_d}),$$

where $\mathsf{O}^{rat,gr}_{\mathbf{Z}_e \wr S_a}(\vec{\nu}_e(a))_{\mathsf{b}_a}$ and $\mathsf{O}^{rat,gr}_{\mathbf{Z}_m \wr S_a}(\vec{\nu}_m(a))_{\mathsf{c}_d}$ are respectively graded lifts of the categories $\mathsf{O}^{rat}_{\mathbf{Z}_e \wr S_a}(\vec{\nu}_e(a))_{\mathsf{b}_a}$ and $\mathsf{O}^{rat}_{\mathbf{Z}_m \wr S_d}(\vec{\nu}_m(d))_{\mathsf{c}_d}$.

Now we finish the proof of Corollary 5. Write $W_{\mathbb{G},\mathbb{L},\lambda} = \mathbf{Z}_e \wr S_a$ and $W_{\mathbb{G},\mathbb{M},\mu} = \mathbf{Z}_m \wr S_d$. It is enough to construct equivalences

$$\mathsf{D}^b(\mathsf{O}^{rat}_{W_{\mathbb{G},\mathbb{L},\lambda}}(\vec{\nu}_e)_{\mathsf{b}}) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{O}^{rat}_{W_{\mathbb{G},\mathbb{L},\lambda}}(\vec{\nu}^\dagger_e(a))_{\mathsf{b}^\dagger_a})$$
$$\mathsf{D}^b(\mathsf{O}^{rat}_{W_{\mathbb{G},\mathbb{M},\mu}}(\vec{\nu}_m)_{\mathsf{c}}) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{O}^{rat}_{W_{\mathbb{G},\mathbb{M},\mu}}(\vec{\nu}^\dagger_m(d))_{\mathsf{c}^\dagger_d})$$

such that $\mathsf{b}^\dagger_a$, *resp.* $\mathsf{c}^\dagger_d$, is a block of $\mathsf{Rep}_{\mathbf{C}}(H_{\mathbf{Z}_e \wr S_a, m, \vec{s}^\dagger})$ for some $\vec{s} \in \mathbf{Z}^e$, *resp.* a block of $\mathsf{Rep}_{\mathbf{C}}(H_{\mathbf{Z}_m \wr S_d, e, \vec{r}^\dagger})$ for some $\vec{r}^\dagger \in \mathbf{Z}^m$, and $\mathsf{b}^\dagger_a, \mathsf{c}^\dagger_d$ correspond to each other in Theorem 7.5; and $\vec{\nu}^\dagger_e$, *resp.* $\vec{\nu}^\dagger_m$, is a $\mathcal{T}_{e,m,\vec{s}^\dagger}$-charge, *resp.* $\mathcal{T}_{m,e,\vec{r}^\dagger}$-charge.

By [W, Thm. B(3)], the action of affine permutations such as $\tilde{w}_{m,e,m+\ell_\mu}$ on the sets $\Pi^e \times \mathbf{Z}^e$ categorify to strong categorical actions of the corresponding affine braid groups, in which braids define functors of the form

$$\mathsf{O}^{rat}_{\vec{s},\vec{\nu}} \xrightarrow{\sim} \mathsf{O}^{rat}_{\vec{s}^\dagger,\vec{\nu}^\dagger}.$$

These functors necessarily preserve blocks, so it remains to show that the input parameters $\vec{\nu}_e, \mathsf{b}, \vec{\nu}_m, \mathsf{c}$ give rise to output parameters $\vec{\nu}^\dagger_e, \mathsf{b}^\dagger_a, \vec{\nu}^\dagger_m, \mathsf{c}^\dagger_d$ such that $\mathsf{b}^\dagger_a$ and $\mathsf{c}^\dagger_d$ match under Υ^e_m. This is what the commutativity of the diagram in Theorem 4 provides.

Remark 7.6. Above, $\vec{s}^\dagger$ and $\vec{\nu}^\dagger_e$ can be related more explicitly to $\vec{\nu}_e$. From the discussion in Section 6.2, we see that the charged e-partitions in the image of

$$\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda}) \to \Pi^e \times \mathbf{Z}^e$$

all have a common e-charge, say $\vec{s} \in \mathbf{Z}^e$. Next, from the definition of $\tilde{w}_{e,m,s}$ in (7.2), we see that the action of $\tilde{w}_{e,m,e+\ell_\lambda}$ on charged e-partitions descends to an action on e-charges. Therefore, we can write $\vec{s}^\dagger = \tilde{w}_{e,m,e+\ell_\lambda}(\vec{s})$. Finally, we can check that $\vec{\nu}_e$ and $\vec{s}^\dagger$ together determine a $\mathcal{T}_{e,m,\vec{s}^\dagger}$-charge $\vec{\nu}_e^\dagger$.

8. The Exceptional Groups

8.1. To conclude the paper, we explain how to check the numerical predictions of Conjecture 1 for the Hecke specializations that were either determined in [L-Cox, L78] or conjectured in [BM93] for generic simple groups of exceptional type. As in the rest of the paper, we will exclude the Suzuki type ${}^2C_2$ and the Ree types ${}^2G_2, {}^2F_4$.

8.2. **Φ-Harish-Chandra Series.** By Corollary 2.7, it suffices to study the Φ_e-Harish-Chandra series in the cases where e is a singular number for $\mathbb{G}$.

Below we list the singular numbers $e > 1$ for each exceptional type, which can be deduced from the table in [C, §2.9]. For the term *twisted Coxeter number*, see Section 2.8. It is the usual Coxeter number except for the twisted types ${}^3D_4$ and ${}^2E_6$, where it is given by [S74, Table 10].

type	nontrivial singular numbers	twisted Coxeter number
${}^3D_4$	$2, 3, 6$	12
G_2	$2, 3$	6
F_4	$2, 3, 4, 6, 8$	12
E_6	$2, 3, 4, 5, 6, 8, 9$	12
${}^2E_6$	$2, 3, 4, 6, 8, 10, 12$	18
E_7	$2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14$	18
E_8	$2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24$	30

For maximal tori and Φ_1-split cuspidal pairs, the associated Harish-Chandra series and complex reflection groups were determined in [L-Cox, L78]. Note that for Φ_2-split maximal tori, the relative Weyl groups can also be determined via Ennola duality, and for Φ_e-split maximal tori of the generic simple exceptional groups with $e > 2$, they are given by [BMM93, Table 3]. For all other Φ-cuspidal pairs of these groups, they can be determined from combining [BMM93, Table 1], [BM93, Table 8.1], and Ennola duality (Section 6.5).

Observe that the tables of [BMM93] use Shephard–Todd notation to label the complex reflection groups. We will also use it in what follows.

8.3. **Specializations.** As stated in Section 3, the putative specializations $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ of Conjecture 3.3 were determined for Φ_1-cuspidal pairs in [L78, Table II], for Coxeter maximal tori in [L-Cox, (7.3)], and for maximal tori of type $[w_0 f]$ in [DMR, Th. 5.4.1]. For almost all other Φ_e-cuspidal pairs, a formula for $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ compatible with generic-degree data was conjectured by Broué–Malle in [BM93] or by Malle in [M], either explicitly, in the form of tables, or implicitly, via Ennola duality and/or reduction to parabolic subalgebras.

The data in [BM93, M] is organized according to the isomorphism class of the group $W_{\mathbb{G},\mathbb{L},\lambda}$, rather than the tuple $(\mathbb{G}, \mathbb{L}, \lambda)$. In addition, these references use a different system of notation to label the complex reflection groups. Thus, in the tables at the end of this section, we have reorganized the data for the convenience of the reader. For each $(\mathbb{G}, e)$, we list the set $\mathsf{HC}_e(\mathbb{G})$ of Φ_e-cuspidal pairs for $\mathbb{G}$ up to conjugation by $W_{\Gamma_{\mathbb{G}}}$, omitting those of the form $(\mathbb{G}, \rho)$ for some $\rho \in \mathrm{Uch}(\mathbb{G})$, since Conjecture 4.1 is trivial for these pairs. For each pair, we state the isomorphism class of the group $W_{\mathbb{G},\mathbb{L},\lambda}$ in Shephard–Todd notation, the parameters of the specialization $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$, and the reference we used for the latter, possibly in conjunction with Ennola duality. The references are abbreviated as follows:

- "E" refers to Ennola.
- [L-Fin] refers to (7.3) of *ibid.*
- [L78] refers to Table II of *ibid.* Lusztig uses Iwahori's sign conventions, *cf.* Example 3.5; in our tables, we have flipped the signs.
- "BM, _" refers to [BM93, _].
- In type E_6, "[L78] + E" means that we use Ennola to derive the prediction for $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ from the corresponding entry for ${}^2E_6$ in [L78, Table II].
- In type E_8, "BM/M" means that the given entry or its Ennola dual can be derived from [BM93, 180] and [M, Table 9].

We have fixed an apparent typo in [BM93, 180] for $\mathbb{G}$ of type E_7 and $e = 3, 6$.

The only case where we have been unable to derive a conjecture for $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ from the literature is when $\mathbb{G}$ is of type E_7 and $e = 4$ and $W_{\mathbb{G},\mathbb{L},\lambda} = G_{4,1,2}$.

8.4. **Blocks.** Suppose that $C = W_{\mathbb{G},\mathbb{L},\lambda}$, and that $H_{\mathbb{G},\mathbb{L},\lambda}(x)$ is the base change of $H_C(\vec{u})$ along the specialization $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ described above. Then, in the notation of Section 4, the blocks of $\mathsf{Rep}_{\mathbf{Q}_{cyc}}(H_{\mathbb{G},\mathbb{L},\lambda}(\zeta_m))$ can be determined from:

- [LM, Thm. 2.11] when $C = G_{e,1,a} := \mathbf{Z}_e \wr S_a$, as in Section 6.
- [GP, Appendix F] and [GJ, §7.2] when C is an irreducible Weyl group of split exceptional type.
- [CM, §3] when $C = G_4, G_5, G_8, G_9, G_{10}, G_{16}$. See also Section 1.3 of *ibid.*

8.5. **What We Can Verify.** For each generic simple exceptional group $\mathbb{G}$, and certain pairs of singular numbers (e, m), the information in Sections 8.2–8.4 makes it possible to compare the following numbers as we run over $[\mathbb{L}, \lambda] \in \mathsf{HC}_e(\mathbb{G})$ and $[\mathbb{M}, \mu] \in \mathsf{HC}_m(\mathbb{G})$:

(1) The sizes of the sets $\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda, \mathbb{M}, \mu) = \mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda) \cap \mathrm{Uch}(\mathbb{G}, \mathbb{M}, \mu)$.
(2) The sizes of the $(\Phi_m, \mathcal{S}_{\mathbb{G},\mathbb{L},\lambda})$-blocks of $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})$, where $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ is the Φ_e-specialization described in Section 8.3.
(3) The analogue of (2) with $e, m, (\mathbb{M}, \mu)$ in place of $m, e, (\mathbb{L}, \lambda)$.

We have verified directly that:

Proposition 8.1. *In each case where the data of Sections 8.3–8.4 are sufficient, (1)–(3) are consistent with the existence of the partitions and bijections predicted by Conjecture 4.1.*

In the cases where the references listed in Section 8.4 give the decomposition numbers of the blocks of the Hecke algebras, we can even verify Conjecture 4.3. We will return to this point in a sequel.

Below, we provide the details of the complete verification for types G_2 and F_4. For each pair (e, m), we list the possibilities for $[\mathbb{L}, \lambda] \in \mathsf{HC}_e(\mathbb{G})$ and $[\mathbb{M}, \mu] \in \mathsf{HC}_m(\mathbb{G})$. We omit all cases where $\mathbb{L} = \mathbb{G}$ or $\mathbb{M} = \mathbb{G}$, since the associated Φ-Harish-Chandra series would be singletons, rendering Conjecture 4.1 trivial. For each choice of $[\mathbb{L}, \lambda]$ and $[\mathbb{M}, \mu]$, we state a partition of $|\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda, \mathbb{M}, \mu)|$ that simultaneously gives the sizes of the $(\Phi_m, \mathcal{S}_{\mathbb{G},\mathbb{L},\lambda})$-blocks occuring in $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu}$ and those of the $(\Phi_e, \mathcal{S}_{\mathbb{G},\mathbb{M},\mu})$-blocks occuring in $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda}$.

For the other generic simple exceptional groups, we have prepared similar lists and tables for many pairs of singular numbers. They are available upon request.

8.6. **Type G_2.** Here, all Φ-cuspidal pairs with $\mathbb{L} \neq \mathbb{G}$ arise from taking $\mathbb{L}$ to be a maximal torus $\mathbb{T}$. Below, we just state the order $|\mathbb{T}|$.

e	$\mathbb{T}$	$W_{\mathbb{G},\mathbb{T}}$	$\mathcal{S}_{\mathbb{G},\mathbb{T},1}(u_{\mathcal{C},j})$	reference
1	Φ_1^2	W_{G_2}	$1, -x; 1, -x$	Ex. 3.5
2	Φ_2^2	W_{G_2}	$1, x; 1, x$	E
3	Φ_3	$\mathbf{Z}_6$	$1, \pm x, x^2, -\zeta_3 x, -\zeta_3^2 x$	E
6	Φ_6	$\mathbf{Z}_6$	$1, \pm x, x^2, \zeta_3 x, \zeta_3^2 x$	[L-Cox]

8.6.1. *Verifying Conjecture 4.1 for G_2.* For each pair of distinct singular numbers (e, m), the preceding table shows there is a unique choice of $[\mathbb{L}, \lambda] \in \mathsf{HC}_e(\mathbb{G})$ and $[\mathbb{M}, \mu] \in \mathsf{HC}_m(\mathbb{G})$ such that $\mathbb{L}, \mathbb{M} \neq \mathbb{G}$.

- If $(e, m) = (1, 2)$, then $|\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda, \mathbb{M}, \mu)| = 4$ and the partition is trivial: $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu}$ is a single $(\Phi_m, \mathcal{S}_{\mathbb{G},\mathbb{L},\lambda})$-block and $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda}$ is a single $(\Phi_e, \mathcal{S}_{\mathbb{G},\mathbb{M},\mu})$-block.
- If $(e, m) = (1, 3), (1, 6), (2, 3), (2, 6)$, then $|\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda, \mathbb{M}, \mu)| = 3$ and the partition is trivial.
- If $(e, m) = (3, 6)$, then the partition of $|\mathrm{Uch}(\mathbb{G}, \mathbb{L}, \lambda, \mathbb{M}, \mu)|$ is $2 + 2$.

8.7. **Type F_4.** We omit λ when $\mathbb{L}$ is a torus, since $\lambda = 1$, and in other cases where disambiguation is not needed. The notation $\Phi_e^a.B_2$ means we extend a generic almost-simple group of type B_2 by a torus of order Φ_e^a. The notations $\phi_{1^2,-}$ and $\phi_{-,2}$ are based on [C, §13.8].

e	$\mathbb{L}$	λ	$W_{\mathbb{G},\mathbb{L},\lambda}$	$\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}(u_{\mathcal{C},j})$	reference
1	Φ_1^4		W_{F_4}	$1, -x; 1, -x$	Ex. 3.5
1	$\Phi_1^2.B_2$		W_{BC_2}	$1, -x^3; 1, -x^3$	[L78]
2	Φ_2^4		W_{F_4}	$1, x; 1, x$	E
2	$\Phi_2^2.B_2$		W_{BC_2}	$1, x^3; 1, x^3$	E
3	Φ_3^2		G_5	$1, x, x^2$	BM, 5.9
4	Φ_4^2		G_8	$1, \pm x, x^2$	BM, 5.12
4	$\Phi_4.B_2$	$\phi_{1^2,-}$	$\mathbf{Z}_4$	$1, \pm x^3, x^6$	BM, 8.1
4	$\Phi_4.B_2$	$\phi_{-,2}$	$\mathbf{Z}_4$	$1, \pm x^3, x^6$	BM, 8.1
6	Φ_6^2		G_5	$1, -x, x^2$	BM, 5.9
8	Φ_8		$\mathbf{Z}_8$	$1, x^2, \pm x^3, \pm \zeta_4 x^3, x^4, x^6$	BM, 8.1
12	Φ_{12}		$\mathbf{Z}_{12}$	$1, \pm x, \pm x^2, \pm x^3, x^4, \pm \zeta_4 x^2, \zeta_3 x^2, \zeta_3^2 x^2$	[L-Cox]

8.7.1. *Verifying Conjecture 4.1 for F_4.* First, we handle the cases where there is a unique choice of $[\mathbb{L},\lambda] \in \mathsf{HC}_e(\mathbb{G})$ and $[\mathbb{M},\mu] \in \mathsf{HC}_m(\mathbb{G})$ such that $\mathbb{L},\mathbb{M} \neq \mathbb{G}$.

- If $(e,m) = (1,3)$, then the partition of $|\mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu)|$ is $9+9$.
- If $(e,m) = (1,8)$, then $|\mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu)| = 5$ and the partition is trivial.
- If $(e,m) = (3,6)$, then the partition of $|\mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu)|$ is $6+6$.
- If $(e,m) = (3,8)$, then $|\mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu)| = 3$ and the partition is trivial.
- If $(e,m) = (3,12),(4,8)$, then the partition of $|\mathrm{Uch}(\mathbb{G},\mathbb{L},\lambda,\mathbb{M},\mu)|$ is $3+3$.
- If $(e,m) = (4,12)$, then the partition is $3+3+3$.
- If $(e,m) = (8,12)$, then the partition is $2+2$.

For $(e,m) = (1,2)$, we get the following table with $[\mathbb{L},\lambda]$ along the rows and $[\mathbb{M},\mu]$ along the columns:

	$(\Phi_2^4,1)$	$(\Phi_2^2 B_2,-)$
$(\Phi_1^4,1)$	18	4
$(\Phi_1^2 B_2,-)$	4	1

(That is, all four partitions are trivial.) For $(e,m) = (1,4),(1,6),(1,12)$, we get, in order, the following tables:

	$(\Phi_2^4,1)$	$(\Phi_4 B_2,\phi_{1^2,-})$	$(\Phi_4 B_2,\phi_{-,2})$
$(\Phi_1^4,1)$	9	3	3
$(\Phi_1^2 B_2,-)$	3	1	1

	$(\Phi_6^2,1)$
$(\Phi_1^4,1)$	13
$(\Phi_1^2 B_2,-)$	4

	$(\Phi_{12},1)$
$(\Phi_1^4,1)$	5
$(\Phi_1^2 B_2,-)$	3

We can check that the cases where $e=2$ have the same sizes and partitions as the Ennola-dual cases where $e=1$. Finally, for $(e,m)=(3,4)$, we get the table:

	$(\Phi_2^4,1)$	$(\Phi_4 B_2,\phi_{1^2,-})$	$(\Phi_4 B_2,\phi_{-,2})$
$(\Phi_3^2,1)$	6	3	3

8.8. **The Other Exceptional Types.**

8.8.1. *${}^3D_4$.* As in type G_2, all Φ-cuspidal pairs with $\mathbb{L} \neq \mathbb{G}$ arise from taking $\mathbb{L}$ to be a maximal torus $\mathbb{T}$.

e	$\mathbb{T}$	$W_{\mathbb{G},\mathbb{T}}$	$\mathcal{S}_{\mathbb{G},\mathbb{T},1}(u_{\mathcal{C},j})$	reference
1	$\Phi_1^2\Phi_3$	W_{G_2}	$1,-x;1,-x^3$	[L78]
2	$\Phi_2^2\Phi_6$	W_{G_2}	$1,x;1,x^3$	E
3	Φ_3^2	G_4	$1,x,x^2$	BM, §5.6
6	Φ_6^2	G_4	$1,-x,x^2$	E
12	Φ_{12}	$\mathbf{Z}_4$	$1,\pm x^3,x^6$	[L-Cox]

8.8.2. *E_6.* The notation is similar to that of Section 8.7. The notations $\phi_{2^2},\phi_2,\phi_{1^2}$ are based on [C, §13.8], while the notation ${}^3D_4[-1]$ is based on [C, §13.9].

e	$\mathbb{L}$	λ	$W_{\mathbb{G},\mathbb{L},\lambda}$	$\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}(u_{\mathcal{C},j})$	reference
1	Φ_1^6		W_{E_6}	$1, -x$	Ex. 3.5
1	D_4		W_{A_2}	$1, -x^4$	[L78]
2	$\Phi_1^2\Phi_2^4$		W_{F_4}	$1, x; 1, -x^2$	[L78] + E
2	$\Phi_2.A_5$		$\mathbf{Z}_2$	$1, x^9$	[L78] + E
3	Φ_3^3		G_{25}	$1, x, x^2$	BM, §5.16
3	$\Phi_3.{}^3D_4$	${}^3D_4[-1]$	$\mathbf{Z}_3$	$1, x^4, x^8$	BM, §8.1
4	$\Phi_4^2\Phi_1^2$		G_8	$1, -x, x^2, -x^3$	BM, §5.12
4	$\Phi_1\Phi_4.{}^2A_3$	ϕ_{2^2}	$\mathbf{Z}_4$	$1, x^3, x^6, x^9$	BM, §8.1
5	$\Phi_1\Phi_5.A_1$	ϕ_2	$\mathbf{Z}_5$	$1, x^3, x^4, x^6, x^{12}$	BM, §8.1
5	$\Phi_1\Phi_5.A_1$	ϕ_{1^2}	$\mathbf{Z}_5$	$1, x^6, x^8, x^9, x^{12}$	BM, §8.1
6	$\Phi_6^2\Phi_3$		G_5	$1, -x, x^2; 1, x^2, x^4$	BM, §5.9
8	$\Phi_1\Phi_2\Phi_8$		$\mathbf{Z}_8$	$1, \pm x^3, x^4, x^5, \pm x^6, x^9$	BM, §8.1
9	Φ_9		$\mathbf{Z}_9$	$1, x^2, x^3, x^4, \zeta_3 x^4, \zeta_3^2 x^4, x^5, x^6, x^8$	BM, §8.1
12	$\Phi_3\Phi_{12}$		$\mathbf{Z}_{12}$	$1, x, \pm x^2, \pm x^3, \pm x^4, x^5, x^6, \zeta_3 x^3, \zeta_3^2 x^3$	[L-Cox]

8.8.3. ${}^2E_6$. By Ennola duality, this table can be derived from the table for E_6 by substituting $-x$ for x everywhere.

8.8.4. E_7. The notation is similar to that of Sections 8.7 and 8.8.2, but for further concision, we write the Ennola-dual cases in the same rows, omitting some of their data. The question marks "**?**" indicate the cases where we could not derive a prediction for $\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}$ from [BM93, M].

8.8.5. E_8. The notation is similar to that of Section 8.8.4.

E_7.

e	$\mathbb{L}$	λ	$W_{\mathbb{G},\mathbb{L},\lambda}$	$\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}(u_{\mathcal{C},j})$	reference	e	$\mathbb{L}$	λ
1	Φ_1^7		W_{E_7}	$1, -x$	Ex. 3.5	2	Φ_2^7	
1	D_4		W_{BC_3}	$1, -x^4; 1, -x$	[L78]	2	$\Phi_2^3.D_4$	
1	E_6		$\mathbf{Z}_2$	$1, -x^9$	[L78]	2	$\Phi_2.E_6$	
4	$\Phi_4^2.A_1^3$	ϕ_2^3	G_8	$1, \pm x, -x^4$	BM, §5.12			
4	$\Phi_4^2.A_1^3$	$\phi_{1^2}^3$	G_8	$1, \pm x^3, -x^4$	BM, §5.12			
4	$\Phi_4^2.A_1^3$	$\phi_2^2\phi_{1^2}$	$G_{4,1,2}$	**?**				
4	$\Phi_4^2.A_1^3$	$\phi_2\phi_{1^2}^2$	$G_{4,1,2}$	**?**				
6	$\Phi_6^3\Phi_2$		G_{26}	$1, -x, x^2; 1, x^3$	BM, 180	3	$\Phi_3^3\Phi_1$	
6	$\Phi_2\Phi_6.{}^3D_4$	$\phi_{2^11^1}$	$\mathbf{Z}_6$	$1, x, x^4, x^5, x^8, x^9$	BM, §8.1	3	$\Phi_1\Phi_3.{}^3D_4$	${}^3D_4[-1]$
6	$\Phi_6.{}^2A_5$	$\phi_{4.2}$	$\mathbf{Z}_6$	$1, \pm x, x^3, x^5, x^8$	BM, §8.1	3	$\Phi_3.A_5$	$\phi_{4^12^1}$
6	$\Phi_6.{}^2A_5$	$\phi_{2^21^2}$	$\mathbf{Z}_6$	$1, x^3, x^5, \pm x^7, x^8$	BM, §8.1	3	$\Phi_3.A_5$	$\phi_{2^21^2}$
8	$\Phi_8{}^2.D_4$	ϕ_2^2	$\mathbf{Z}_8$	$1, x^2, \pm x^3, \pm x^5, x^6, x^{12}$	BM, §8.1			
8	$\Phi_8{}^2.D_4$	$\phi_2\phi_{1^2}$	$\mathbf{Z}_8$	$1, \pm x, x^4, x^6, \pm x^7, x^{10}$	BM, §8.1			
8	$\Phi_8{}^2.D_4$	$\phi_{1^2}^2$	$\mathbf{Z}_8$	$1, x^6, \pm x^7, \pm x^9, x^{10}, x^{12}$	BM, §8.1			
8	$\Phi_8{}^2.D_4$	$\phi_{1^2}\phi_2$	$\mathbf{Z}_8$	$1, \pm x^3, x^4, x^6, \pm x^9, x^{10}$	BM, §8.1			
10	$\Phi_1\Phi_5.{}^2A_2$	ϕ_3	$\mathbf{Z}_{10}$	$1, -1, x, \pm x^2, \pm x^3, x^4, x^6, x^9$	BM, §8.1	5	$\Phi_1\Phi_5.A_2$	ϕ_3
10	$\Phi_1\Phi_5.{}^2A_2$	$\phi_{2^11^1}$	$\mathbf{Z}_{10}$	$1, x, x^3, x^4, \pm\zeta_4 x^{\frac{9}{2}}, x^5, x^6, x^8, x^9$	BM, §8.1	5	$\Phi_1\Phi_5.A_2$	$\phi_{2.1}$
10	$\Phi_1\Phi_5.{}^2A_2$	ϕ_{1^3}	$\mathbf{Z}_{10}$	$1, x^3, x^5, \pm x^6, \pm x^7, x^8, \pm x^9$	BM, §8.1	5	$\Phi_1\Phi_5.A_2$	ϕ_{1^3}
12	$\Phi_{12}.A_1(q^3)$	ϕ_2	$\mathbf{Z}_{12}$	$1, \pm x, x^2, \zeta_3 x^2, \zeta_3^2 x^2, \pm x^3, x^4, \pm x^5, x^8$	BM, §8.1			
12	$\Phi_{12}.A_1(q^3)$	ϕ_{1^2}	$\mathbf{Z}_{12}$	$1, \pm x^3, x^4, \pm x^5, x^6, \zeta_3 x^6, \zeta_3^2 x^6, \pm x^7, x^8$	BM, §8.1			
14	$\Phi_2\Phi_{14}$		$\mathbf{Z}_{14}$	$1, x^2, \pm x^3, \pm x^4, \pm\zeta_4 x^{\frac{9}{2}}, \pm x^5, \pm x^6, x^7, x^9$	BM, §8.1	7	$\Phi_1\Phi_7$	
18	$\Phi_2\Phi_{18}$		$\mathbf{Z}_{18}$	$1, x, \pm x^2, \pm x^3, \pm x^4, \pm x^5, \pm\zeta_4 x^{\frac{7}{2}}, \zeta_3 x^3, \zeta_3 x^4, \zeta_3^2 x^3, \zeta_3^2 x^4, x^6, x^7$	[L-Cox]	9	$\Phi_1\Phi_9$	

E_8.

e	$\mathbb{L}$	λ	$W_{\mathbb{G},\mathbb{L},\lambda}$	$\mathcal{S}_{\mathbb{G},\mathbb{L},\lambda}(u_{\mathcal{C},j})$	reference	e	λ
1	Φ_1^8		W_{E_8}	$1, -x$	Ex. 3.5	2	
1	$\Phi_1^4.D_4$		W_{F_4}	$1, -x^4; 1, -x$	[L78]	2	
1	$\Phi_1^2.E_6$		W_{G_2}	$1, -x^9; 1, -x$	[L78]	2	
1	$\Phi_1.E_7$		$\mathbf{Z}_2$	$1, -x^{15}$	[L78]	2	
4	Φ_4^4		G_{31}	$1, x^2$	BM, 180		
4	$\Phi_4^2.D_4$	$\phi_{3,1}$	G_8	$1, -1, -x, x^5$	BM, §5.12		
4	$\Phi_4^2.D_4$	$\phi_{123,013}$	G_8	$1, -x^4, \pm x^5$	BM, §5.12		
4	$\Phi_4^2.D_4$	$\phi_{013,2}$	G_8	$1, -x, -x^4, x^5$	BM, §5.12	4	$\phi_{012,3}$
6	Φ_6^4		G_{32}	$1, -x, x^2$	BM, 180	3	
6	$\Phi_6^2.{}^3D_4$	$\phi_{2,1}$	G_5	$1, -x, x^2; 1, x^4, x^8$	BM, §5.9	3	${}^3D_4[-1]$
6	$\Phi_6.{}^2E_6$	$\phi'_{9,6}$	$\mathbf{Z}_6$	$1, -1, x^2, x^5, x^7, x^{10}$	BM, §8.1	3	
6	$\Phi_6.{}^2E_6$	$\phi''_{9,6}$	$\mathbf{Z}_6$	$1, x^3, x^5, x^8, \pm x^{10}$	BM, §8.1	3	
6	$\Phi_6.{}^2E_6$	$\phi''_{6,6}$	$\mathbf{Z}_6$	$1, x, \pm x^5, x^9, x^{10}$	BM, §8.1	3	
8	Φ_8^2	1	G_9	$1, x^2, x^4, x^6; 1, x^4$	BM/M		
8	$\Phi_8.{}^2D_4$	$\phi_{13,-}$	$\mathbf{Z}_8$	$\pm 1, x, x^3, x^5, \pm x^6, x^{15}$	BM, §8.1		
8	$\Phi_8.{}^2D_4$	$\phi_{0123,13}$	$\mathbf{Z}_8$	$1, \pm x^9, x^{10}, x^{12}, x^{14}, \pm x^{15}$	BM, §8.1		
8	$\Phi_8.{}^2D_4$	$\phi_{023,1}$	$\mathbf{Z}_8$	$1, x^3, x^5, -x^6, x^9, -x^{10}, x^{12}, x^{15}$	BM, §8.1	8	$\phi_{012,3}$
8	$\Phi_8.{}^2D_4$	$\phi_{123,0}$	$\mathbf{Z}_8$	$1, x^3, -x^5, x^6, -x^9, x^{10}, x^{12}, x^{15}$	BM, §8.1	8	$\phi_{013,2}$
10	Φ_{10}^2	1	G_{16}	$1, -x, x^2, -x^3, x^4$	BM/M	5	
10	$\Phi_{10}.{}^2A_4$	$\phi_{3^12^1}$	$\mathbf{Z}_{10}$	$\pm 1, \pm x^3, x^4, \pm x^6, x^7, x^9, x^{12}$	BM, §8.1	5	
10	$\Phi_{10}.{}^2A_4$	$\phi_{2^21^1}$	$\mathbf{Z}_{10}$	$1, x^3, x^5, \pm x^6, x^8, \pm x^9, \pm x^{12}$	BM, §8.1	5	
12	Φ_{12}^2	1	G_{10}	$1, x^3, -x^3, x^6; 1, -x^2, x^4$	BM/M		
12	$\Phi_{12}.{}^3D_4$	$\phi'_{1,3}$	$\mathbf{Z}_{12}$	$1, \zeta_3, \zeta_3^2, \pm x, x^2, \pm x^3, \pm x^5, x^6, x^{10}$	BM, §8.1		
12	$\Phi_{12}.{}^3D_4$	$\phi''_{1,3}$	$\mathbf{Z}_{12}$	$1, x^4, \pm x^5, \pm x^7, x^8, \pm x^9, x^{10}, \zeta_3 x^{10}, \zeta_3^2 x^{10}$	BM, §8.1		

12	$\Phi_{12}.{}^3D_4$	$\phi_{2,2}$	$\mathbf{Z}_{12}$	$1, -x, x^2, x^4, \pm x^5, \zeta_3 x^5, \zeta_3^2 x^5, x^6, x^8, -x^9, x^{10}$	BM, §8.1	12	${}^3D_4[-1]$
14	$\Phi_2\Phi_{14}.A_1$	ϕ_2	$\mathbf{Z}_{14}$	$1, \pm x^3, \pm x^5, \pm x^6, x^7, \pm\zeta_4 x^{\frac{15}{2}}, x^8, \pm x^9, x^{15}$	BM, §8.1	7	
14	$\Phi_2\Phi_{14}.A_1$	ϕ_{1^2}	$\mathbf{Z}_{14}$	$1, \pm x^6, x^7, \pm\zeta_4 x^{\frac{15}{2}}, x^8, \pm x^9, \pm x^{10}, \pm x^{12}, x^{15}$	BM, §8.1	7	
18	$\Phi_{18}.{}^2A_2$	ϕ_3	$\mathbf{Z}_{18}$	$1, -1, x, x^2, \zeta_3 x^2, \zeta_3^2 x^2, \pm\zeta_4 x^{5/2}, \pm x^3, \pm x^4, x^5, \zeta_3 x^5, \zeta_3^2 x^5, \pm x^6, x^{10}$	BM, §8.1	9	
18	$\Phi_{18}.{}^2A_2$	$\phi_{2^1 1^1}$	$\mathbf{Z}_{18}$	$1, x, \pm x^3, \pm x^4, \pm x^5, \pm\zeta_3 x^5, \pm\zeta_3^2 x^5, \pm x^6, \pm x^7, x^9, x^{10}$	BM, §8.1	9	
18	$\Phi_{18}.{}^2A_2$	ϕ_{1^3}	$\mathbf{Z}_{18}$	$1, \pm x^4, x^5, \zeta_3 x^5, \zeta_3^2 x^5, \pm x^6, \pm x^7, \pm\zeta_4 x^{\frac{15}{2}}, x^8, \zeta_3 x^8, \zeta_3^2 x^8, x^9, \pm x^{10}$	BM, §8.1	9	
20	Φ_{20}		$\mathbf{Z}_{20}$	$1, \pm x^3, x^4, \pm x^5, \pm x^6, \pm x^7, x^8, \pm x^9, \pm\zeta_4 x^6, \zeta_5 x^6, \zeta_5^2 x^6, \zeta_5^3 x^6, \zeta_5^4 x^6, x^{12}$	BM, §8.1		
24	Φ_{24}		$\mathbf{Z}_{24}$	$1, x^2, \pm x^3, \pm x^4, \pm x^5, \pm x^6, \pm x^7, x^8, x^{10}, \pm\zeta_4 x^5, \zeta_3 x^4, \pm\zeta_3 x^5, \zeta_3 x^6,$ $\zeta_3^2 x^4, \pm\zeta_3^2 x^5, \zeta_3^2 x^6$	BM, §8.1		
30	Φ_{30}		$\mathbf{Z}_{30}$	$1, x, \pm x^2, \pm x^3, \pm x^4, \pm x^5, \pm x^6, \pm x^7, x^8, \pm\zeta_4 x^{7/2}, \pm\zeta_4 x^{9/2}, \zeta_3 x^3, \pm\zeta_3 x^4,$ $\pm\zeta_3 x^5, \zeta_3^2 x^3, \pm\zeta_3^2 x^4, \pm\zeta_3^2 x^5, \zeta_5 x^4, \zeta_5^2 x^4, \zeta_5^3 x^4, \zeta_5^4 x^4$	[L-Cox]	15	

References

[BGK] F. M. Bleher, M. Geck, W. Kimmerle. Automorphisms of Generic Iwahori–Hecke Algebras and Integral Group Rings of Finite Coxeter Groups. *J. Algebra*, **197** (1997), 615–655.

[BL] P. Boixeda Alvarez & I. Losev. Affine Springer Fibers, Procesi Bundles, and Cherednik Algebras. With an appendix by P. Boixeda Alvarez, O. Kivinen, I. Losev. Preprint (2021). arXiv:2104.09543

[BDR] C. Bonnafé, O. Dudas, R. Rouquier. Translation by the Full Twist and Deligne–Lusztig Varieties. *J. Algebra*, **558** (2020), 129–145.

[B] M. Broué. Reflection Groups, Braid Groups, Hecke Algebras, Finite Reductive Groups. *Current Developments in Mathematics* (2000), 1–107.

[BM93] M. Broué & G. Malle. Zyklotomische Heckealgebren. Représentations unipotentes génériques et blocs des groupes réductifs finis. *Astérisque*, **212** (1993), 119–189.

[BM97] M. Broué & J. Michel. Sur certains éléments réguliers des groupes de Weyl et les variétiés de Deligne–Lusztig associées. *Finite Reductive Groups: Related Structures and Representations*. Ed. M. Cabanes. Birkhäuser Boston (1997), 73–139.

[BMM93] M. Broué, G. Malle, J. Michel. Generic Blocks of Finite Reductive Groups. *Représentations unipotentes génériques et blocs des groupes réductifs finis. Astérisque* tome 212 (1993), 7–92.

[BMM99] M. Broué, G. Malle, J. Michel. Toward Spetses I. *Transform. Groups*, **4**(2–3) (1999), 157–218.

[BMR] M. Broué, G. Malle, R. Rouquier. Complex Reflection Groups, Braid Groups, Hecke Algebras. *J. reine angew. Math.*, **500** (1998), 127–190.

[C] R. W. Carter. *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*. Wiley Classics Library Edition. John Wiley & Sons (1993).

[CVX1] T. H. Chen, K. Vilonen, T. Xue. On the Cohomology of Fano Varieties and the Springer Correspondence. *Adv. Math.*, **318** (2017), 515–533.

[CVX2] T. H. Chen, K. Vilonen, T. Xue. Springer Correspondence for the Split Symmetric Pair in Type *A*. *Compos. Math.*, **154**(11) (2018), 2403–2425.

[CVX3] T. H. Chen, K. Vilonen, T. Xue. Hessenberg Varieties, Intersections of Quadrics, and the Springer Correspondence. *Trans. Amer. Math. Soc.*, **373**(4) (2020), 2427–2461.

[Ch] I. Cherednik. Double Affine Hecke Algebras and Macdonald's Conjectures. *Ann. of Math. (2)*, **141**(1) (1995), 191–216.

[CM] M. Chlouveraki & H. Miyachi. Decomposition Matrices for d-Harish-Chandra Series: The Exceptional Rank Two Cases. *LMS J. Comput. Math.*, **14** (2011), 271–290.

[Chm] T. Chmutova. Representations of the Rational Cherednik Algebras of Dihedral Type. *J. Algebra*, **297**(2) (2006), 542–565.

[ChuM] J. Chuang & H. Miyachi. Hidden Hecke Algebras and Koszul Duality. Preprint (2012). https://www.math.nagoya-u.ac.jp/~miyachi/preprints/lrkszl21.pdf

[CR] C. W. Curtis and I. Reiner. *Methods of Representation Theory: With Applications to Finite Groups and Orders*. Vol. I. John Wiley & Sons (1981).

[DM06] F. Digne & J. Michel. Endomorphisms of Deligne–Lusztig Varieties. *Nagoya Math. J.*, **183** (2006), 35–103.

[DM14] F. Digne & J. Michel. Parabolic Deligne–Lusztig Varieties. *Adv. Math.*, **257** (2014), 136–218.

[DM20] F. Digne & J. Michel. Representations of Finite Groups of Lie Type. 2nd Edition. *London Mathematical Society Student Texts*, **95**, London Mathematical Society (2020).

[DMR] F. Digne, J. Michel, R. Rouquier. Cohomologie des variétés de Deligne–Lusztig. *Adv. Math.*, **209** (2007), 749–822.

[D] V. G. Drinfeld. Degenerate Affine Hecke Algebras and Yangians. *Func. Anal. Appl.*, **20** (1986), 58–60.

[Du] O. Dudas. Cohomology of Deligne–Lusztig Varieties for Unipotent Blocks of $\mathrm{GL}_n(q)$. *Rep. Theory*, **17** (2013), 647–662.

[E] P. I. Etingof. Proof of the Broué–Malle–Rouquier Conjecture in Characteristic Zero (after I. Losev and I. Marin–G. Pfeiffer). *Arnold Math. J.*, **3**(3) (2017), 445–449.

[EG] P. Etingof & V. Ginzburg. Symplectic Reflection Algebras, Calogero–Moser Space, and Deformed Harish-Chandra Homomorphism. *Invent. math.*, **147** (2002), 243–348.

[GJ] M. Geck & N. Jacon. *Representations of Hecke Algebras at Roots of Unity*. Algebra and Applications, Vol. 15. Springer-Verlag (2011).

[GP] M. Geck & G. Pfeiffer. *Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras*. Clarendon Press, Oxford (2000).

[G] T. Gerber. Heisenberg Algebra, Wedges and Crystals. *J. Algebr. Comb.*, **49** (2019), 99–124.

[GGOR] V. Ginzburg, N. Guay, E. Opdam, R. Rouquier. On the Category O for Rational Cherednik Algebras. *Invent. math.*, **154** (2003), 617–651.

[HL] R. B. Howlett & G. I. Lehrer. Induced Cuspidal Representations and Generalised Hecke Rings. *Invent. math.*, **58** (1980), 37–64.

[I] N. Iwahori. On the Structure of a Hecke Ring of a Chevalley Group over a Finite Field. *J. Faculty of Sci., Univ. of Tokyo*, **10** (1964), 215–236.

[J] G. D. James. The Decomposition Matrices of $\mathrm{GL}_n(q)$ for $n \leq 10$. *Proc. Lond. Math. Soc.*, **3**(2), 225–265.

[JK] G. D. James & A. Kerber. *The Representation Theory of the Symmetric Group*. Encyclopedia of Mathematics and its Applications, **16**, Addison-Wesley Publishing Co., Reading, MA (1981).

[KL] D. Kazhdan & G. Lusztig. Fixed Point Varieties on Affine Flag Manifolds. *Israel J. Math.*, **62**(2) (1988), 129–168.

[L] I. Losev. Proof of Varagnolo–Vasserot Conjecture on Cyclotomic Categories $\mathcal{O}$. *Sel. Math. New Ser.*, **22** (2016), 631–668.

[L-Fin] G. Lusztig. On the Finiteness of the Number of Unipotent Classes. *Invent. math.*, **34** (1976), 201–213.

[L-Cox] G. Lusztig. Coxeter Orbits and Eigenspaces of Frobenius. *Invent. math.*, **38** (1976), 101–159.

[L77] G. Lusztig. Irreducible Representations of Finite Classical Groups. *Invent. math.*, **43** (1977), 125–175.

[L78] G. Lusztig. *Representations of Finite Chevalley Groups*. Conference Board of the Mathematical Sciences. Number 39. American Mathematical Society (1978).

[L84] G. Lusztig. *Characters of Reductive Groups over a Finite Field*. Princeton University Press (1984).

[L90] G. Lusztig. Green Functions and Character Sheaves. *Ann. of Math. (2)*, **131**(2) (Mar., 1990), 355–408.

[L96] G. Lusztig. Affine Weyl Groups and Conjugacy Classes in Weyl Groups. *Transform. Groups*, Vol. 1, Nos. 1 & 2 (1996), 83–97.

[LY17] G. Lusztig & Z. Yun. $\mathbf{Z}/m$-Graded Lie Algebras and Perverse Sheaves, I. *Represent. Theory*, **21** (2017), 277–321.

[LY18] G. Lusztig & Z. Yun. $\mathbf{Z}/m\mathbf{Z}$-Graded Lie algebras and Perverse Sheaves, III: Graded Double Affine Hecke Algebra. *Represent. Theory*, **22** (2018), 87–118.

[LM] S. Lyle and A. Mathas. Blocks of Cyclotomic Hecke Algebras, *Adv. Math.*, **216** (2007), no. 2, 854–878.

[M] G. Malle. Degrés relatifs des algèbres cyclotomiques associées aux groupes de réflexions complexes de dimension deux. In *Finite Reductive Groups: Related Structures and Representations*. Proceedings of an International Conference held in Luminy, France. Ed. M. Cabanes. Progress in Mathematics, Vol. 141. Birkhäuser (1997), 311–332.

[OY] A. Oblomkov & Z. Yun. Geometric Representations of Graded and Rational Cherednik Algebras. *Adv. Math.*, **292** (2016), 601–706.

[R08] R. Rouquier. q-Schur Algebras and Complex Reflection Groups. *Moscow Math. J.*, **8**(1) (January-March 2008), 119–158.

[RSVV] R. Rouquier, P. Shan, M. Varagnolo, E. Vasserot. Categorifications and Cyclotomic Rational Double Affine Hecke Algebras. *Invent. math.*, **204** (2016), 671–786.

[SVV] P. Shan, M. Varagnolo, E. Vasserot. Koszul Duality of Affine Kac–Moody Algebras and Cyclotomic Rational Double Affine Hecke Algebras. *Adv. Math.*, **262** (2014), 370–435.

[S74] T. A. Springer. Regular Elements of Finite Reflection Groups. *Invent. math.*, **25** (1974), 159–198.

[S76] T. A. Springer. Trigonometric Sums, Green Functions of Finite Groups and Representations of Weyl Groups. *Invent. math.*, **36** (1976), 173–207.

[T] M. Q. Trinh. From the Hecke Category to the Unipotent Locus. Preprint (2021). arXiv:2106.07444

[U] D. Uglov. Canonical Bases of Higher-Level q-Deformed Fock Spaces and Kazhdan–Lusztig Polynomials. *Physical Combinatorics*. Ed. M. Kashiwara and T. Miwa. Springer, New York (2000).

[VV] M. Varagnolo & E. Vasserot. Finite-Dimensional Representations of DAHA and Affine Springer Fibers: The Spherical Case. *Duke Math. J.*, **147**(3) (2009), 439–540.

[VX] K. Vilonen & T. Xue. Character Sheaves for Graded Lie Algebras: Stable Gradings. *Adv. Math.*, **417**, 108935 (2023), 59 pp.

[W] B. Webster. Rouquier's Conjecture and Diagrammatic Algebra. *Forum Math. Sigma*, Vol. 5, e27 (2017), 71 pp.

Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139

Email address: mqt@mit.edu

School of Mathematics and Statistics, University of Melbourne, VIC 3010, Australia

Email address: ting.xue@unimelb.edu.au