Higgs Bundles and Global Springer Theory

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§1 Ngô (2008)

Let $G = \mathrm{SL}_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

Centralizer group scheme:

$$I = \{ (\gamma, g) \in \mathfrak{g} \times G \mid \text{Ad}(g)\gamma = \gamma \}$$

For any field $F$ and $\gamma \in \mathfrak{g}(F)$, we say that:

- $\gamma$ is \textit{regular} iff $\dim I_\gamma$ is minimal. In this case, $I_\gamma$ is commutative.
- $\gamma$ is \textit{regular semisimple} iff $I_\gamma$ is a torus.

Let $\mathfrak{g}^{rs} \subseteq \mathfrak{g}^{\text{reg}} \subseteq \mathfrak{g}$ be the corresponding loci.
Let \( c = \mathbb{A}^{n-1} \parallel S_n \simeq \text{Spec } \mathbb{C}[e_2, \ldots, e_n] \).

The Chevalley map

\[ \chi : \mathfrak{g} \to c \]

sends a matrix \( \gamma \) to the tuple \( a = (a_i)_{i=2}^n \) given by

\[ \det(t - \gamma) = t^n + a_2 t^{n-2} + \cdots + a_{n-1} t + a_n. \]

Let \( c^o \) be the locus where this polynomial is separable.

**Lem** \( \chi|_{\mathfrak{g}^{\mathfrak{reg}}} : \mathfrak{g}^{\mathfrak{reg}} \to c \) is surjective.

**Lem** \( \mathfrak{g}^{rs} = \chi^{-1}(c^o) \).

**Lem** \( I|_{\mathfrak{g}^{\mathfrak{reg}}} \) descends to \( c \): There’s a smooth group scheme \( J \) over \( c \) and

\[ \chi^* J|_{\mathfrak{g}^{\mathfrak{reg}}} \sim I|_{\mathfrak{g}^{\mathfrak{reg}}}. \]

It extends to a morphism \( \chi^* J \to I \).

Explicitly, if \( \gamma \in \mathfrak{g}(F) \) and \( \chi(\gamma) = a \), then:

\[ J_a = \left\{ f \in (F[t]/a(t))^{\times} \left| \prod_{\lambda \in \mathbb{C}} f(\lambda) = 1 \right. \right\} \]

and \( J_a \to I_\gamma \) sends \( f \mapsto f(\gamma) \).

**Ex** If \( \mathfrak{g} = \mathfrak{sl}_2 \), then \( \chi \simeq \det : \mathfrak{sl}_2 \to \mathbb{A}^1 \).

\( J(C) \) is a family of \( \mathbb{C}^\times \)'s degenerating to \( \mathbb{C}^+ \times \{ \pm 1 \} \).
Since $J$ is a commutative group scheme, $BJ$ forms a commutative group stack over $c$.

The fiberwise action

$$\chi^*BJ = B(\chi^*J) \rightrightarrows BI$$

over $g$ descends to a fiberwise action

$$BJ \rightrightarrows \chi^*BI = [g/G]$$

over $c$.

It is simply transitive on the regular loci of the fibers.

The geometry of this action underlies the geometry of both affine Springer fibers and Hitchin fibers.

**Interlude** Suppose $H \rightrightarrows X$ and $H \rightrightarrows V$. Recall:

- An $H$-bundle $E \to X$ is *principal* iff it trivializes over an fpqc cover of $X$.
- The *associated bundle* $V_E \to X$ is defined by

$$V_E = (E \times V)/H$$

as an fpqc quotient.

Principal $H$-bundles are classified by maps $X \to BH$.

**Ex** Suppose $L \to X$ is a line bundle.

Its frame bundle $L^x \to X$ is a principal $G_m$-bundle such that $L = (\mathbb{A}^1)_L^x$. 
Suppose $X$ is integral, separated, noetherian, and $\hat{\mathcal{O}}_{X,v} \simeq \mathbb{C}[x]$ for all $v \in X(\mathbb{C})$.

An *$L$-twisted $G$-Higgs bundle* on $X$ is a section of $\left[ g/G \right]_{L^\times} \to X$,

where $G_m \acts \left[ g/G \right]$ by scaling $g$. Equivalent to $(E, \theta)$ with:

- $E \to X$ a principal $G$-bundle.
- $\theta$ a global section of $g_E \otimes L \to X$.

Since $G = \text{SL}_n$, this is equivalent via $V = (\mathbb{A}^n)_E$ to:

- $V \to X$ a rank-$n$ vector bundle with $\text{det}(V)$ trivial.
- $\theta$ a traceless global section of $\text{End}(V) \otimes L$.

The map $\chi : g \to c$ sends:

scaling action $G_m \acts g$
\[ \downarrow \]
weighted action $G_m \acts c = \text{Spec} \mathbb{C}[e_i]_{i=2}^n$

The weights are $c \cdot e_i = c^i e_i$.

So $\chi$ induces a *Hitchin morphism* $h : \mathcal{M} \to \mathcal{A}$, where

\[ \mathcal{M} = \mathcal{M}_{X,L} = H^0(X, \left[ g/G \right]_{L^\times}), \]
\[ \mathcal{A} = \mathcal{A}_{X,L} = H^0(X, c_{L^\times}) \]
\[ = \bigoplus_{i=2}^n H^0(X, L^\otimes i). \]

Intuitively, $h(V, \theta)$ lists coefficients of $\text{det}_L(t - \theta)$. 
The fiberwise action $BJ \curvearrowright [g/G]$ over $\mathfrak{c}$ is equivariant with respect to the $G_m$-actions.

Therefore, $\mathcal{P} \curvearrowright \mathcal{M}$ over $\mathcal{A}$, where

$$\mathcal{P} = \mathcal{P}_X := H^0(X, (BJ)_{L\times})$$

is called the (relative) Picard stack.

**Motivation** If $X$ is a nice curve and $a \in \mathcal{A}$ is also nice, then:

- $\mathcal{P}_a$ parametrizes line bundles of a fixed degree on a certain branched cover $X_a \to X$.
- $\mathcal{M}_a$ is a certain compactification of $\mathcal{P}_a$.

We say that $X_a$ is the spectral curve of $a$.

**Global Picture** Let $X$ be a smooth proper curve. Fix $a = (a_i)_{i=2}^n \in \mathcal{A} = \bigoplus_{i=2}^n H^0(L^i)$.

Let $y$ be a vertical coordinate on $L$, and let

$$X_a = \{y^n + a_2y^{n-2} + \cdots + a_{n-1}y + a_n = 0\} \subseteq L.$$ 

Let $\mathcal{A}^{\bullet}$, resp. $\mathcal{A}^{\circ}$, be the locus in $\mathcal{A}$ where $X_a$ is integral, resp. reduced.

**Lem** If $a \in \mathcal{A}^{\bullet}$, then $\mathcal{M}_a$ is proper.

**Lem** If $X$ has genus zero and $a \in \mathcal{A}^{\circ}$, then

$$\mathcal{P}_a \simeq Pic^d(X_a) \quad \text{and} \quad \mathcal{M}_a \simeq \overline{Pic}^d(X_a),$$

where $d = \binom{n}{2} \deg L$. 


Local Picture  For all $v \in X(C)$, let
\[ \hat{X}_v = \text{Spec } \hat{O}_v \quad \text{and} \quad \hat{X}_v^o = \text{Spec } \hat{F}_v. \]
Abbreviate $a_v = a|_{\hat{X}_v}$ and $L_v = L|_{\hat{X}_v}$.

**Prop**  If $a \in A^\otimes(C)$ and $\gamma \in \chi^{-1}(a_v)$, then
\[ [\mathcal{P}_{\hat{X}_v,a_v} \backslash \mathcal{M}_{\hat{X}_v,\hat{O}_v,a_v}] \simeq [\mathcal{P}_\gamma \backslash \mathcal{M}_\gamma] \]
where
\[ \mathcal{M}_\gamma = \{ g \in G(\hat{F}_v)/G(\hat{O}_v) \mid \text{Ad}(g^{-1})\gamma \in g_L \times (\hat{O}_v) \}, \]
\[ \mathcal{P}_\gamma = I_\gamma(\hat{F}_v)/J_{a_v}(\hat{O}_v), \]
given the structure of $C$-ind-schemes.

Note: $\mathcal{M}_\gamma$ is a (spherical) affine Springer fiber.

**Proof sketch**

The fpqc quotient $G(\hat{F}_v)/G(\hat{O}_v)$ classifies $(E, \iota)$ with:
- $E \to \hat{X}_v$ a principal $G$-bundle.
- $\iota : E|_{\hat{X}_v^o} \sim \to G \times \hat{X}_v^o$.

$\mathcal{M}_\gamma$ classifies $(E, \theta, \iota)$ with:
- $(E, \theta) \in \mathcal{M}_{\hat{X}_v,\hat{O}_v,a_v}$.
- $\iota : E|_{\hat{X}_v^o} \sim \to G \times \hat{X}_v^o$ such that $\iota(\theta) = \gamma$.

$\mathcal{P}_\gamma$ classifies $(E', \iota')$ with:
- $E' \to \hat{X}_v$ a principal $J_{a_v}$-bundle.
- $\iota' : E'|_{\hat{X}_v^o} \sim \to I_\gamma \times \hat{X}_v^o$. 
Local to Global  Suppose $L$ admits a square root. It defines a *Kostant section*

$$c_{L×} → [g^{\text{reg}}/G]_{L×},$$

which in turn induces a gluing map

$$\prod_{a(v)\notin c_{L×}^o} \mathcal{M}_{\gamma_v} → \mathcal{M}_{X,L,a}$$

for any $a ∈ A^\diamondsuit(C)$ and $\gamma_v ∈ \chi^{-1}(a_v)$.

**Thm (Ngô)** If $a ∈ A^\star(C)$, then any square root of $L$ induces an algebraic homeomorphism

$$\mathcal{P}_{X,a} × \prod_{a(v)\notin c_{L×}^o} \mathcal{M}_{\gamma_v} \simeq \mathcal{M}_{X,L,a}.$$

**Ex** Let $G = SL_2$ and $X = \mathbb{P}^1$ and $L = \mathcal{O}(2)$. Then

$$\mathcal{A} = H^0(X,L^\otimes 2) = H^0(\mathbb{P}^1,\mathcal{O}(4)).$$

Fix a coordinate $[x : 1]$ on $X$. Spectral curves look like

$$X_a = \{y^2 + a(x) = 0\} ⊆ L,$$

where $\deg a(x) = 4$.

If $a(x) = x^3$, then

$$\mathcal{M}_a = \overline{\text{Pic}^1(X_a)} ≃ X_a × \mathbb{P} \mu_2,$$

$$\mathcal{P}_a = \text{Pic}^1(X_a) ≃ G_a,$$

$$\mathcal{M}_{\gamma_0} × \mathcal{M}_{\gamma_\infty} = \mathbb{P}^1 × pt,$$

$$\mathcal{P}_{\gamma_0} × \mathcal{P}_{\gamma_\infty} = G_a × 1.$$

Note: $\overline{\text{Pic}^1(X_a)} ≃ X_a × \mathbb{P} \mu_2$ for general $a ∈ A^\star(C)$.
§2 Yun (2011)

Z. Yun’s global Springer action fits into a table:

<table>
<thead>
<tr>
<th>point</th>
<th>Springer fibers</th>
</tr>
</thead>
<tbody>
<tr>
<td>disk $\hat{X}_v$</td>
<td>affine Springer fibers $\mathcal{M}_{\gamma_v}$</td>
</tr>
<tr>
<td>compact surface $X$</td>
<td>parabolic Hitchin fibers $\tilde{\mathcal{M}}_a$</td>
</tr>
</tbody>
</table>

Full statement involves a graded $\mathbb{C}$-algebra

$$D^{\text{trig}} = \text{Sym} (V_{KM} \oplus \mathbb{C}) \otimes \mathbb{C} [W^{\text{aff}}].$$

By a Springer action, we really mean a morphism

$$D^{\text{trig}} \rightarrow \bigoplus_i \text{End}^2 (\tilde{h}_\bullet \mathbb{C}),$$

where $\tilde{h}_\bullet$ is a parabolic version of $h_\bullet$.

Here, $V_{KM} = X^* (T_{KM}) \otimes \mathbb{C}$, where

$$T_{KM} = G^{cen}_m \times T \times G^{rot}_m$$

is the maximal torus of a certain Kac–Moody group

$$G_{KM} = \hat{L}G \rtimes G^{rot}_m.$$

Explicitly:

- $T \subseteq G$ is a maximal torus.
- $LG$ is the loop group given by $LG(\mathbb{C}) = G(\mathbb{C}(x))$ on points, and

$$1 \to G^{cen}_m \to \hat{L}G \to LG \to 1$$

is the central extension formed by the frame bundle of its determinant line bundle.

- $G^{rot}_m$ acts on $LG$ and $\hat{L}G$ by scaling $x$. 

$\mathbb{V}$
Fix a Borel $B \supseteq T$. Gives simple roots

$$\Delta = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Phi^* \subseteq X^*(T)$$

and affine simple roots

$$\Delta^{\text{aff}} = \{\alpha_0\} \cup \Delta \subseteq X^*(T \times G^\text{rot})$$.

We have Weyl groups

$$W = \langle s_\alpha \rangle_{\alpha \in \Delta},$$
$$W^{\text{aff}} = \langle s_\alpha \rangle_{\alpha \in \Delta^{\text{aff}}} \cong \mathbb{Z}\Phi_* \rtimes W.$$

Note: Since $G = \text{SL}_n$, we have $\mathbb{Z}\Phi_* = X_*(T)$.

We will use $W^{\text{aff}} \acts on V_{\text{KM}}$ to define $D^{\text{trig}}$.

Let $u$ be an indeterminate.

The *trigonometric DAHA in the sense of Yun* is

$$D^{\text{trig}} = \text{Sym}(V_{\text{KM}} \oplus \mathbb{C}\langle u \rangle) \otimes \mathbb{C}[W^{\text{aff}}]$$

under this ring structure:

- $\mathbb{C}[W^{\text{aff}}]$ and $\text{Sym}(\cdots)$ are subalgebras.
- $u$ commutes with everything.
- For all $\xi \in V_{\text{KM}}$ and $\alpha \in \Delta^{\text{aff}}$, we have
  $$s_\alpha \xi - s_\alpha \xi s_\alpha = \langle \xi, \alpha^\vee \rangle u.$$

The grading is:

- $\deg w = 0$ for $w \in W^{\text{aff}},$
- $\deg \xi = 2i$ for $\xi \in \text{Sym}^i(\cdots)$. 
Write $X^*(G_m^{rot}) = Z\delta_{rot}$. For any $c \in \mathbb{C}$, we set

$$D_c^{trig} = D^{trig}/(u + c\delta_{rot}).$$

Still graded!

**Rem** The usual trig DAHA is $D^{trig}/(\delta_{rot} - 1)$ (up to sign??). Filtered, not graded!

**Rem** The subalgebra of $D^{trig}$ or $D_c^{trig}$ generated by

$$\text{Sym}(V \oplus \mathbb{C}\langle u \rangle) \otimes \mathbb{C}[W],$$

where $V = X^*(T) \otimes \mathbb{C}$, is Lusztig’s graded AHA.

To get the $W$-part of the global Springer action, we must extend the Hitchin morphism $h$.

Let $f : \tilde{\mathfrak{g}} \to \mathfrak{g}$ be the Springer morphism, and let the top square below be cartesian:

$$
\begin{array}{ccc}
\tilde{M} & \longrightarrow & [\tilde{\mathfrak{g}}/G]_{L \times} \\
\downarrow & & \downarrow f \\
\mathcal{M} \times X & \xrightarrow{\text{eval}} & [\mathfrak{g}/G]_{L \times} \\
\downarrow \text{h x id} & & \downarrow \chi \\
\mathcal{A} \times X & \xrightarrow{\text{eval}} & \mathfrak{c}_{L \times}
\end{array}
$$

Note that $[\tilde{\mathfrak{g}}/G] \simeq [\mathfrak{b}/B]$, where $\mathfrak{b} = \text{Lie}(B)$.

Let $\tilde{h} : \tilde{M} \to \mathcal{A} \times X$ be the composition.
To construct

\( \text{D}^{\text{trig}} \rightarrow \bigoplus_i \text{End}^{2i}(\tilde{h}^\bullet \mathbb{C}) \),

we need to describe the actions of

- \( w \in \text{W}^{\text{aff}} \).
- \( \xi \in \text{X}^*(\text{G}_m^{\text{cen}} \times T \times \text{G}_m^{\text{rot}}) \oplus \mathbb{Z}u \).

The \( \text{W}^{\text{aff}} \)-action is built up via induction on \( s_\alpha \)'s, just like in affine Springer theory.

As for the lattice, we’ll construct a map

\( \tilde{L} : \text{X}^*(T_{\text{KM}}) \oplus \mathbb{Z}u \rightarrow \text{Pic}(\tilde{M}) \),

then let \( \xi \rightsquigarrow \tilde{h}^\bullet \mathbb{C} \) via cupping with \( \tilde{h}^\bullet c_1(\tilde{L}(\xi)) \).

Let \( \text{Bun}^B_G = (\text{Bun}_G \times X) \times_{B_G} B \). In each case,

\[ \tilde{L}(\xi) = K|_{\tilde{M}} \]

for some map \( \tilde{M} \rightarrow \text{Bun}^B_G \rightarrow Z \) and \( K \in \text{Pic}(Z) \).

Write \( \text{X}^*(\text{G}_m^{\text{rot}}) = \mathbb{Z}\delta_{\text{rot}} \) and \( \text{X}^*(\text{G}_m^{\text{cen}}) = \mathbb{Z}\delta_{\text{cen}} \).

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( Z )</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi \in \text{X}^*(T) )</td>
<td>( B )</td>
<td>( K(\xi) )</td>
</tr>
<tr>
<td>( \delta_{\text{rot}} )</td>
<td>( X )</td>
<td>( \omega_X )</td>
</tr>
<tr>
<td>( \delta_{\text{cen}} )</td>
<td>( \text{Bun}_G )</td>
<td>( \omega_{\text{Bun}_G} )</td>
</tr>
<tr>
<td>( u )</td>
<td>( X )</td>
<td>( L )</td>
</tr>
</tbody>
</table>

Above, \( \xi \mapsto K(\xi) \) under \( \text{X}^*(T) \sim \rightarrow \text{Pic}(B) \).

\textbf{Thm (Yun)} (\( \ast \)) is well-defined for \( \text{deg}(L) \geq 2g_X \).

(This condition ensures Ngô’s “support theorem”.)
Rem (**) descends to $D^{\text{trig}}_c = D^{\text{trig}}/(u + c\delta_{\text{rot}})$ iff

$$L \otimes \omega_X^c = \mathcal{O}_X.$$ 

This forces $c = -\deg(L)/(2g_X - 2)$.

Rem For all $(a, v) \in A \times X$, we get

$$D^{\text{trig}} \rhd H^*(\tilde{M}_{a, v}, \mathbb{C})$$

by pullback and base-change.

But since $\omega_X$ and $L$ trivialize upon pullback to $v$, the action factors through $D^{\text{trig}}/(\delta_{\text{rot}}, u)$.

To get interesting actions on fibers, need orbifold $X$ and equivariant cohomology.

§3 Oblomkov–Yun (2014)

Let $G^{(m)}_m \rhd \mathbb{A}^2$ with weights $(m, 1)$. Then

$$X_m := [(\mathbb{A}^2 - (0, 0))/G^{(m)}_m]$$

is a weighted projective line in which $\infty$ has stabilizer $\mu_m$ and no other points are stacky.

Simultaneously,

- $G^{\text{rot}}_m \rhd X_m$ via $t \cdot [x : z] = [tx : z]$.
- $G^{\text{dil}}_m \rhd g, \epsilon$ and $\chi$ is $G^{\text{dil}}_m$-equivariant.

So for any $L \in \text{Pic}(X_m) \simeq \frac{1}{m}\mathbb{Z}$, we have

$$G^{\text{rot}}_m \times G^{\text{dil}}_m \rhd \mathcal{M}_{X_m, L}, \tilde{\mathcal{M}}_{X_m, L}, \mathcal{A}_{X_m, L}$$

and $\tilde{h} : \tilde{\mathcal{M}} \to \mathcal{A}$ is equivariant.
Fix $c = d/m$ in lowest terms. Define $\mathbb{G}_m(c)$ as the subtorus acting on $a = (a_i)_i \in A$ by
\[ t^d \cdot a_i(x : z) = a_i(t^m x : z) \]
The points of
\[ A_c := A^{\mathbb{G}_m(c)} = \mathbb{C}\langle x^{ic} z^i(\deg(L)-c)m \rangle_{i=2}^n \]
are said to be homogeneous of slope $c$.

**Thm (OY)** There are graded actions
\[ D^{\text{trig}} \to \text{End}^2_{\mathbb{G}_m} \times \text{G}_{\text{dil}} (\tilde{h}^\heartsuit C), \]
\[ D^{\text{trig}}_c \to \text{End}^2_{\mathbb{G}_m(c)} (\tilde{h}^\heartsuit C), \]
where $\tilde{h}^\heartsuit C$, $\tilde{h}^\heartsuit C$ are viewed as ind-complexes.

**Cor** $D^{\text{trig}}_c \sim H^*_{\mathbb{G}_m(c)}(\tilde{\mathcal{M}}_{a,0})$ for $a \in A_c(C)$.

There’s also a rational degeneration of this story. The **rational DAHA in the sense of Yun** is
\[ D^{\text{rat}} = \text{Sym}(V \oplus V^\vee \oplus C\langle u, \delta_{\text{rot}} \rangle) \otimes \mathbb{C}[W] \]
under a graded ring structure we won’t state. Let $D^{\text{rat}}_c = D^{\text{rat}} / (u + c\delta_{\text{rot}})$.

**Thm (OY)** If $m = n$, the Coxeter number, then:
- $A^\heartsuit_c = A^\clubsuit_c$.
- There’s a graded action
  \[ D^{\text{rat}}_c \to \text{End}^2_{\mathbb{G}_m(c)} (\text{gr}^P \tilde{h}^\heartsuit C), \]
  where $P_{\leq *}$ is the perverse filtration on $\tilde{h}^\heartsuit C$.

**Cor** In this case, $D^{\text{rat}}_c \sim \text{gr}^P H^*_{\mathbb{G}_m(c)}(\tilde{\mathcal{M}}_{a,0})$. 
**Ex** Take $a = (0, \ldots, 0, x^d) \in A_c(\mathbb{C})$, where $d$ is coprime to $n$.

Here, $\mathcal{M}_{a,0} \simeq \overline{Pic}^{d(n-1)/2} \left\{ y^n + x^d = 0 \right\}$ and $\widetilde{\mathcal{M}}_{a,0}$ is a “flagged” version.

Oblomokov–Yun:

\[
\begin{align*}
D_{c}^{\text{trig}} & \simeq H_{G_m(c)}^{*}(\widetilde{\mathcal{M}}_{a,0}), \\
D_{c}^{\text{rat}} & \simeq \text{gr}^P H_{G_m(c)}^{*}(\widetilde{\mathcal{M}}_{a,0}).
\end{align*}
\]

If we specialize $\delta_{\text{rot}} \to 1$ in the latter, then we get the *spherical simple module* of the usual rDAHA!

Garner–Kivinen have an alternate construction that does not rely on the perverse filtration.

*Thank you for listening.*