2024 RESEARCH STATEMENT

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0. My goal is to understand patterns in algebraic varieties arising not from algebraic geometry alone, but from paradigms in knot theory, Lie theory, and algebraic combinatorics.

In knot theory, there is a tension between *diagrammatic* link invariants, which depend on a choice of planar diagram for the link, and *intrinsic* invariants which do not.

In Lie theory, much of the structure and representation theory of a reductive algebraic group is controlled by a finite group called its Weyl group. The fundamental group of the space of free orbits of a Weyl group W on its reflection representation is called the braid group Br_W. There is a tension between the *diagrammatic* presentation of Br_W, using generators and relations derived from W, and its *intrinsic* meaning as a fundamental group.

A Weyl group defines a family of integers called its rational Catalan numbers. The basic problem of Catalan combinatorics is to construct explicit sets counted by these numbers and bijections between them. There is a tension between *noncrossing* constructions, which depend on choosing a Coxeter element in the Weyl group, and *nonnesting* ones which do not.

Each tension above reflects, and is reflected by, a tension between two kinds of algebro-geometric objects. Situations where the two sides of a tension combine to produce surprising phenomena are the theme of my research.

input data are diagrammatic braids	input data *define* intrinsic links or braids
braid versions of Steinberg and Richardson varieties, Betti moduli spaces	Hilbert/Quot schemes of plane curve singularities, Dolbeault moduli spaces, affine Springer fibers

1. If p is a singularity on a complex algebraic plane curve $C \subseteq \mathbf{C}^2$, then the intersection of the curve with a small 3-sphere around p will be a nontrivial link in the 3-sphere. A decade-old conjecture of Oblomkov–Rasmussen–Shende predicts a formula for a powerful diagrammatic invariant of this link, called its Khovanov–Rozansky (KhR) homology, from the structure of the formal coordinate ring of (C, p): more precisely, from the virtual weight polynomials of its *Hilbert schemes* $\mathcal{H}^\ell_{C,p}$. By definition, $\mathcal{H}^\ell_{C,p}$ is an algebraic variety whose points parametrize codimension-ℓ ideals of $\hat{\mathcal{O}}_{C,p}$. The force of the conjecture is that the diagrammatic invariant knows the intrinsic geometry of these varieties. Before my work, no results had captured all three gradings on KhR, beyond the "trivial" cusps $y^2 = x^d$ with d odd.

When (C, p) takes the form $y^n = x^d$, the link is the positive (n, d) torus knot. Its KhR homology can be computed using the combinatorics of $d \times n$ Dyck paths. From a close study of these cases, Kivinen and I discovered that the KhR side could be matched more easily, up to a change of variables, with the virtual weight polynomials of certain *Quot schemes* $\mathcal{Q}^\ell_{C,p}$ [KT]. The points of $\mathcal{Q}^\ell_{C,p}$ parametrize $\hat{\mathcal{O}}_{C,p}$-submodules of the normalization of $\hat{\mathcal{O}}_{C,p}$. Using tools from q, t-symmetric function theory, developed to study rational Dyck paths, we proved a Quot analogue of the ORS conjecture—keeping all three gradings—for all singularities $y^n = x^d$ with d coprime to or dividing n. Our results show that it is productive to attack ORS by fissioning it into two conjectures: our Quot analogue, and a *purely algebro-geometric* identity of the form

$$(\diamondsuit) \qquad \sum_{\ell \geq 0} q^\ell \chi(\mathcal{Q}^\ell_{C,p}, q^{\frac{1}{2}} t) \stackrel{?}{=} \sum_{\ell \geq 0} q^\ell \chi(\mathcal{H}^\ell_{C,p}, t), \qquad \text{where } \chi(-, t) \text{ means virtual weight polynomials.}$$

It is remarkable that ($\diamondsuit$), while inspired by knot theory, does not involve links at all. We have now proven (a refinement of) it—hence the original ORS conjecture—for the cusps $y^3 = x^d$ with $3 \nmid d$.

2. In 2013, Shende had sketched a different program to prove ORS via nonabelian Hodge theory, the study of certain canonical, *non-algebraic* homeomorphisms between *algebraic* moduli spaces inspired by gauge theory. Namely, the polynomials $\chi(\mathcal{H}^{\ell}_{C,p}, t)$ are known to be encoded in the cohomology of a single moduli space of $\hat{\mathcal{O}}_{C,p}$-modules, known as the compactified Jacobian $\overline{\mathcal{J}}_{C,p}$, by means of a *perverse* filtration $\mathsf{P}_{\leq *}$ defined using the theory of perverse sheaves. The $\mathsf{P} = \mathsf{W}$ phenomenon in nonabelian Hodge theory matches perverse filtrations on the cohomology of *Dolbeault moduli spaces* with weight filtrations $\mathsf{W}_{\leq *}$ on the cohomology of *Betti moduli spaces*. Shende speculated that the KhR homology of any positive braid closure could be extracted from the Betti side of such a correspondence.

By relating the Quot schemes $\mathcal{Q}^{\ell}_{C,p}$ to $\overline{\mathcal{J}}_{C,p}$, Kivinen and I showed that ($\diamondsuit$) would yield an elementary definition of the bigrading on $\mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*(\overline{\mathcal{J}}_{C,p})$, not requiring sheaf theory.

In work from 2021 [T21], I confirmed Shende's vision by constructing, for any positive braid β on n strands, an algebraic variety $\mathcal{Z}(\beta)$ with an action of GL_n and an explicit embedding

$$(\heartsuit) \qquad \mathsf{KhR}(\hat{\beta}) \subseteq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,\mathrm{GL}_n}(\mathcal{Z}(\beta)), \qquad \text{where } \hat{\beta} \text{ is the link closure of } \beta.$$

Above, $\mathrm{H}^*_{c,\mathrm{GL}_n}$ is equivariant cohomology with compact support. The points of $\mathcal{Z}(\beta)$ are defined in terms of sequences of flags in $\mathbf{C}^n$ and unipotent matrices. The crossings in a planar diagram of β are used to constrain the relative positions of consecutive flags.

For the trivial braid, $\mathcal{Z}(\beta)$ is the Steinberg variety studied in Lie theory. The proof of ($\heartsuit$) involves generalizing the S_n-action on its cohomology due to Springer. Prior results by Galashin–Lam only captured part of KhR, as they worked with Richardson-type varieties not incorporating unipotent data or Springer theory: The idea to introduce these was mine. I also used $\mathcal{Z}(\beta)$ to give the first algebro-geometric explanation of a meridian-twisting identity for KhR due to Gorsky–Hogancamp–Mellit–Nakagane: a purely knot-theoretic fact [T22].

Recent work of Bezrukavnikov–Boixeda-Alvarez–McBreen–Yun has shown how (torus bundles over) my quotient stacks $\mathcal{Z}(\beta)/\mathrm{GL}_n$ can be interpreted as *wild* Betti moduli spaces, where the tuple of flags arises from the Stokes structure of an irregular connection. In future work, I will show that when $\hat{\beta}$ is the (d, n) torus knot, $\mathcal{Z}(\beta)/\mathrm{GL}_n$ deformation retracts onto (a parabolic version of) the compactified Jacobian for $y^n = x^d$ via a wild nonabelian Hodge correspondence. This will give the first concrete demonstration that the Betti-vs-Dolbeault dichotomy in nonabelian Hodge theory manifests the diagrammatic-vs-intrinsic dichotomy in the theory of braids.

3. Much of the story above generalizes to any reductive algebraic group G. Its Weyl group W generalizes S_n, the Weyl group of GL_n; the braid group Br_W generalizes the braid group on n strands. The compactified Jacobian of $y^n = x^d$ generalizes to a space called a *homogeneous affine Springer fiber (ASF)*. I have found new uses for the latter in combinatorics and Lie theory.

In special cases, the homogeneous ASF admits a paving by affine spaces, and the paving strata form a *nonnesting* set counted by a rational Catalan number of W. Galashin–Lam–Williams and I have found a Betti counterpart of this phenomenon [GLTW]. We construct disjoint *Deodhar-type cells* in a braid Richardson variety that conjecturally corresponds to the ASF under nonabelian Hodge. Using ideas from my work on $\mathcal{Z}(\beta)$, we show that they form a *noncrossing* set counted by the same Catalan number. This settled enumerative conjectures dating to 2012. We hope that nonabelian Hodge methods will show how to construct an explicit noncrossing-to-nonnesting bijection.

Separately, the cohomology of a homogeneous ASF is often endowed with a bimodule structure: a Springer-type action of a double affine Hecke algebra and a monodromic braid action. Xue and I have found evidence that these bimodules manifest a wide-ranging family of block dualities in the modular representation theory of G [TX]. The braid actions bear a strange resemblance to Hecke actions on Deligne–Lusztig representations of the finite reductive groups $G(\mathbf{F}_q)$, constructed by means of generators and relations. This seems to be yet another incarnation of the diagrammatic-vs-intrinsic dichotomy.

Hecke action defined diagrammatically	braid action defined intrinsically by monodromy
finite field $\mathbf{F}_q$	field of Laurent series over $\mathbf{C}$
Deligne–Lusztig version of Steinberg variety	homogeneous affine Springer fiber
(cyclotomic Hecke, cyclotomic Hecke)	(double affine Hecke, cyclotomic braid group)
(q, q)	$(1, e^{2\pi i/m})$

This theory also leads to conjectures for the structure of the cohomology of the ASF, which I expect will illuminate its nonabelian Hodge theory.

References

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