THOM COBORDISM THEOREM

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1. Introduction

In this short notes we will explain the remarkable theorem of Thom that classifies cobordisms. This theorem was originally proven by René Thom in [] but we will follow a more modern exposition. We want to give a precise idea of the everything involved but we won't give complete details at some points. Let's define the concept of cobordism.

Definition 1. *Let M, N be two smooth closed n-manifolds. We say that M, N are cobordant if there exists a compact $(n+1)$-manifold W such that $\partial W = M \sqcup N$. This is an equivalence relation.*

Given a space X and $f : M \to X$ we call the pair (M, f) a singular manifold. We say that (M, f) and (N, g) are cobordant if there is a cobordism W between M and N and $f \sqcup g$ extends to $F : W \to X$.

We denote by $\mathcal{N}_n(X)$ the set of cobordism classes of singular n-manifolds (M, f). In particular $\mathcal{N}_n \equiv \mathcal{N}_n(*)$ is the set of cobordism sets.

We'll already state the two amazing theorems by Thom. The first gives a homotopy-theoretical interpretation of $\mathcal{N}_n$ and the second actually computes it.

Theorem 1. *$\mathcal{N}_*$ is a generalized homology theory associated to the Thom spectrum MO, i.e.*

$$\mathcal{N}_n(X) = \operatorname*{colim}_{k\to\infty} \pi_{n+k}(MO(k) \wedge X_+).$$

Theorem 2. *We have an isomorphism of graded $\mathbb{Z}/2$-algebras*

$$\pi_* MO \cong \mathbb{Z}/2[x_i : \, 0 < i \neq 2^\ell - 1] = \mathbb{Z}/2[x_2, x_4, x_5, x_6, \ldots]$$

where x_i has grading i.

We'll later define what the Thom spectrum is and the (graded) algebra structures on both $\pi_* MO$ and $\mathcal{N}_*$. For now we consider a few examples of low dimension

Example 1. In particular we have

$$\mathcal{N}_0 = \mathbb{Z}/2 \, , \mathcal{N}_1 = 0 \, , \mathcal{N}_2 = \mathbb{Z}/2 \, , \mathcal{N}_3 = 0 \, , \mathcal{N}_4 = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

For $\mathcal{N}_0$ this is clear as closed 0-manifolds are finite collection of points which are null-cobordant if and only if they are an even number. $\mathcal{N}_1 = 0$ is also clear since closed 1-manifolds are disjoint unions of S^1's which are null-cobordant. From the classification theorem of surfaces the only possible non-trivial class is $[\mathbb{R}P^2]$. For $n = 3$ we get the already non-trivial result that any closed 3-manifold is null-cobordant.

Thom's paper was published in 1954 and it was ground breaking. It introduced the notion of transversality, which will be of great importance in the first part. Also, the computation of $\pi_* MO$ used the method Serre had developed just a couple of years before to compute the homology of Eilenberg-Maclane spaces. Thom was awarded the Fields medal in 1958 for this work.

The concept of cobordism had a great influence in mathematics, in particular in geometry and topology, and in physics. We list here a few influences:

(1) Cobordisms, and their more refined version of h-cobordisms, have a central role in the development of surgery theory in the 60s by Milnor, Smale, Hirsch and others. This led to great results in high dimensional topology, such as the generalized Poincaré conjecture in dimension bigger then 5.
(2) Cobordism groups, such as $\pi_* MO$ or its complex analogue $\pi_* MU$, are naturally universal formal group laws.
(3) Cobordisms played a role in the famous Atiyah-Singer index theorem.
(4) Topological quantum field theories (TQFT) are monoidal functors from the category of cobordisms (where the objects are manifolds and the morphisms cobordisms) to $\mathrm{Vect}_{\mathbb{C}}$. These are relevant in quantum physics and had incredible applications in geometry. 2-TQFT's correspond to Frobenius algebras. Moreover certain 3-TQFTs and 4-TQFTs led to invariants of knots and 4-manifolds, respectively, that proved the existence of exotic smooth structures in $\mathbb{R}^4$.

More detailed explanations of these can also be found in [**?**].

There is some extra structure we can give to $\mathcal{N}_*$, which we now describe. The disjoint union of manifolds induced a sum on $\mathcal{N}_n(X)$. Moreover the cartesian product induced a graded product in $\mathcal{N}_n$; more generally we get a map

$$\mathcal{N}_n(X) \otimes \mathcal{N}_m(Y) \to \mathcal{N}_{n+m}(X + Y).$$

Since $2M$ is null-cobordant (consider the cobordism $W = M \times I$) actually $\mathcal{N}_n$ is a $\mathbb{Z}/2$-algebra.

We also have functionality. A map $f : X \to Y$ induced $\mathcal{N}_n(X) \to \mathcal{N}_n(Y)$ by $[M, g] \mapsto [M, f \circ g]$. We also have a map from $\mathcal{N}_n(X)$ to $H_n(X; \mathbb{Z}/2)$ defined by $[M, g] \mapsto g_*[M]$ where $[M] \in H_n(M; \mathbb{Z}/2)$ is the fundamental class. This map turns out to be a map of generalized homology theories.

2. The Pontrjagin-Thom isomorphism

In this section we prove theorem 1. Before we enter in the actual proof we'll prove a lemma that will only be useful later but we want to use to motivate some of the ideas:

Lemma 3. *Let V, N be manifolds, V compact, and let $A \subseteq N$ be a closed submanifold. If $f, g : V \to N$ are smooth functions such that $f \pitchfork A$, $g \pitchfork A$ and $f \simeq g$, then $f^{-1}(A)$ and $g^{-1}(A)$ are cobordant.*

Proof. Let $H : V \times I \to N$ be a homotopy between $f = H_0$ and $g = H_1$. We can perturb H without changing H_0 and H_1 so that H is smooth and $H \pitchfork A$ (see [**?**]). Then $H^{-1}(A) \subseteq V \times [0,1]$ is a cobordism between $f^{-1}(A)$ and $g^{-1}(A)$. □

This means that given V, N, A as above we get a map

$$[V, N] \to \mathcal{N}_n$$

where $n = \dim V - \dim N + \dim A$.

Now we would like to chose V, N, A in a way that we can cover every cobordism class $[M]$ by such maps. The first restriction is that we need to be able to embed M in V, so we will chose $V = S^{n+k}$ with large enough k. The embedding of M in S^{n+k} extends to an embedding of the total space of its normal bundle $E(\nu)$.We would be very happy if we could map S^{n+k} to $E(\nu)$ and then M would be just the pre-image of the zero section. Of course this is not possible. Thom's idea was simply to add an ∞ point to $E(\nu)$, creating the so called Thom space, and map $S^{n+k} \setminus E(\nu)$ to ∞.

2.1. Vector bundles and Thom spaces.

Given a vector bundle η we denote by $E(\eta)$ and $B(\eta)$ its total and base spaces. Recall that k-vector bundles are classified by their maps to the universal bundle $E(\gamma^k) \to BO(k)$. The classifying space $BO(k)$ can be obtained as a colimit of Grassmanians, i.e.

$$BO(k) = \operatorname*{colim}_{\ell \to \infty} \mathrm{Gr}_k(\mathbb{R}^{k+\ell}).$$

Moreover the universal bundle $E(\gamma^k)$ is the colimit of the tautological k bundle γ^k_ℓ over $\mathrm{Gr}_k(\mathbb{R}^{k+\ell})$; this bundle has total space

$$E(\gamma^k_\ell) = \{(P, x) \in \mathrm{Gr}_k(\mathbb{R}^{k+\ell}) \times \mathbb{R}^{k+\ell} : x \in P\}.$$

Let's now define the Thom space of a bundle which, as we explained, will play a central role.

Definition 2. *Let η be a vector bundle. Then we defined its Thom space* $\mathrm{Th}(\eta) = E(\eta)_+$ *as the* 1*-point compactification of the total space.*

We use here the convention that the one point compactification of a compact X is just X with a disjoint base-point. An equivalent construction of the Thom space when $M = B(\eta)$ is compact is the following: considering a metric defined in the fibers, and let $D(\eta), S(\eta)$ be the disk and sphere bundles, respectively, with respect to this metric. Then $\mathrm{Th}(\eta) = D(\eta)/S(\eta)$. Note that the Thom space is not a manifold as it has a singularity at ∞.

We note a few properties of the Thom space that will be useful. First, it's easy to see that $(S \times T)_+ = S_+ \wedge T_+$. It follows from this that if η_1, η_2 are two bundles then $\mathrm{Th}(\eta_1 \times \eta_2) = \mathrm{Th}\,\eta_1 \wedge \mathrm{Th}\,\eta_2$. In particular, if we denote by $\boldsymbol{k}_M$ the trivial k-bundle over M, then

$$\mathrm{Th}(\eta \oplus \mathbf{1}_M) = \mathrm{Th}(\eta \times \mathbf{1}_*) = \mathrm{Th}(\eta) \wedge S^1 = \Sigma \mathrm{Th}(\eta).$$

Example 2. It follows from the above that

$$\mathrm{Th}(\boldsymbol{k}_M) = \Sigma^k \mathrm{Th}(\mathbf{0}_M) = \Sigma^k X_+.$$

Note also that if $\eta_1 \to \eta_2$ is a morphism of vector bundles that's an isomorphism (or monomorphism) on the fibers we have an induced map $\mathrm{Th}(\eta_1) \to \mathrm{Th}(\eta_2)$.

2.2. **Pontrjagin-Thom construction.** We're now ready to define a map $\mathcal{N}_*(X) \to \pi_*(MO \wedge X_+)$. Let M be a closed n-manifold and $f : M \to X$. For k large enough (more precisely $k > n$) there is an embedding $M \hookrightarrow S^{n+k}$ by Whitney's theorem. Moreover this embedding is unique up to isotopy if $k > n+1$:

Theorem 4 (Reference)**.** *If M has dimension n then any two embedding $M \hookrightarrow S^m$ with $m \geq 2n+2$ are isotopic.*

Consider the normal bundle $E(\nu) \hookrightarrow M$. The embedding of M in S^{n+k} extends to an embedding of $E(\nu)$ into a tubular neighbourhood of M. This is a k-vector bundle, so it admits a map (which we also call ν) to the universal bundle:

$$\begin{array}{ccccc} S^{n+k} & \hookleftarrow & E(\nu) & \xrightarrow{\bar{\nu}} & E(\gamma^k) \\ & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ & & M & \xrightarrow{\nu} & BO(k) \end{array}$$

To incorporate the information of f we create the following bundle morphism:

$$\begin{array}{ccc} E(\nu) & \xrightarrow{(\bar{\nu}.f\pi)} & E(\gamma^k) \times X = E(\gamma^k \times \mathbf{0}_X) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ M & \xrightarrow{(\nu.f)} & BO(k) \times X \end{array}$$

This induces a map of Thom spaces

$$\mathrm{Th}(\nu) \to \mathrm{Th}(\gamma^k \oplus \mathbf{0}_X) = \mathrm{Th}(\gamma^k) \wedge X_+.$$

Definition 3. *Denote $MO(k) = \mathrm{Th}(\gamma^k)$ the universal Thom space where γ^k is the universal k-bundle over $BO(k)$.*

Composing with the collapse map $S^{n+k} \to \mathrm{Th}(\nu)$, that is the identity restricted to $E(\nu)$ and sends $S^{n+k} \setminus E(\nu)$ to ∞, gives a map $S^{n+k} \to MO(k) \wedge X_+$. We denote by $\alpha_k(M, f) \in \pi_{n+k} MO(k)$ the homotopy class of this map. This map α_k, for $k > n$, is precisely the Thom isomorphism! First we need to show that this is indeed a well defined map from $\mathcal{N}_n(X)$ to $\pi_{n+k} MO(k)$.

Lemma 5. *If (M, f) and (N, g) are cobordant singular manifolds then $\alpha_k(M, f) = \alpha_k(N, g)$.*

Proof. It's enough to prove that if (M, f) is null-cobordant then $\alpha_k(M, f) = 0$. Let (W, F) be a null-cobordism, i.e. $M = \partial W$ and $f = F_M$. It's a theorem in differential topology that the embedding $M \hookrightarrow S^{n+k}$ extends to a neat embedding $W \hookrightarrow D^{n+k+1}$. Neat means that $W \cap S^{n+k} = \partial W$ and $T_x W \not\subseteq T_x S^{n+k}$.

Let $\widetilde{\nu}$ be the normal bundle of W in D^{n+k+1}. We can embed (by neatness) $E(\widetilde{\nu})$ in D^{n+k+1} in a way that $E(\nu) = E(\widetilde{\nu}) \cap S^{n+k}$.

Then the following diagram commutes:

$$\begin{array}{ccccc} S^{n+k} & \longrightarrow & \mathrm{Th}(\nu) & \longrightarrow & MO(k) \wedge X_+ \\ \downarrow & & \downarrow & \nearrow_{\widetilde{\nu} \wedge F} & \\ D^{n+k+1} & \longrightarrow & \mathrm{Th}(\widetilde{\nu}) & & \end{array}$$

The upper composition is (a representative of) $\alpha_k(M, f)$, so since it factors through D^{n+k+1} its homotopy class is trivial. $\square$

2.3. MO **as a (ring) spectrum.** We can regard MO as a spectrum. Consider the classifying map of the $(k+1)$-bundle $E(\gamma^k) \times \mathbb{R} \to BO(k)$. This bundle map induces a map on Thom spaces

$$\Sigma MO(k) = \mathrm{Th}\left(\gamma^k \oplus \mathbf{1}_{BO(k)}\right) \to MO(k+1).$$

These maps make MO a spectrum. We claim that the classes $\alpha_k(M, f)$ are actually stable. Suppose that $M \hookrightarrow \mathbb{R}^{n+k} \hookrightarrow S^{n+k+1}$. Then we get an embedding of M in $\mathbb{R}^{n+k} \times \mathbb{R} = \mathbb{R}^{n+k+1} \subseteq S^{n+k+1}$. It's clear that the normal bundle of this embedding is now $\nu \oplus \mathbf{1}_M$. We can check that the following diagram commutes:

$$\begin{array}{ccccc} S^{n+k+1} & \longrightarrow & \mathrm{Th}(\nu \oplus \mathbf{1}_M) & \longrightarrow & MO(k+1) \wedge X_+ \\ \cong\uparrow & & \cong\uparrow & & \uparrow \\ \Sigma S^{n+k} & \longrightarrow & \Sigma\mathrm{Th}(\nu \oplus \mathbf{1}_M) & \longrightarrow & \Sigma MO(k) \wedge X_+ \end{array}$$

Now the upper composition is $\alpha_{k+1}(M, f)$ and the lower composition is $\Sigma\alpha_k(M, f)$. Thus these maps arrange to give an element

$$\alpha(M, f) \in \pi_n MO = \operatorname*{colim}_{k \to \infty} \pi_{n+k} MO(k).$$

Moreover we remark that MO is a ring spectrum. Indeed, the classifying map of $\gamma^k \times \gamma^\ell$ induces a map of thom spaces

$$MO(k) \wedge MO(\ell) = \mathrm{Th}(\gamma^k \times \gamma^\ell) \to MO(k+\ell).$$

These maps are easily seen to be stable, so they induce a map of spectra $MO \wedge MO \to MO$. This map makes MO a ring spectrum, and hence also induces a ring structure on $\pi_* MO$ with product defined as

$$\pi_* MO \otimes \pi_* MO \cong \pi_*(MO \wedge MO) \to \pi_* MO.$$

We leave it as an exercise to the reader to check that the Pontrjagin-Thom map is really a ring homomorphism with the product structure we just gave $\pi_* MO$ and the ring structure of $\mathcal{N}_*$ induced by cartesian product of manifolds.

2.4. **The inverse map.** We now construct an inverse map $\beta : \pi_n(MO \wedge X_+) \to \mathcal{N}_n(X)$. Suppose we are given a class $[g] \in \pi_{n+k} MO(k)$, that is, a map $g : S^{n+k} \to MO(k) \wedge X_+$. Consider the composition

$$g_1 : S^{n+k} \xrightarrow{g} MO(k) \wedge X_+ = \frac{MO(k) \times X}{\infty \times X} \xrightarrow{\pi_1} MO(k).$$

Since $MO(k)$ is the colimit of the cell complexes $\mathrm{Th}(\gamma^k_\ell)$ and S^{n+k} is compact, g_1 must factor through $\mathrm{Th}(\gamma^k_\ell)$ for some ℓ large enough. By perturbing g slightly we may assume that g_1 is smooth in a neighbourhood of $g_1^{-1}\left(\mathrm{Gr}_k(\mathbb{R}^{k+\ell})\right) \subseteq S^{n+k}$ and that $g_1 \pitchfork \mathrm{Gr}_k(\mathbb{R}^{k+\ell}) \supseteq MO(k)$.

Now we let $M = g_1^{-1}\left(\mathrm{Gr}_k(\mathbb{R}^{k+\ell})\right)$, which is a closed manifold of dimension n by transversality. Note that this is the same as defining $M = g^{-1}(\mathrm{Gr}_k(\mathbb{R}^{k+\ell}) \times X)$. Now we define f as the composition

$$f : M \xrightarrow{g} \mathrm{Gr}_k(\mathbb{R}^{k+\ell}) \times X \xrightarrow{\pi_2} X.$$

Finally define $\beta([g]) = (M, f)$. One can see that this is well (i.e. doesn't depend on the representative g chosen) using lemma 3.

We sketch here a proof that these are indeed inverse maps, and hence we finish the proof of theorem 1. We begin with $\beta \circ \alpha = \mathrm{id}_{\mathcal{N}_*}$. Let (M, f) be a singular manifold and let g be a representative of $\alpha_k(M, f)$. As before, g factors through $\mathrm{Th}(\gamma^k_\ell) \wedge X_+$. Now look at the commutative diagram

$$\begin{tikzcd}
S^{n+k} \arrow[r] & \mathrm{Th}(\nu) \arrow[r] & \mathrm{Th}(\gamma^k_\ell) \wedge X_+ \arrow[r] & \mathrm{Th}(\gamma^k_\ell) \wedge X_+ & \\
 & M \arrow[u, hook] \arrow[r] & \mathrm{Gr}_k(\mathbb{R}^{k+\ell}) \times X \arrow[u, hook] \arrow[r] & BO(k) \times X \arrow[u, hook] \arrow[r, "\pi_2"] & X
\end{tikzcd}$$

The composition at the top is g. It's clear from this diagram that $M = g^{-1}(\mathrm{Gr}_k(\mathbb{R}^{k+\ell}) \times X)$ and that the composition at the bottom is f, hence by the construction of *beta* we have $\beta([g]) = (M, f)$.

Now we prove (or give an idea) of $\alpha \circ \beta = \mathrm{id}_{\pi_*}$. Chose $[g] \in \pi_{n+k} MO(k)$ as before. Since $g_1 \pitchfork \mathrm{Gr}_k(\mathbb{R}^{k+\ell})$, g_1 induces a morphism bundle from the normal bundle of $M \subseteq S^{n+k}$ to the normal bundle of $\mathrm{Gr}_k(\mathbb{R}^{k+\ell}) \subseteq \mathrm{Th}(\gamma_\ell^k)$. Note that $E(\gamma_k^\ell) \subseteq \mathrm{Th}(\gamma_\ell^k)$ is a tubular neighborhood of $\mathrm{Gr}_k(\mathbb{R}^{k+\ell})$, and so the normal bundle of $\mathrm{Gr}_k(\mathbb{R}^{k+\ell})$ is isomorphic to the tautological bungle γ_ℓ^k. Since the space of tubular neighbourhood is contractible, we can assume without lost of generality that g_1 maps $E(\nu) \subseteq S^{n+k}$ to $E(\gamma_\ell^k)$. But then g is the collapse map we used to define α, hence it's clear from the construction of α that $\alpha(M, f) = [g]$.

3. Computation of $\pi_* MO$

In this section we' ll prove theorem 2 and compute $\pi_* MO$. By the previous question this is the same as the algebra of cobordism classes of manifolds, but now we have an entirely homotopy theoretical problem. The reason why we can actually compute $\pi_* MO$ is because it decomposes as a wedge of Eilenberg-MacLane spectra.

Theorem 6. *The spectrum MO is a generalized EM spectrum, i.e.*

$$MO \simeq \bigvee_i \Sigma^{\alpha_i} K(\mathbb{Z}/2)$$

where $K(\mathbb{Z}/2)$ is the Eilenberg-MacLane spectrum.

We will show this by proving that the cohomology of MO is a sum of (shifted) copies of $H^* K(\mathbb{Z}/2)$. We denote $\mathcal{A} = H^* K(\mathbb{Z}/2)$, known as the Steenrod algebra. Actually there is a natural action of $\mathcal{A}$ on $H^* MO$ and we'll see that this makes $H^* MO$ a free $\mathcal{A}$-algebra. To do this we need to understand $\mathcal{A}$, $H^* MO$ and the action; actually we'll look at the dual coaction.

In this entire subsection, if not explicit, everything is assumed to have $\mathbb{Z}/2$ coefficients (homology, cohomology, tensor products, duals, etc.).

3.1. Thom isomorphism.

The first tool we need is Thom isomorphism, relates the cohomology of a Thom space and the cohomology of the base.

Theorem 7. *Let X be a paracompact space and let η be a k-bundle over X. Then there is an isomorphism*

$$\widetilde{H}^{j+k}(\mathrm{Th}(\eta); \mathbb{Z}/2) \cong H^j(X; \mathbb{Z}/2).$$

A way to describe Thom isomorphism is the following. The map $(\mathrm{id}, \pi) : E(\eta) \to E(\eta) \times X$ induced a map

$$\mathrm{Th}(\eta) \to \mathrm{Th}(\eta) \wedge X_+.$$

Applying cohomology we get a map

$$H^* X \otimes \widetilde{H}^* \mathrm{Th}(\eta) \to \widetilde{H}^* \mathrm{Th}(\eta)$$

that makes $\widetilde{H}^*\mathrm{Th}(\eta)$ a module over H^*X. Thom theorem says that this module is free of rank 1, generated by a cohomology class $u \in H^n\mathrm{Th}(\eta)$ which is the image of $1 \in H^0X$ via the Thom isomorphism. This class u is called the Thom class of the bundle and its pullback to X is called the Euler class $e = e(\eta)$ of the bundle.

In the particular case of the universal k bundle γ^k we get an isomorphism $\widetilde{H}^{n+k}MO(K) \cong H^nBO(k)$. Moreover this isomorphism is stable in the sense that the following diagram commutes:

$$\begin{array}{ccc} H^nBO(k) & \xrightarrow{\cong} & \widetilde{H}^{n+k}MO(K) \\ \uparrow & & \uparrow \\ H^nBO(k+1) & \xrightarrow{\cong} & \widetilde{H}^{n+k+1}MO(K+1) \end{array}$$

where the left arrow is induced by the inclusion $BO(k) \hookrightarrow BO(k+1)$ and the left arrow by the structure maps of MO as a spectrum. Hence, if we denote by $BO = \operatorname*{colim}_{k\to\infty} BO(k)$ we get a stable isomorphism

$$H^nMO = \varprojlim \widetilde{H}^{n+k}MO(k) \cong \varprojlim H^nBO(k) \cong H^nBO.$$

Note that the cohomology of a spectrum E is defined by $H^nE = \lim_{\longleftarrow} \widetilde{H}^{n+k}E(k)$ and that the last isomorphism comes from the fact that cohomology sends limits to colimits.

3.2. **Steenrod algebra.** We need to understand the Steenrod algebra $\mathcal{A} = H^*K(\mathbb{Z}/2)$. Recall that by Yoneda's lemma and Brown representability there is a correspondence between natural transformations H^p to H^q and

$$[K(\mathbb{Z}/2,p), K(\mathbb{Z}/2,q)] = H^q(K(\mathbb{Z}/2,p);\mathbb{Z}/2).$$

Hence an element of $\mathcal{A}$ gives cohomology operations that are (in some sense we'll make precise when we talk about Steenrod squares) stable. Hence $\mathcal{A}$ acts on the cohomology H^*X of any space. Because this action is stable, it also acts on the cohomology of any spectrum.

We describe $\mathcal{A}$ in terms of the set of generators formed by the Steenrod squares. Steenrod squares are certain (natural) cohomology operations $\mathrm{Sq}^k_n : H^nX \to H^{n+k}X$ which obey the following properties:

(1) $\mathrm{Sq}^0_n = \mathrm{id}$.
(2) $\mathrm{Sq}^n_n x = x^2$ for $x \in H^nX$.
(3) If we write $\mathrm{Sq} = \sum_{k\geq 0} \mathrm{Sq}^k$ then $\mathrm{Sq}(xy) = \mathrm{Sq}(x)\mathrm{Sq}(y)$ (Cartan formula)
(4) $\mathrm{Sq}^k_n = 0$ for $k < 0$ and $k > n$.
(5) The following diagram commutes

$$\begin{array}{ccc} \widetilde{H}^m X & \xrightarrow{\mathrm{Sq}^k_m} & \widetilde{H}^{m+1}\Sigma X \\ \downarrow \cong & & \downarrow \cong \\ \widetilde{H}^{m+k} X & \xrightarrow{\mathrm{Sq}^k_m} & \widetilde{H}^{m+k+1}\Sigma X \end{array}$$

This property is called stability.

(6) If $i < 2j$ we have

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k\geq 0} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k.$$

These are Adem relations.

The stability property means that we can regard Sq^k as an element in $\mathcal{A}$. Even more, we have the following:

Theorem 8. *The Steenrod squares generate $\mathcal{A} = H^*K(\mathbb{Z}/2)$. More precisely, the action of $\{\mathrm{Sq}^k\}$ on $H^*K(\mathbb{Z}/2)$ induces an isomorphism*

$$\mathbb{Z}/2\langle \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots\rangle/\text{Adem relations} \cong H^*K(\mathbb{Z}/2) = \mathcal{A}.$$

Moreover one can identify a basis of the Steenrod algebra. If $I = (i_1, \ldots, i_n)$ we write

$$\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \ldots \mathrm{Sq}^{i_n}.$$

Proposition 9. *We say that $I = (i_1, \ldots, i_n)$ is admissible if $i_j \geq 2i_{j+1}$. Then*

$$\{\mathrm{Sq}^I : I \text{ is admissible}\}$$

is a basis of $\mathcal{A}$.

Actually $\mathcal{A}$ has a Hopf algebra structure. Its coproduct is induced by the map $K(\mathbb{Z}/2)\times K(\mathbb{Z}/2) \to K(\mathbb{Z}/2)$ induced by the H-space structure on $K(\mathbb{Z}/2, n)$. On generators the coproduct is written as

$$\mathrm{Sq}^n \mapsto \sum_{i+j=k} \mathrm{Sq}^i \otimes \mathrm{Sq}^j.$$

3.3. **Stiefel-Whitney classes.** Given a n-bundle $E(\eta) \to X$ recall that $\widetilde{H}^*\mathrm{Th}(\eta)$ is freely generated by the Thom class $u \in H^n\mathrm{Th}(\eta)$ as an H^*X-module. Hence this means that for any k there is a uniquely defined $w_k(\eta) \in H^k X$ such that

$$\mathrm{Sq}^k u = w_k(u)u.$$

The properties of the Steenrod squares assure that w_k is natural (i.e. $w_k(f^*\eta) = f^*w_k(\eta)$), that $w_k = 0$ for $k > n$ and that

$$w(\eta_1 \times \eta_2) = w(\eta_1) \times w(\eta_2)$$

where we write $w(\eta) = \sum_{k\geq 0} w_k(\eta)$.

We'll use the Stiefel-Whitney classes of the universal bundle to understant the cohomology of BO. We begin by looking how these things look in $BO(1) = \mathbb{R}P^\infty$.

Recall that the cohomology of $\mathbb{R}P^\infty$ is isomorphic (as a ring) to a polynomial ring $\mathbb{Z}/2[x]$ with $|x| = 1$. Since

$$\mathrm{Sq}\, x = \mathrm{Sq}^1 x + \mathrm{Sq}^0 x = x^2 + x$$

and by Cartan formula

$$\mathrm{Sq}\, x^i = x^i (x+1)^i.$$

Hence $\mathrm{Sq}^j x^i = \binom{i}{j} x^{i+j}$.

Consider now the universal line bundle γ^1 over $\mathbb{R}P^\infty$. Note that $S(\gamma^1)$ is a double cover of $\mathbb{R}P^\infty$ and is not trivial, so $S(\gamma^1) = S^\infty \simeq *$ is contractible. Thus the quotient map $\mathbb{R}P^\infty \simeq D(\gamma^1) \to \mathrm{Th}(\gamma^1)$ is a homotopy equivalence or, equivalently, the inclusion of the zero section in the Thom space is a homotopy equivalence. Clearly the Thom class of γ^1 is $x \in H^*\mathrm{Th}(\nu) \cong \mathbb{Z}/2[x]$.

3.4. **Cohomology of $BO(n)$.** We now use the Stiefel-Whitney classes to describe the cohomology of $BO(n)$. We do this by looking at the classifying map of the bundle $(\gamma^1)^n$:

$$\begin{tikzcd} E(\gamma^1)^n \arrow[r] \arrow[d] & E(\gamma^n) \arrow[d] \\ (\mathbb{R}P^\infty)^n \arrow[r] & BO(n) \end{tikzcd}$$

Note that $H^*(\mathbb{R}P^\infty)^n \cong \mathbb{Z}/2[x]^{\otimes n} \cong \mathbb{Z}/2[x_1, \dots, x_n]$. By Cartan formula for the Stiefel-Whitney classes

$$w((\gamma^1)^n) = (1 + x_1) \dots (1 + x_n) = \sum_{k=0}^{n} \sigma_k(x_1, \dots, x_n)$$

where σ_k is the k elementary symmetric polynomial. In particular the Euler class of $(\gamma^1)^n$ is $x_1 \dots x_n \in H^n((\mathbb{R}P^\infty)^n)$. By naturality of the Stiefel-Whitney classes and the above discussion the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z}/2[w_1, \dots, w_n] \arrow[r, "\beta"] \arrow[d] & H^*BO(n) \arrow[d] \\ \mathbb{Z}/2[x_1, \dots, x_n] \arrow[r, "\cong"] & H^*(\mathbb{R}P^\infty)^n \end{tikzcd}$$

Here β is the map that sends the formal symbol w_k to $w_k(\gamma^n)$, the left map sends w_k to $\sigma_k(x_1, \dots, x_n)$ and the right map is induced by the classifying map of $(\gamma^1)^n$.

Proposition 10. *The map $\beta : \mathbb{Z}/2[w_1, \dots, w_n] \to H^*BO(n)$ is a (graded) ring isomorphism, where $|w_i| = i$.*

Proof. Note that the left map in the above diagram is injective since the elementary symmetric polynomials are algebraically independent.

To prove surjectivity we consider the Schubert cell decomposition of $BO(n)$ and we note that the number of cells of dimension ℓ is the same as the dimension of $\mathbb{Z}/2[w_1, \ldots, w_n]$ in degree ℓ. Indeed both numbers are equal to

$$\# = \{(s_1, \ldots, s_d) : s_j \in \{1, \ldots, n\}, s_1 + \ldots + s_d = \ell\}. \quad \square$$