Chern filtrations for moduli of 1-dim sheaves and bundles

Let M be one of the following:

(A) $M_{r,d}(\Sigma)$ = moduli space of semistable bundles on smooth projective curve, rk $= r$, deg $= d$.

(B) $M^H_{\beta,\chi}(S)$ = moduli of H-semistable 1-dim sheaves on del Pezzo S, $c_1(F)=\beta$, $\chi(F)=\chi$.

(B') $M^{Higgs}_{r,d}(\Sigma)$

F is called semistable (stable) if $\mu(F') \leq \mu(F)$ ($<$) for any $F' \subseteq F$, where

$$\mu(F) = \begin{cases} \dfrac{\deg F}{\operatorname{rk} F} & (A) \\ \dfrac{\chi(F)}{c_1(F)\cdot H} & (B) \end{cases}$$

depends on ample divisor H

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When st $=$ ss, in both (A) and (B), M is smooth projective of dim

$$1-\chi(v,v) = \begin{cases} r^2(g-1)+1 & \text{(A)} \\ \beta^2+1 & \text{(B)} \end{cases}$$

[From now on, assume st $=$ ss always. In (A) $\iff \gcd(r,d)=1$]

We might want to study top/ invariants of M, e.g. $b_i(M)$ or (more refined) ring structure on $H^*(M)$.

<u>Today</u>: Study the Chern filtration on M

("Intermediate invariants between Betti #'s and cohomology ring")

Let $X = \Sigma$ or S, $\mathbb{F}$ the universal bundle/sheaf

$[\mathbb{F}|_{\{F\}\times X} \simeq F]$

$$\begin{tikzcd} & \mathbb{F} \arrow[d, dash] & \\ & M\times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ M & & X \end{tikzcd}$$

Define "descendents" for $k \geq 0$, $\gamma \in H^*(X)$

$$\mathrm{ch}_k(\gamma) = \begin{cases} p_*(\mathrm{ch}_k(\mathcal{F})\, q^*\gamma) & (A) \\ p_*(\mathrm{ch}_{\underbrace{k+1}_{\text{codim of the sheaves}}}(\mathcal{F})\, q^*\gamma) & (B) \end{cases} \in H^*(M)$$

<u>Fact</u>: $\mathrm{ch}_k(\gamma)$ generate $H^*(M)$ as an algebra.

<u>Def</u>: The Chern filtration on $H^*(M)$ is

$$C_k H^*(M) = \mathrm{Span}\left\{ \mathrm{ch}_{k_1}(\gamma_1) \dots \mathrm{ch}_{k_\ell}(\gamma_\ell) \;\middle|\; k_1 + \dots + k_\ell \leq k \right\}$$

Let $\mathrm{gr}_j^C H^*(M) = C_j H^*(M) / C_{j-1} H^*(M)$.

$\mathrm{gr}_\bullet^C H^*(M) = \bigoplus_{j \geq 0} \mathrm{gr}_j^C H^*(M)$ is a bigraded algebra $<$ cohomological grading, Chern grading

$$\Omega_M(q,t) = \sum_{i,j\geq 0} q^j t^i \dim\left(gr_j^C H^{i+j}(M)\right).$$

"refined Poincaré polynomial"

$q = t \rightsquigarrow$ "Poincaré polynomial"

Betti #'s	Refined Poincaré	Associated gr algebra	Cohomology ring
$\Omega_M(t,t)$ ⟵	$\Omega_M(q,t)$ ⟵	$gr_\bullet^C H^*(M)$ ⟵	$H^*(M)$

<u>Slogan</u>: $gr_\bullet^C H^*(M)$ is better behaved / simpler than $H^*(M)$

§ 1-dim sheaves & Gopakumar-Vafa invariants (GV)

X Calabi-Yau 3-fold $(X = K_S,\ S$ del Pezzo$)$

Different curve counting theories on X:

GW / DT / PT / GV

Pardon Toda, Bridgeland

* GV defined mathematically by Maulik–Toda
* Hardest to define and compute
* Makes some properties more transparent ("integrality", "finiteness")
* Agrees with naive curve counts when defined.

When $X = K_S$, S del Pezzo, $n_{g,\beta}$ (think of them as counting genus g curves in class $\beta \in H_2(X) \simeq H_2(S) \simeq H^2(S)$) is

$$\underbrace{(-1)^{d^2+1} \sum_{j \geq 0} q^j \sum (-1)^i \dim \left(\mathrm{gr}_j^P H^{i+j}(M^H_{\beta,\chi}(S)) \right)}_{\Omega^P_M(q,-1)}$$

$$= \sum_{g \geq 0} n_{g,\beta} (q^{1/2} + q^{-1/2})^{2g}.$$

where $P_\bullet H^*(M^H_{\beta,\chi}(S))$ is the perverse filtration associated to the Hilbert–Chow morphism

$$\begin{aligned} M^H_{\beta,\chi} &\longrightarrow |\mathcal{O}_S(\beta)| \\ F &\longmapsto \operatorname{supp} F \end{aligned}$$

Thm (Maulik-Shen): $\dim\left(gr_j^P H^{i+j}(M^H_{\beta,\chi}(S))\right)$ does not depend on χ, H.

Full $\Omega^P_M(q,t)$ is (conjecturally) related to refined GW/PT invariants.

§ P = C

[Kononov-Pi-Shen, Kononov-Pi-M-Lim]

Conjecture: $P_\bullet H^*(M^H_{\beta,\chi}(S)) = C_\bullet H^*(M^A_{\beta,\chi}(S))$.

Evidence: 1) If we replace S with $T^*\Sigma$ then this is true by the proofs of $P=C=W$ conjecture [Maulik-Shen, Hausel-Mellit-Minets-Schiffmann]

2) Conjecture holds up to some explicit cohomological degree $N(\beta)$ (e.g. $S=\mathbb{P}^2$, $N(d)=2d-4$, $\dim M = d^2+1$)

[Maulik-Shen-Yin, Pi-Shen-Si-Zhang]

3) Conjecture holds for $\mathbb{P}^2$, $\beta = dH$ with $d \leq 5$
[Li - M - P.]

P=C has strong implications for $C_\bullet$:

<u>Def</u>: The top Chern degree of M is the only N s.t.
$$gr_N^C H^{2\dim(M)}(M) \neq 0.$$

1) $P=C \Rightarrow$ top Chern degree of $M^H_{\beta,\chi}(S)$ is
$$N = 2(\dim M - \dim|\mathcal{O}(\beta)|) = 2 + \beta(\beta + K_S)$$

2) (relative) Hard Lefschetz symmetries:

- $\bar{\Omega}(q,t) = q^{-N/2} \cdot t^{-\dim|\mathcal{O}(\beta)|}\, \Omega(q,t)$ is invariant under $q \leftrightarrow q^{-1}$ and $t \leftrightarrow t^{-1}$.
- $gr_j^C H^*(M) \otimes gr_{N-j}^C H^*(M) \to \mathbb{Q}$ is non-degenerate.

3) $\dim gr^C_j H^*(M^H_{\beta,\chi}(S))$ does not depend on χ, H.

Let $\mathbb{D} = \mathbb{Q}\left[\left\{ ch_k(\gamma) \;\middle|\; \begin{array}{l} k \geq 1 \\ \gamma \in H^*(S) \end{array} \right\}\right]$ ← bigraded algebra

$$\begin{array}{c} \mathbb{D} \\ \big\downarrow \xi \\ H^*(M) \end{array}$$

"realization homomorphism"

1) $\Leftrightarrow \int_M D = 0 \;\forall D$ with $\deg^C D < N$

and $\exists D$ with $\deg^C D = N$ s.t. $\int_M D \neq 0$ (easy)

2) $\Rightarrow gr^C_\bullet H^*(M)$ is determined as an algebra from knowing $\int_M D$ for $\deg^C D = N$.

$$\operatorname{Ker}\left(\mathbb{D} \xrightarrow{\xi^{gr}} gr^C_\bullet H^*(M)\right) = \operatorname{Ker}\left(\langle\,,\,\rangle^{gr}\right)$$

$$\langle D, D' \rangle^{gr} = \begin{cases} \int_M \xi(D \cdot D'), & \deg^C D + \deg^C D' = N \\ 0 & \text{otherwise} \end{cases}$$

↑ Pairing on $\mathbb{D}$

Conjecture: $\int_{M^H_{\beta,\chi}(S)} D$ does not depend on χ, H

if $\deg^C D = N$.

Evidence: • Holds for $M_{5,1}(\mathbb{P}^2)$, $M_{5,2}(\mathbb{P}^2)$.

• Assuming 1), we can prove independence of H.

• We might have an idea of how the proof should go...

Conjecture + 2) $\Rightarrow$ $\mathrm{gr}^C_\bullet H^*(M^H_{\beta,\chi}(S))$ does not depend on χ, H as algebra

Contrasts with:

Thm: If $H^*(M_{d,\chi}(\mathbb{P}^2)) \cong H^*(M_{d,\chi'}(\mathbb{P}^2))$ then
(L-M-P) $\chi \equiv \pm\chi' \bmod d$.

§ The curve case

Fix Σ curve, $\Lambda \in \mathrm{Pic}^d \Sigma$

$N_{r,d}$ = ss bundles on Σ of rank r, $\det = \Lambda$.

Everything is known about $H^*(N_{r,d})$:

* Betti numbers: Atiyah–Bott, Harder–Narasimhan, Zagier, ...

* Relations among generators

  Rank 2: Zagier, Kirwan, etc.

  Higher rank: Earl–Kirwan give recursive algorithm

* Integrals of taut/ classes

  Rank 2: Thaddeus

  Higher rank: Jeffrey–Kirwan

Formula in rk 2: $H^*(N_{2,1})$ is generated by

$\alpha \in H^2(N)$, $\psi_1, \dots, \psi_{2g} \in H^3(N)$, $\beta \in H^4(N)$ s.t.

$c_2(\mathcal{F}) = 2\alpha \otimes pt + 4 \sum \psi_i \otimes e_i - \beta \otimes 1 \in H^4(N \times \Sigma)$

$\gamma := 2\sum_{i=1}^{g} \psi_{i+g}\psi_i \in H^6(N)$

Thaddeus formula:

$$\int_{N_{2,1}} \alpha^m \beta^n \gamma^p = (-1)^{g-1-p} \frac{m!\, g!}{q!\,(g-p)!} 2^{2g-p-2} (2^q-2) B_q$$

0 if $q<0$

for $m+2n+3p = 3g-3$, where $q = m+p-g+1$.

This vanishing is secretely about the Chern filtration!

<u>Thm</u> (Earl-Kirwan, L-M-P)

The top Chern degree of $N_{r,d}$ is

$N = (r+2)(r-1)(g-1)$

[implies Newstead vanishing: polynomials in $ch_k(pt)$ vanish in cohomology deg $> r(r-1)(g-1)$

e.g. $\beta^i\gamma^j = 0$ if $i+j \geq g$ in $H^*(N_{2,1})$]

Moreover, $\int_{N_{r,d}} D$ does not depend on d if $\deg^c D = N$.

Ex: $r=2$

$$\deg^C(\alpha^m \beta^n \gamma^p) = 2m + 2n + 4p$$
$$= \underbrace{2g-2}_{\text{top Chern degree.}} + q$$

In Top Chern degree ($q=0$) the formula simplifies drastically!

Q: Is the pairing

$$\mathrm{gr}^c_j H^*(N) \otimes \mathrm{gr}^c_{N-j} H^*(N) \to \mathbb{Q}$$

non-degenerate?

Yes for $r=2$.

Yes for $r=3,4$ and low genus.

No in general! Would contradict the fact that Betti numbers of $N_{r,d}$ depend on d (e.g. $N_{5,1}$ vs. $N_{5,2}$)

Speculation: Yes for $d=1$?

§ Rank 2

We understand $C_\bullet H^*(N_{2,1})$ really well.

__Thm__: $\Omega_{N_{2,1}}(q,t) = \dfrac{(1+q^2t)^{2g} - q^{2g}(1+t)^{2g}}{(1-q^2)(1-q^2t^2)}$.

__Q__: General formula for $\Omega_{N_{r,d}}(q,t)$ refining known Poincaré?

We have conjecture for $r=3$, but not in general.

However:

__Conj__: $\Omega_{N_{r,d}}(q, \underset{\substack{\uparrow \\ \text{"GV specialization"}}}{t=-1}) = \prod_{k=2}^{r} (1-(-q)^k)^{2g-2}$.

__Rmk__: $\bar{\Omega}_{N_{2,1}}(q,t) = q^{-2(g-1)} t^{-(g-1)} \Omega_{2,1}(q,t)$

has $\mathbb{Z}/2 \times \mathbb{Z}/2$ symmetries generated by

$\bar{\Omega}(q^{-1}, t^{-1}) = \bar{\Omega}(q,t)$ ($\Leftrightarrow$ non-degeneracy of the pairing)

$\bar{\Omega}(q, q^{-2}t^{-1}) = \bar{\Omega}(q,t)$ (weird...)

Thm (L-M-P): There is an explicit $sl_2 \times sl_2$ that "categorifies" this $\mathbb{Z}/2 \times \mathbb{Z}/2$ - symmetry.

Idea: $\mathbb{D} = \mathbb{Q}[\alpha, \beta, \psi_1, \dots, \psi_{2g}] \xrightarrow{\rho^{gr}} grH^*(N_{2,1})$

$sl_2 \times sl_2$ representation given by explicit differential operators in α, β, ψ_i.

We prove that these operators preserve $\ker(\rho^{gr}) = \ker(\langle , \rangle^{gr})$ by actually showing that they are (anti)-self-adjoint wrt $\langle , \rangle^{gr}$, using the formula for $\int_{N_{2,1}} \mathbb{D}$ when $\deg^c \mathbb{D} = N = 2g-2$

Explicit formulas for sl_2 triples $(e_\alpha, h_\alpha, f_\alpha)$
$(e_\beta, h_\beta, f_\beta)$

$e_\alpha = \alpha$
$h_\alpha = 2\alpha \frac{\partial}{\partial \alpha} + \sum \psi_i \frac{\partial}{\partial \psi_i} - (g-1)$

$$f_\alpha = -\alpha \frac{\partial^2}{\partial \alpha^2} + (g-1)\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \alpha} \sum \psi_i \frac{\partial}{\partial \psi_i} - \frac{\beta}{4} \sum \frac{\partial}{\partial \psi_i} \frac{\partial}{\partial \psi_{i+g}}$$

$(e_\beta, h_\beta, f_\beta)$ obtained by switching $\alpha \leftrightarrow \beta$.

[Weird fact: $gr^c_\bullet H^*(N_{2,1})$ is symmetric wrt $\alpha \leftrightarrow \beta$].

<u>Thm</u> (L-M-P): $\mathcal{N}^{\leq d}_{2,1}$ = stack of v.b. rk=2, det = 1 that do not admit subbundles of deg > d.

$$\Omega_{\mathcal{N}^{\leq d}_{2,1}}(q,t) = \frac{(1+q^2t)^{2g} - q^{2g+4d}(1+t)^{2g}}{(1-q^2)(1-q^2t^2)}$$

The proof involves a fairly delicate study of

$$\operatorname{Ker}\left(H^*(\mathcal{N}^{\leq d}_{2,1}) \twoheadrightarrow H^*(\mathcal{N}^{\leq d-1}_{2,1})\right).$$

Fun application

$\ker(\varphi^{gr} : \mathbb{D} \to gr^C_\bullet H^*(N_{2,1}))$ is the smallest ideal containing all $\mathbb{D}$ with $\deg^C > 2g^2$

closed under f_α, f_β.