THE FLUX HOMOMORPHISM

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1. Introduction

This paper is concerned with the Flux homomorphism as a tool to understand the group of Hamiltonian symplectomorphisms in a symplectic manifold (M, ω). We will follow very closely the exposition in the book Introduction of Symplectic Topology by McDuff and Salamon ([7]).

In the second section we talk in general terms about the groups of symplectomorphisms and Hamiltonian symplectomorphisms, and try to frame these in the theory of (infinite dimensional) Lie groups, in particular thinking of their Lie algebras; note that rigorously it's true that the Hamiltonian symplectomorphisms are a Lie group, but this is a very non-trivial fact that we will discuss only briefly in section 4.

Section 3 is the main part of the text; we define the flux homomorphism and prove its main properties. This homomorphism is a priori defined in the universal cover of the group of symplectomorphisms and its relevance is the fact that the Hamiltonian symplectomorphisms form the kernel of Flux.

In section 4 we talk briefly about the Flux conjecture, a very important problem which was solved in 2006 by K. Ono; he proved that the group of Hamiltonian symplectomorphisms is C^1-closed in the group of symplectomorphisms. We show what is the relation with the flux homomorphism. It's still an open problem wheter the Hamiltonian symplectomorphisms are actually C^0-closed.

At last, in section 5 we mention (whithout much detail) some other results concerning the group of Hamiltonian symplectomorphism, regardind its simplicity in the compact case and the Calabi homomorphism in the non-compact case.

2. The groups of Symplectomorphisms and Hamiltonian symplectomorphisms

Recall that given a closed manifold M its group of diffeomorphisms $\mathrm{Diff}(M)$ with the C^1 topology is a (Fréchet, infinite dimensional) Lie group (see [9]). It has Lie algebra $\chi(M)$, the space of vector fields on M with Lie bracket given by $-[\cdot,\cdot]^1$ where $[\cdot,\cdot]$ is the usual Lie bracket of vector fields. Indeed if $\{\psi_t\}$ is a path in $\mathrm{Diff}(M)$ such that $\psi_0 = \mathrm{id}_M$ the vector field $x \mapsto \frac{d\varphi_t(x)}{dt}_{|t=0}$ is tangent to $\{\psi_t\}$

[1] In this paper we are using the usual definition of Lie Bracket $[X, Y] = \mathcal{L}_X Y$, which is different from the one used in [**?**].

at $t = 0$. Given such a path (that is, an isotopy of diffeomorphisms), we call its infinitesimal generator the time dependent vector field X_t which is defined by

$$\frac{d\psi_t}{dt} = X_t \circ \psi_t.$$

The exponential of $X \in \chi(M)$ is the time-1 map of the flow of X, that is, $\exp(X) = \varphi_1^X$.

Given a symplectic manifold (M, ω) we are interested in studying the subgroup of symplectomorphisms $\mathrm{Symp}(M, \omega) \subseteq \mathrm{Diff}(M)$. This is a closed Lie subgroup[2] and has corresponding Lie sub-algebra the algebra of symplectic vector fields

$$\chi^{\mathrm{symp}}(M, \omega) = \{X \in \chi(M) : \iota(X)\omega \text{ is closed}\} \subseteq \chi(M).$$

To give a local model to $\mathrm{Symp}(M, \omega)$ we use Weinstein's Lagrangian theorem to identify sufficiently C^1-small symplectomorphisms with closed forms, giving then a coordinate chart for a neighborhood of id. This is done as follows: we consider the symplectic manifold $(M \times M, \omega \oplus (-\omega))$. Its diagonal $\Delta = \{(x, x) : x \in M\} \cong M$ is a Lagrangian submanifold, so by Weinstein's theorem there is a symplectomorphism $\Phi : \mathcal{N}(\Delta) \to \mathcal{N}(M_0)$ from a neighborhood of Δ to a neighborhood of the zero section $M_0 \subseteq T^*M$. For a sufficiently small C^1 neighborhood $\mathcal{U}$ of the identity in $\mathrm{Symp}(M, \omega)$ for every $\psi \in \mathcal{U}$ we have that $\mathrm{graph}(\psi) \subseteq \mathcal{N}(\Delta)$ and there is a closed form $\sigma \in \Omega^1(M)$ such that $\mathrm{graph}(\sigma) = \Phi(\mathrm{graph}(\psi))$ (here we think of σ as a section $M \to T^*M$). Thus we define a coordinate chart

$$\mathcal{C} : \mathcal{U} \to \mathcal{V} \subseteq \{\text{closed 1-forms on } M\}.$$

This shows that $\mathrm{Symp}(M, \omega)$ is a Lie group locally modeled by $\{\text{closed 1-forms}\} \subseteq \Omega^1(M)$. In particular it is locally path connected.

An important subgroup of the symplectomorphism group is the group of Hamiltonian symplectomorphisms.

Definition 1. *A vector field is said to be Hamiltonian if $\iota(X)\omega$ is exact, that is, $\iota(X)\omega = dH$ for some $H \in C^\infty(M, \mathbb{R})$. We denote the set of Hamiltonian vector fields by $\chi^{\mathrm{ham}}(M, \omega)$.*

We say that an isotopy $\{\psi_t\}$ is a Hamiltonian isotopy if its infinitesimal generator X_t is Hamiltonian for every t. A symplectomorphism ψ is said to be Hamiltonian if there is a Hamiltonian isotopy $\{\psi_t\}$ such that $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$. We denote by $\mathrm{Ham}(M, \omega)$ *the group of Hamiltonian symplectomorphisms.*

Note that ψ being a Hamiltonian symplectomorphism means that it's the time-1 map of the flow generated by a time dependent Hamiltonian H_t. This shows that Arnold's conjecture is essentially a question about Hamiltonian symplectomorphisms: it bounds from below the number of fixed points of a Hamiltonian symplectomorphism.

[2]The fact that it's closed in the C^1 topology comes straightorwardly from the definition. Eliashberg-Gromov rigidity theorem tells the non-trivial fact that it's also C^0-closed.

Note that $\chi^{\text{ham}}(M,\omega)$ is the kernel of the map $f : \chi^{\text{symp}}(M,\omega) \to H^1(M;\mathbb{R})$ defined by

$$X \mapsto [\iota(X)\omega] \in H^1(M;\mathbb{R}).$$

Proposition 1. *The map f above defined is a Lie algebra homomorphism (where $H^1(M;\mathbb{R}) \cong \mathbb{R}^{\beta_1}$ has the trivial Lie bracket).*

In particular χ^{ham} is a Lie ideal of χ^{symp}.

Proof. Let $X, Y \in \chi^{\text{symp}}(M,\omega)$. Recall that we have the identity $\iota([X,Y]) = [\mathcal{L}_X, \iota(Y)]$. Hence, we compute as follows:

$$\begin{aligned}\iota([X,Y])\omega &= \mathcal{L}_X\iota(Y)\omega - \iota(Y)\mathcal{L}_X\omega = d\left(\iota(X)\iota(Y)\omega\right) + \iota(X)d(\iota(Y)\omega)\\ &= d\left(\iota(X)\iota(Y)\omega\right)\end{aligned}$$

We used that $\mathcal{L}_X\omega = d\iota(Y)\omega = 0$ (because X and Y are symplectic) and Cartan's magic formula. Hence $\iota([X,Y])\omega$ is exact and so $f([X,Y]) = 0$ for any $X, Y \in \chi^{\text{symp}}(M,\omega)$. □

The fact that $\chi^{\text{ham}}(M,\omega)$ is a Lie sub-algebra suggests that $\text{Ham}(M,\omega)$ is the Lie subgroup corresponding to it (and actually, since it's a Lie ideal, we would expect $\text{Ham}(M,\omega)$ to be a normal subgroup). However in infinite dimensional Lie groups the correspondence between Lie sub-algebras and Lie subgroups does not work in general, so it's not even obvious by now that $\text{Ham}(M,\omega)$ is a subgroup. The following proposition shows this is true.

Proposition 2. *The set $\text{Ham}(M,\omega) \subseteq \text{Symp}(M,\omega)$ is a normal and path-connected subgroup.*

Proof. We can check that if $\{\psi_t\}$, $\{\phi_t\}$ are Hamiltonian isotopies generated by H_t and G_t, respectively, then $\{\psi_t \circ \psi_t\}$ is generated by $H_t + G_t \circ \psi_t^{-1}$ and $\{\psi_t^{-1}\}$ is geneated by $-H_t \circ \psi_t$; moreover if φ is any symplectomorphism then $\{\varphi^{-1} \circ \psi_t \circ \varphi\}$ is generated by $H_t \circ \varphi$. The fact that $\text{Ham}(M,\omega)$ is path-connected follows from the definition. □

Remark 1. *If we have a non-compact symplectic manifold (M,ω) we can recover most of what was said if we replace the groups of diffeomorphisms, symplectomorphisms, etc. by their compactly supported variations. For instance*

$$\text{Diff}^c(M) = \{\psi \in \text{Diff}(M) : \psi \text{ has compact support}\}.$$

The support of a diffeomorphism is the closure of $\{x : \psi(x) \neq x\}$. In this case the infinitesimal generators all have compact support as well and hence the map f is defined not onto $H^1(M;\mathbb{R})$ but onto $H^1_c(M;\mathbb{R})$, the compactly supported cohomology.

The idea to study the group of Hamiltonian symplectomorphisms is the following: we know that $\chi^{\text{ham}}(M,\omega)$ is the kernel of a Lie algebra homomorphism f which we described before, and "morally" $\chi^{\text{ham}}(M,\omega)$ is the Lie algebra of $\text{Ham}(M,\omega)$.

Hence we'd like to integrate f to a homomorphism of Lie groups. By Lie's second theorem (see [8]) we can integrate f to a homomorphism $\widetilde{\mathrm{Symp}}_0(M,\omega) \to H^1(M;\mathbb{R})$ where $\widetilde{\mathrm{Symp}}_0(M,\omega)$ is the universal cover of the connected component of the identity of $\mathrm{Symp}(M,\omega)$.

This universal cover can be described as equivalence classes of symplectic isotopies up to homotopies with fixed end points. More precisely

$$\widetilde{\mathrm{Symp}}_0(M,\omega) = \{\psi \in C^\infty([0,1], \mathrm{Symp}(M,\omega)) : \psi_0 = \mathrm{id}\}/\sim$$

where $\{\psi_t^0\} \sim \{\psi_t^1\}$ if there is a (C^∞) homotopy $s \mapsto \{\psi_t^s\}$ with fixed endpoints, i.e. such that $\psi_0^s = \mathrm{id}$ and $\psi_1^s = \psi$ for any $s \in [0,1]$.

The projection $\widetilde{\mathrm{Symp}}_0 \to \mathrm{Symp}$ is given by $\{\psi_t\} \mapsto \psi_1$. We can give a group structure to $\widetilde{\mathrm{Symp}}_0$ in two (equivalent) ways, either by defining the product of isotopies with pointwise composition or with concatenation. That is

$$[\psi_t] \cdot [\varphi_t] = [\eta_t] = [\theta_t]$$

where $\eta_t = \psi_t \circ \varphi_t$ and

$$\theta_t = \begin{cases} \psi_{2t} & \text{if } 0 \le t \le 1/2 \\ \psi_1 \circ \varphi_{2t-1} & \text{if } 1/2 \le t \le 1. \end{cases}$$

3. The Flux Homomorphism

With the ideas before in mind, we define now the Flux homomorphism.

Definition 2. *Given a symplectic isotopy $\{\psi_t\}$ we define its flux as*

$$\mathrm{Flux}\left(\{\psi_t\}\right) = \int_0^1 [\iota(X_t)\omega]\, \mathrm{d}t \in H^1(M;\mathbb{R})$$

where X_t is the (symplectic) vector defined by

$$\frac{d\psi_t}{dt} = X_t \circ \psi_t.$$

We'll see shortly that Flux is invariant with respect to homotopies with fixed end points, and hence descends to a map $\widetilde{\mathrm{Symp}}_0(M,\omega) \to H^1(M;\mathbb{R})$, and moreover that this map is a homomorphism, called the flux homomorphism. Moreover this homomorphism integrates f; indeed this is something general about integrating Lie algebra morphisms $f : \mathfrak{g} \to \mathbb{R}^n$

Another way to write the flux homomorphism is by regarding $H^1(M;\mathbb{R})$ as $\mathrm{Hom}(\pi_1(M), \mathbb{R})$ via identifying $[\alpha] \in H^1(M;\mathbb{R})$ with the homomorphism

$$\pi_1(M) \ni [\gamma] \mapsto \int_\gamma \alpha = \int_{S^1} \gamma^*\alpha \in \mathbb{R}.$$

Under this identification, the flux homomorphism sends a path $\{\psi_t\}$ to the homomorphism

$$[\gamma] \mapsto \int_0^1 \int_0^1 \omega(X_t(\gamma(s)), \dot{\gamma}(s))\, \mathrm{d}s.$$

Given a symplectic isotopy $\{\psi_t\}$ and a loop γ consider the map $\beta : S^1 \times [0,1] \to M$ defined by $\beta(s,t) = \psi_t^{-1}(\gamma(s))$. Geometrically β is a cylinder in M which is obtained by starting at the boundary circle γ and flowing along the isotopy $\{\psi_t^{-1}\}$. Differentiating the identity $\gamma(s) = \psi_t(\beta(s,t))$ in the s and t variables we get, respectively,

$$\dot{\gamma}(s) = (d\psi_t)\frac{\partial \beta}{\partial s} \text{ and } 0 = (d\psi_t)\frac{\partial \beta}{\partial t} + X_t(\psi_t(\beta(t,s))).$$

The second identity simplifies to $X_t(\gamma(s)) = -(d\psi_t)\frac{\partial\beta}{\partial t}$. Since $\psi_t^*\omega = \omega$ we now have:

$$\begin{aligned}
\mathrm{Flux}(\{\psi_t\})\gamma &= \int_0^1 \int_0^1 \omega\left((d\psi_t)\frac{\partial \beta}{\partial s}, (d\psi_t)\frac{\partial \beta}{\partial t}\right) \mathrm{d}s \\
&= \int_0^1 \int_0^1 \omega\left(\frac{\partial \beta}{\partial s}, \frac{\partial \beta}{\partial t}\right) \mathrm{d}s \\
&= \int_{S^1\times[0,1]} \beta^*\omega.
\end{aligned}$$

Proposition 3. *The flux of an isotopy $\{\psi_t\}$ doesn't depend on the class of $\{\psi_t\}$ up to homotopies with fixed end points. Moreover the induced map* $\mathrm{Flux} : \widetilde{\mathrm{Symp}}_0 \to H^1(M;\mathbb{R})$ *is a homomorphism.*

Proof. Let $u \mapsto \{\psi_t^u\}$ be a homotopy with fixed end-points id and $\psi = \psi_1^u$. Let $\beta_u : S^1 \times [0,1] \to M$ be given by $\beta_u(s,t) = (\psi_t^u)^{-1}(\beta(s))$ (this is the cylinder above considered). Let S_u be the chain (in the singular complex of M) defined by β_u. Then, since ψ has fixed end-points, we have that ∂S_u is the same for every u and is given by the difference between the chains defined by $\psi^{-1} \circ \gamma$ and γ. Thus $S_u - S_0$ is closed for every 0. But $S_1 - S_0$ is homologous to $S_0 - S_0$, which is trivial, via $u \mapsto S_u - S_0$. Hence

$$\begin{aligned}
\mathrm{Flux}(\{\psi_t^1\})\gamma - \mathrm{Flux}(\{\psi_t^1\})\gamma &= \int_{S^1\times[0,1]} (\beta^1)^*\omega - \int_{S^1\times[0,1]} (\beta^0)^*\omega \\
&= \langle [\omega], S_1 - S_0 \rangle = 0.
\end{aligned}$$

This shows invariance under homotopies with fixed endpoints. For the homomorphism part, suppose that $\{\psi_t\}, \{\varphi_t\}$ are Hamiltonian isotopies and let $\{\theta_t\}$ be (a smooth reparametrization of) the concatenation of $\{\psi_t\}$ and $\{\varphi_t\}$. Then the cylinder $(s,t) \mapsto \theta_t^{-1}(\gamma(s))$ is the union of the cylinders $(s,t) \mapsto \psi_t^{-1}(\gamma(s))$ and $(s,t) \mapsto \varphi_t^{-1}(\psi_1^{-1}(\gamma(s)))$. Thus

$$\mathrm{Flux}(\{\theta_t\})\gamma = \mathrm{Flux}(\{\psi_t\})\gamma + \mathrm{Flux}(\{\varphi_t\})(\psi_1^{-1}\circ\gamma) = \mathrm{Flux}(\{\psi_t\})\gamma + \mathrm{Flux}(\{\varphi_t\})\gamma.$$

The last equality follows from the fact that γ and $\psi_1 \circ \gamma$ are the same in the fundamental group since they are homotopic via $t \mapsto \psi_t \circ \gamma$. $\square$

Note that if $X \in \chi^{\mathrm{symp}}(M, \omega)$ and ψ_t is the flow of X then $\mathrm{Flux}(\{\psi_t\}) = [\iota(X)\omega]$. Since ω is non-degenerate for any $[\alpha] \in H^1(M; \mathbb{R})$ there is X such that $\iota(X)\omega = \alpha$ (and since α is closed X is symplectic). This shows that the flux homomorphism is surjective.

We now want to understand the kernel of Flux. Recall that Flux is supposed to integrate the map $X \mapsto [\iota(X)\omega]$, which has kernel $\chi^{\mathrm{ham}}(M, \omega)$. Thus we expect the following result:

Theorem 4. *A symplectic isotopy $\{\psi_t\}$ has flux 0 if and only if it's homotopic with fixed end points to a Hamiltonian isotopy.*

In particular ψ is a Hamiltonian symplectomorphism if and only if there is symplectic isotopy $\{\psi_t\}$ such that $\psi_0 = \mathrm{id}$, $\psi_1 = \psi$ and $\mathrm{Flux}(\{\psi_t\}) = 0$.

Proof. The only if part is obvious since the flux is invariant with respect to homotopies with fixed end points and a Hamiltonian isotopy has flux 0 because $[\iota(X_t)\omega] = 0$ for the infinitesimal generator X_t of a Hamiltonian isotopy.

For the if part, suppose that $\int_0^1 \iota(X_t)\omega \, \mathrm{d}t = dF$. We divide the proof in two steps:

<u>Step 1</u>: We may assume that $F = 0$.

Indeed let φ_t be the Hamiltonian flow of $-F$. Since the path $t \mapsto \varphi_t \circ \psi$ is Hamiltonian (where $\psi = \psi_1$) it's enough to prove the result for the isotopy

$$t \mapsto \begin{cases} \psi_{2t} & \text{if } t \leq 1/2 \\ \psi \circ \varphi_{2t-1} & \text{if } t > 1/2 \end{cases}.$$

Indeed if we find a Hamiltonian isotopy homotopic to the above concatenation then we can concatenate it with $\{\varphi_{1-t} \circ \psi\}$ to get a Hamiltonian isotopy homotopic with fixed end points to the original one.

Moreover if $\widetilde{X}_t$ is the infinitesimal generator of the concatenation then

$$\int_0^1 \iota\left(\widetilde{X}_t\right) \omega \, \mathrm{d}t = \int_0^{1/2} \iota(2X_{2t})\omega \, \mathrm{d}t + \int_{1/2}^1 (-2dF) \, \mathrm{d}t = 0.$$

This shows we may assume $F = 0$ and hence by the non-degeneracy of ω it follows that $\int_0^1 X_t \, \mathrm{d}t = 0$.

<u>Step 2</u>: The claim is true if $F = 0$, and hence $\int_0^1 X_t \, \mathrm{d}t = 0$.

Take $Y_t = -\int_0^t X_\tau \, \mathrm{d}\tau$. For each t let $s \mapsto \theta_t^s$ be the flow of the (constant) vector field Y_t. Since $Y_0 = Y_1 = 0$ we have $\theta_0^s = \theta_1^s = \mathrm{id}$ for any s and trivially $\theta_t^0 = \mathrm{id}$. Define now $\phi_t = \theta_t^1 \circ \psi_t$ and we claim that this is the desired Hamiltonian isotopy.

Indeed it's homotopic with fixed end points to $\{\psi_t\}$ via $s \mapsto \theta_t^s \circ \psi_t$. Now we have for any $T \in [0,1]$

$$\begin{aligned}\mathrm{Flux}(\{\phi_t\}_{0\leq t\leq T}) &= \mathrm{Flux}(\{\theta_t^1\}_{0\leq t\leq T}) + \mathrm{Flux}(\{\psi_t\}_{0\leq t\leq T})\\ &= \mathrm{Flux}(\{\theta_T^s\}_{0\leq s\leq 1}) + \mathrm{Flux}(\{\psi_t\}_{0\leq t\leq T})\\ &= [\iota(Y_t)\omega] + \int_0^T [\iota(X_t)\omega]\,\mathrm{d}t = 0.\end{aligned}$$

The first equality comes from the fact that Flux is a homomorphism and the second from the invariance with respect to homotopies with fixed end points by noticing that $\{\theta_t^1\}_{0\leq t\leq T}$ and $\{\theta_T^s\}_{0\leq s\leq 1}$ are homotopic. From the equality follows that ϕ_t is Hamiltonian: if Z_t is its infinitesimal generator then $\int_0^T [\iota(Z_t)\omega]\,\mathrm{d}t = 0$ for every $T \in [0,1]$, and thus $[\iota(Z_t)\omega] = 0$. $\square$

Remark 2. *Since the construction is canonical, it defines a map*

$$\begin{aligned}\mathcal{H} : &\{\text{symplectic isotopies with Flux} = 0 \text{ from } 1 \text{ to } \psi\} \to\\ &\{\text{hamiltonian isotopies from } 1 \text{ to } \psi\}\end{aligned}$$

such that $\mathcal{H}(\{\psi_t\})$ is homotopic with fixed end points to $\{\psi_t\}$. Moreover it can be seen from the construction that if $\{\psi_t\}$ is already Hamiltonian then the homotopy is via isotopies consisting of Hamiltonian symplectomorphisms (indeed θ_t^s is Hamiltonian for every $s, t \in [0,1]$); we will see later in theorem 9 that these isotopies must then be Hamiltonian isotopies. This remark will later be useful in the proof of proposition 10.

Note that this result can be used to show the following slightly more general fact (which is called deformation lemma in [11]).

Corollary 5. *Any symplectic isotopy $\{\psi_t\}$ is homotopic with fixed end points to $\{\widetilde{\psi}_t\}$ such that $[\iota\left(\widetilde{X}_s\right)\omega] = \mathrm{Flux}(\{\psi_t\})$ doesn't depend on s.*

We now turn to understanding how the flux behaves near the identity, in particular in the neighborhood $\mathcal{U} \subseteq \mathrm{Symp}(M,\omega)$ of id_M we defined earlier in which we can identify symplectomorphisms with closed forms. In order to do this we compute the flux in a exact symplectic manifold (as is the case of T^*Q).

Lemma 6. *If $\omega = -d\lambda$ (i.e. M is exact) and $\{\psi_t\}$ is a compactly supported symplectic isotopy then*

$$\mathrm{Flux}(\{\psi_t\})) = [\lambda - \psi_1^*\lambda].$$

Proof. We use Cartan magical formula to compute as follows

$$\begin{aligned}[\iota(X_t)\omega] &= -[\iota(X_t)d\lambda] = -[\mathcal{L}_{X_t}\lambda] = -\left[\frac{d}{ds}_{|s=t}\left(\psi_s \circ \psi_t^{-1}\right)^* \lambda\right] \\ &= -\left[\left(\psi_t^{-1}\right)^* \frac{d}{ds}_{|s=t}\psi_s^*\lambda\right] = -\left[\frac{d}{ds}_{|s=t}\psi_s^*\lambda\right].\end{aligned}$$

Note that in the third equality we used that X_t is the vector field tangent to $s \mapsto \psi_s \circ \psi_t^{-1}$ at $s = t$ and in the fourth we used that $\psi_t^{-1} \simeq \mathrm{id}$. Now integrating from $t = 0$ to $t = 1$ gives the desired result. □

Proposition 7. *Suppose that $\{\psi_t\}$ is a symplectic isotopy such that $\psi_t \in \mathcal{U}$ for every $t \in [0,1]$. Then* $\mathrm{Flux}(\{\psi_t\}) = -[\sigma_1]$ *where $\sigma_t = \mathcal{C}(\psi_t)$.*

In particular $\{\psi_t\}$ is a Hamiltonian isotopy if and only if the forms σ_t are exact.

Proof. Let $\iota : M \to T^*M$ be the inclusion of M as the zero section and let $\iota_\Delta : M \to M \times M$ be the inclusion as the diagonal, that is, $\iota(x) = (x,x)$. Recall that we have the map $\Phi : \mathcal{N}(\Delta) \to T^*M$ defined earlier, which is a symplectomorphism onto its image, and that $\iota = \Phi \circ \iota_\Delta$. Define in a neighborhood of $M_0 \subseteq T^*M$ the local isotopy $\Psi_t = \Phi \circ (\mathrm{id} \times \psi_t) \circ \Phi^{-1}$.

Note that by definition of $\mathcal{C}$ we have $\Phi(\mathrm{graph}(\psi_t)) = \mathrm{graph}(\sigma_t)$, thus there is diffeomorphism $f_t : M \to M$ such that $\Psi_t \circ \iota = \sigma_t \circ f_t$. The diffeomorphism f_t is given by $f_t(x) = \pi(\Phi(x, \psi_t(x)))$ where $\pi : T^*M \to M$ is the canonical projection. We compute now the flux using lemma 6. Recall that T^*M is a symplectic manifold with symplectic form $-d\lambda_{\mathrm{can}}$.

$$\begin{aligned}\mathrm{Flux}(\{\psi_t\}) &= \iota_\Delta^*\mathrm{Flux}(\{\mathrm{id} \times \psi_t\}) = \iota_\Delta^*\Phi^*\mathrm{Flux}(\{\Psi_t\}) = \iota^*\mathrm{Flux}(\{\Psi_t\}) \\ &= \iota^*[\lambda_{\mathrm{can}} - \Psi_1^*\lambda_{\mathrm{can}}] = -[\iota^*\Psi_1^*\lambda_{\mathrm{can}}] = -[f_1^*\sigma_1^*\lambda_{\mathrm{can}}] \\ &= -[\sigma_1^*\lambda_{\mathrm{can}}] = -[\sigma_1].\end{aligned}$$

Besides lemma 6 we used here that $\iota^*\lambda_{\mathrm{can}} = 0$ (since M_0 is a Lagrangian submanifold of T^*M), that $f_1 \simeq f_0 = \mathrm{id}$ and that $\sigma^*\lambda_{\mathrm{can}} = \sigma$ for any 1-form σ on M. This shows the result we wanted. □

In order to study the obstruction to a symplectomorphism (in the connected component of the identity) being Hamiltonian we would like to descend the flux homomorphism to Symp_0. Hence, we consider the group Γ_ω defined by

$$\Gamma_\omega = \mathrm{Flux}(\pi_1(\mathrm{Symp}_0(M,\omega))) \subseteq H^1(M;\mathbb{R}).$$

Here $\pi_1(\mathrm{Symp}_0(M,\omega)) \subseteq \widetilde{\mathrm{Symp}_0(M,\omega)}$ denotes the pre-image of the identity with respect to the projection $\mathrm{Symp}_0 \to \widetilde{\mathrm{Symp}_0}$. Now the flux homomorphism descends to

$$\mathrm{Flux} : \mathrm{Symp}_0(M,\omega) \to H^1(M;\mathbb{R})/\Gamma_\omega.$$

Remark 3. *Suppose that* $\{\psi_t\} \in \pi_1(\mathrm{Symp}(M,\omega))$ *is a loop, that is* $\psi_1 = \psi_0 = \mathrm{id}$. *Then the map* β *we defined earlier by* $\beta(s,t) = \psi_t^{-1}(\gamma(s))$ *is actually a map from the torus since* $\beta(1,s) = \beta(0,s) = \gamma(s)$*, and thus*

$$\mathrm{Flux}(\{\psi_t\})\gamma = \int_{S^1 \times S^1} \beta^* \omega \in P_\omega$$

where $P_\omega = \langle [\omega], H^2(M;\mathbb{Z}) \rangle$ *is the set of possible symplectic areas of closed surfaces embedded in* M*.*

Hence $\mathrm{Flux}(\{\psi_t\}) \in H^1(M; P_\omega)$ *for any loop* $\{\psi_t\}$*, that is,* $\Gamma_\omega \subseteq H^1(M; P_\omega)$*. In particular* Γ_ω *is countable because* P_ω *is a countable subset of* $\mathbb{R}$*.*

Note now that proposition 7 says that if ψ is the endpoint of a small Hamiltonian isotopy (more precisely a Hamiltonian isotopy $\{\psi_t\}$ such that $\psi_t \in \mathcal{U}$ for every t) then $[\sigma] = [\mathcal{C}(\psi)] = -\mathrm{Flux}(\{\psi_t\}) = 0$. However, we can have a small Hamiltonian symplectomorphism ψ for which there are no small Hamiltonian isotopies connecting it to the identity. In that case σ doesn't have to be exact, but the following proposition says it must belong to Γ_ω.

Proposition 8. *If* $\psi \in \mathcal{U} \subseteq \mathrm{Symp}_0(M,\omega)$ *(that is,* ψ *is* C^1*-close to the identity) and* $\sigma = \mathcal{C}(\psi) \in \Omega^1(M)$ *then*

$$\psi \in \mathrm{Ham}(M,\omega) \text{ if and only if } [\sigma] \in \Gamma_\omega.$$

Proof. The main point here is the following: an isotopy with small end-point can be concatenated with a small isotopy to give a loop. To prove the only if part suppose that $\psi = \psi_1$ is the end point of a Hamiltonian isotopy $\{\psi_t\}$. Let $\varphi_t \in \mathcal{U}$ be such that $\mathcal{C}(\varphi_t) = t\sigma$, so in particular $\varphi_1 = \psi_1$ and thus $\varphi_t^{-1} \circ \psi_t$ is a loop. Hence, because $\{\psi_t\}$ is Hamiltonian and by proposition 7 we have

$$\Gamma_\omega \ni \mathrm{Flux}(\{\varphi_t^{-1} \circ \psi_t\}) = -\mathrm{Flux}(\{\varphi_t\}) + \mathrm{Flux}(\{\psi_t\}) = [\sigma].$$

For the if part, suppose that $[\sigma] \in \Gamma_\omega$ and let $\{\theta_t\}$ be a loop such that $\mathrm{Flux}(\{\theta_t\}) = [\sigma]$ and again let $\{\varphi_t\}$ be such that $\mathcal{C}(\varphi_t) = t\sigma$. Then $\mathrm{Flux}(\{\varphi_t \circ \theta_t\}) = 0$ and $\varphi_1 \circ \theta_1 = \varphi_1 = \psi$. By theorem 4 there is a Hamiltonian isotopy homotopic to $\{\varphi_t \circ \theta_t\}$ with fixed endpoints, hence ψ is Hamiltonian. $\square$

This result enables us to prove the following fundamental result about Hamiltonian symplectomorphisms: an isotopy by Hamiltonian symplectomorphisms is Hamiltonian.

Theorem 9. *Every isotopy* $\{\psi_t\}$ *by Hamiltonian symplectomorphisms (i.e. such that* $\psi_t \in \mathrm{Ham}(M,\omega)$ *for every* t*) is a Hamiltonian isotopy.*

Proof. To show that X_{t_0} is Hamiltonian we may consider instead the isotopy $\{\psi_{t+t_0} \circ \psi_{t_0}^{-1}\}$ to prove this only in the case $t_0 = 0$ and $\psi_0 = \mathrm{id}$. Then, for small $t < \epsilon$ we have $\psi_t \in \mathcal{U}$ and since $\psi_t \in \mathrm{Ham}(M,\omega)$ by proposition 8 we have $[\mathcal{C}(\psi_t)] \in \Gamma_\omega$. But Γ_ω is countable, hence $[\mathcal{C}(\psi_t)]$ is constant for $t < \epsilon$. By proposition 7 we now have for $0 \le t < \epsilon$

$$\int_0^t [\iota(X_s)\omega]\,\mathrm{d}s = \mathrm{Flux}(\{\psi_t\}_{0\le s\le t}) = -[\mathcal{C}(\psi_t)] = 0.$$

Taking the derivative at $t = 0$ gives $[\iota(X_0)\omega] = 0$, thus X_0 is Hamiltonian as desired. $\square$

Recall that $\mathrm{Ham}(M,\omega)$ is "morally" the maximal leaf through id tangent to the left-invariant distribution given by the Lie sub-algebra $\chi^{\mathrm{ham}}(M,\omega)$. This last result says that $\mathrm{Ham}(M,\omega)$ is actually tangent to this distribution (at least, for now, only in some sense, because we don't know yet that $\mathrm{Ham}(M,\omega)$ is a manifold).

Proposition 10. *The 3 rows of the following diagram are exact sequences:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi_1(\mathrm{Ham}(M,\omega)) \arrow[r] \arrow[d] & \pi_1(\mathrm{Symp}(M,\omega)) \arrow[r, "\mathrm{Flux}"] \arrow[d] & \Gamma_\omega \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{\mathrm{Ham}}(M,\omega) \arrow[r] \arrow[d] & \widetilde{\mathrm{Symp}_0}(M,\omega) \arrow[r, "\mathrm{Flux}"] \arrow[d] & H^1(M;\mathbb{R}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ham}(M,\omega) \arrow[r] \arrow[d] & \mathrm{Symp}(M,\omega) \arrow[r, "\mathrm{Flux}"] \arrow[d] & H^1(M;\mathbb{R})/\Gamma_\omega \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Proof. We begin by proving that $\widetilde{\mathrm{Ham}}(M,\omega)$ includes in $\widetilde{\mathrm{Symp}_0}(M,\omega)$ (we remark that this part was not shown if [7], although it doesn't appear to be straightforward). This is saying the following: given any isotopies $\{\psi_t^0\}, \{\psi_t^1\} \subseteq \mathrm{Ham}(M,\omega)$ (by the last theorem this is equivalent to $\{\psi_t^0\}, \{\psi_t^1\}$ being Hamiltonian isotopies) which are homotopic with fixed end points via symplectic isotopies are actually homotopic via Hamiltonian isotopies. Indeed, if $s \mapsto \{\psi_t^s\}$ is such a homotopy via symplectic isotopies then consider $s \mapsto \mathcal{H}(\{\psi_t^s\})$ where $\mathcal{H}$ is the map defined in remark 2; note that $\mathrm{Flux}(\{\psi_t^s\}) = \mathrm{Flux}(\{\psi_t^0\}) = 0$ because $\{\psi_t^0\}$ is Hamiltonian, making $\mathcal{H}(\{\psi_t^s\})$ well defined. The homotopy $s \mapsto \mathcal{H}(\{\psi_t^s\})$ is via Hamiltonian isotopies and is between $\mathcal{H}(\{\psi_t^0\})$ and $\mathcal{H}(\{\psi_t^1\})$. Since $\{\psi_t^0\}, \{\psi_t^1\}$ are Hamiltonian, from the construction of $\mathcal{H}$ we get homotopies with fixed end points via Hamiltonian isotopies from $\mathcal{H}(\{\psi_t^0\})$ to $\{\psi_t^0\}$ and from $\mathcal{H}(\{\psi_t^1\})$ to $\{\psi_t^1\}$. This shows the desired.

The fact that the image of $\widetilde{\mathrm{Ham}}$ in $\widetilde{\mathrm{Symp}_0}$ is the kernel of Flux is the content of theorem 4 together with theorem 9; the later says that $\widetilde{\mathrm{Ham}}(M,\omega)$ consists in the classes of Hamiltonian isotopies. The surjectivity of Flux was already shown earlier as well, and this gives the exactness of the second row.

The fact that the first row is exact follows from the exactness of the second one in a straightforward way, and the surjectivity part is just the definition of Γ_ω. At last, the exactness of the third follows from the Nine Lemma considering that the whole diagram commutes and that the vertical sequences are trivially exact. □

Remark 4. *Since Γ_ω is countable $\pi_k(\Gamma_\omega)$ is trivial for $k > 0$. By the long exact sequence of homotopy groups induced by*

$$0 \longrightarrow \Gamma_\omega \longrightarrow H^1(M;\mathbb{R}) \longrightarrow H^1(M;\mathbb{R})/\Gamma_\omega \longrightarrow 0$$

it follows that $\pi_k(H^1(M;\mathbb{R})/\Gamma_\omega)$ is trivial for $k > 1$. Now by the sequence

$$0 \longrightarrow \mathrm{Ham}(M,\omega) \longrightarrow \mathrm{Symp}_0(M,\omega) \longrightarrow H^1(M;\mathbb{R})/\Gamma_\omega \longrightarrow 0$$

we get that the inclusion $\mathrm{Ham}(M,\omega) \hookrightarrow \mathrm{Symp}(M,\omega)$ *induces isomorphisms on π_k for every $k \geq 2$.*

4. The Flux Conjecture

A very important question regarding the group of Hamiltonian diffeomorphisms is whether it's actually a Lie subgroup of $\mathrm{Symp}(M,\omega)$. Note that for finite dimensional Lie groups (or, more generally, Banach ones) this would be automatic. Indeed if the Frobenius theorem was applicable then the left-invariant distribution induced by the Lie sub-algebra $\chi^{\mathrm{ham}}(M,\omega)$ would be integrable and $\mathrm{Ham}(M,\omega)$ (it's involutive because $\chi^{\mathrm{ham}}(M,\omega)$ is a Lie algebra) would be a leaf of this distribution. However, Frobenius theorem doesn't apply for Fréchet manifolds so the problem in this setup is much more subtle.

Remark 5. *The correspondence between Lie sub-algebras and Lie subgroups can be partially recovered when the group is locally exponential, that is, it admits an exponential and the exponential is a local diffeomorphism. Unfortunately, this is not the case of* $\mathrm{Symp}(M,\omega)$ *as the following argument by Leonardo Macarini shows.*

Take $X \in \chi^{\mathrm{ham}}(M,\omega)$ a Hamiltonian autonomous vector field with a 1-periodic orbit $\varphi_t^X(p)$ passing through $p \in M$ and let $f = \varphi_{1/k}^X$ with $k \in \mathbb{Z}$ large enough. Then f has a k-periodic point and for large enough k it's C^1-close to id. *Now perturb f to a symplectomorphism g such that p is still a k-periodic point but is now non-degenerate, i.e. $(dg^k)_x - I$ is invertible.*

Then g (which is arbitrarily C^1-small) can't be the time 1 map of an autonomous vector field, and hence isn't in the image of $\exp : \chi^{\mathrm{symp}}(M,\omega) \to \mathrm{Symp}(M,\omega)$, *showing that* exp *is not locally surjective. Indeed if $g = \varphi_1^Y$ then p is a fixed point of $g^k = \varphi_k^Y$; but it's clear that Y_p is in the kernel of $(d\varphi_k^Y)_p - I$, hence p is not non-degenerate, getting us a contradiction.*

The following proposition shows that the group of Hamiltonian diffeomorphisms being a Lie subgroup is equivalent to Γ_ω being discrete.

Proposition 11. *The following three statements are equivalent:*

(1) $\mathrm{Ham}(M,\omega)$ *is a submanifold of* $\mathrm{Symp}(M,\omega)$*;*
(2) $\mathrm{Ham}(M,\omega)$ *is* C^1*-closed in* $\mathrm{Symp}(M,\omega)$*;*
(3) Γ_ω *is discrete in* $H^1(M;\mathbb{R})$*.*

Proof. Consider again the local model $\mathcal{C}:\mathcal{U}\to\mathcal{V}\subseteq\Omega^1(M)$ for the group of symplectomorphisms. If Γ_ω is discrete we may restrict $\mathcal{U}$ and $\mathcal{V}$ so that the following holds: if $\sigma\in\mathcal{V}$ and $[\sigma]\in\Gamma_\omega$ then $[\sigma]=0$. Then we have $\mathcal{C}(\mathcal{U}\cap\mathrm{Ham}(M,\omega))=\mathcal{V}\cap\{\sigma:[\sigma]=0\}$. This immediately shows that $\mathrm{Ham}(M,\omega)$ is a submanifold (by homogeneity a local chart around id gives local charts everywhere) modeled on the exact 1-forms. Moreover it shows that $\mathcal{U}\cap\mathrm{Ham}(M,\omega)$ is closed in $\mathcal{U}$, and from this follows that $\mathrm{Ham}(M,\omega)$ is closed in $\mathrm{Symp}_0(M,\omega)$. Indeed if $\mathrm{Ham}\ni\psi_n\to\psi\in\mathrm{Symp}_0$ for large enough N we have $\psi\circ\psi_N^{-1}\in\mathcal{U}$ and this is the limit of $\psi_n\circ\psi_N^{-1}\in\mathrm{Ham}(M,\omega)\cap\mathcal{U}$ (again for large n), so $\psi\circ\psi_N^{-1}\in\mathrm{Ham}$ and thus $\psi\in\mathrm{Ham}$. This shows that (3) implies (1) and (2).

Suppose now that (3) does not hold. Then Γ_ω is a non-discrete subgroup of $H^1(M;\mathbb{R})\cong\mathbb{R}^{\beta_1}$, hence Γ_ω is not closed and, moreover, there are arbitrarily small elements in $\mathrm{cl}\,\Gamma_\omega\setminus\Gamma_\omega$. This shows that

$$\mathcal{C}(\mathcal{U}\cap\mathrm{Ham}(M,\omega))=\mathcal{V}\cap\{\sigma:[\sigma]\in\Gamma_\omega\}$$

is the union of an infinite number of accumulating affine subspaces intersected non-trivially with $\mathcal{V}$, thus it's not a submanifold of $\mathcal{V}$ and this shows that $\mathrm{Ham}(M,\omega)$ can't be a submanifold as well. Hence (1) implies (3).

It remains to show that (2) implies (3). Again suppose that (3) doesn't hold. Then we can find a sequence $\sigma_n\in\mathcal{U}$ such that $[\sigma_n]\in\Gamma_\omega$ and $\sigma_n\to\sigma$ with $[\sigma]\notin\Gamma_\omega$. Then let $\psi_n=\mathcal{C}^{-1}(\sigma_n)$, $\psi=\mathcal{C}^{-1}(\sigma)$ and we get $\psi_n\to\psi$, $\psi_n\in\mathrm{Ham}(M,\omega)$ and $\psi\notin\mathrm{Ham}(M,\omega)$. This shows that $\mathrm{Ham}(M,\omega)$ isn't closed and thus that (2) implies (3). □

The fact that any of this statements was true for any closed symplectic manifold (M,ω) was known as the flux conjecture and is now proven by Ono in [10].

Theorem 12 (K. Ono, 2006)**.** *The flux conjecture is true. That is, for any closed symplectic manifold* (M,ω) *the three assertions in 11 are true.*

The proof uses hard methods of modern symplectic topology, namely Floer-Novikov cohomology of paths of symplectomorphisms (non-necessarily Hamitonian). However, the following stronger conjecture is still an open problem:

Conjecture 1. *The subgroup* $\mathrm{Ham}(M,\omega)$ *is* C^0*-closed in* $\mathrm{Symp}_0(M,\omega)$*.*

For some partial results in this direction see [6] and [3].

5. Other results concerning Ham

Here we mention a couple of interesting results results about the group of Hamiltonian symplectomorphisms withouth much detail. A very nice result is the following:

Theorem 13 (Banyaga)**.** *If* (M, ω) *is a closed symplectic manifold then* $\mathrm{Ham}(M, \omega)$ *is simple.*

This theorem was proven in [1] and follows from two results. The first is a general argument due to Epstein (see [4]) that shows that, for certain general groups G of homeomorphisms, $[G, G]$ is simple. This general argument applies for instance to $G = \mathrm{Diff}_0(M)$ and $G = \mathrm{Symp}_0(M, \omega)$. In the case of diffeomorphisms, it's proven (for instance in [5], using a deep result of Herman) that if M is closed then $[\mathrm{Diff}_0(M), \mathrm{Diff}_0(M)] = \mathrm{Diff}_0(M)$. However, in the case of symplectomorphisms it's not true that $\mathrm{Symp}_0(M, \omega)$ is perfect (i.e. equal to its own commutator). Indeed we saw that

$$\mathrm{Symp}_0(M, \omega)/\mathrm{Ham}(M, \omega) \cong H^1(M; \mathbb{R})/\Gamma_\omega$$

which is abelian, hence

$$[\mathrm{Symp}_0(M, \omega), \mathrm{Symp}_0(M, \omega)] \subseteq \mathrm{Ham}(M, \omega).$$

But indeed we have equality, as was proven by Banyaga:

Theorem 14 (Banyaga)**.** *If* (M, ω) *is a closed symplectic manifold then*

$$[\mathrm{Symp}_0(M, \omega), \mathrm{Symp}_0(M, \omega)] = \mathrm{Ham}(M, \omega).$$

This fact, combined with the general argument of Epstein, gives theorem 13.

In the non-compact case a different situation arises. First we have to restrict ourselves to compactly supported diffeomorphisms/symplectomorphisms as for instance $\mathrm{Diff}_0^c(M, \omega)$ is a normal subgroup of $\mathrm{Diff}_0(M, \omega)$. But in this case it's no longer true that $\mathrm{Ham}^c(M, \omega)$ is simple. Indeed we can find another homomorphism from $\mathrm{Ham}^c(M, \omega)$ with non-trivial kernel, known as the Calabi homomorphism (or second Calabi homomorphism, being the Flux the first one).

To construct the Calabi homomorphism, we think again in $\mathrm{Ham}^c(M, \omega)$ is terms of its Lie algebra. This consists of the compactly supported Hamiltonian vector fields X. For such vector field there is $H \in C^\infty(M)$ such that $\iota(X)\omega = dH$; this H is unique as long as we ask that H has compact support. Thus we identify the Lie algebra of $\mathrm{Ham}^c(M, \omega)$ with $C_c^\infty(M)$ with the Poisson bracket defined by

$$\{f, g\} = \omega(X_f, X_g).$$

Then the following map is a Lie algebra homomorphism:

$$H \mapsto \int_M H\omega^n \in \mathbb{R}.$$

Indeed the n-form $\{f,g\}\omega^n$ can be seen to be exact and given by $d(nfdg\wedge\omega^{n-1})$, so $\int_M\{f,g\}\omega^n = 0$. Thus we can integrate this homomorphism to the universal cover $\widetilde{\mathrm{Ham}^c}(M,\omega)$:

Definition 3. *Let $\{\psi_t\}$ be a compactly supported Hamiltonian isotopy generated by the compactly supported (time dependent) Hamiltonian H_t. Then we define*

$$\mathrm{CAL}(\{\psi_t\}) = \int_0^1\int_M H_t\omega^n\,\mathrm{d}t.$$

It can be shown that this is invariant with respect to homotopies with fixed end points, so it gives a map $\widetilde{\mathrm{Ham}^c}(M,\omega)\to\mathbb{R}$. Moreover this map is easily seen to be a surjective homomorphism. In analogy to what we did for the Flux homomorphism we can define Λ_ω to be the image of $\pi_1(\mathrm{Ham}^c(M,\omega))$ under the Calabi homomorphism, and then we can descend it to a homomorphism

$$\mathrm{CAL}:\mathrm{Ham}^c(M,\omega)\to\mathbb{R}/\Lambda_\omega.$$

But now the kernel of the Calabi homomorphism is a normal subgroup of $\mathrm{Ham}^c(M,\omega)$ and it certainly contains its commutator (as $\mathbb{R}/\Lambda_\omega$ is abelian). But actually it really is the commutator and is simple, as Banyaga also shows in [1]:

Theorem 15. *If (M,ω) is a non-compact symplectic manifold then*

$$\ker(\mathrm{CAL}) = [\mathrm{Ham}^c(M,\omega),\mathrm{Ham}^c(M,\omega)]$$

is a simple group.

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