SUMMARIES 18.705 FALL 24

This course mostly follows Eisenbud's book [2]. When I mention a Section, Proposition, Theorem, etc. that does not indicate the reference, it is referring to that book.

Lecture 1 (Sep 4). Logistics. Review of basic language: homomorphism, polynomial ring, quotient rings, maximal/prime/principal ideal, integral domain, unique factorization domain, exact sequences. Motivation to study commutative algebra from number theory. Motivation to study commutative algebra from algebraic geometry.

Lecture 2 (Sep 9). The correspondence between ideals in $k[x_1, \ldots, x_n]$ and algebraic subsets. Radical and radical ideals. The Nullstellensatz (without proof) and its consequences, such as the correspondence between maximal ideals and points on an algebraic set. Spectrum of a ring and functoriality. This is (roughly) the content in Section 5 in [3] (except 5.7) or Section 1.6 in [2].

Lecture 3 (Sep 11). More on functoriality of Spec (and why it fails for Spec_m). Irreducible algebraic sets versus domains, and interpretation of $\text{Spec}A(X)$ as corresponding to irreducible subsets (Sections 5.7, 5.8 in [3]). Basic operations on modules, such as direct sum, product, Hom and tensor (Section 2.2 in [2]); universal properties of these constructions.

Lecture 4 (Sep 16). More on the tensor product (Section 2.2 in [2]): basic properties, exactness and the geometric interpretation of the tensor product of algebras. Localization (Section 2.1): geometric intuition for localization; definition as rings of fractions; examples.

Lecture 5 (Sep 18). The universal property of the localization. Localization of modules as extension of scalars (Lemma 2.4) and flatness of $R[U^{-1}]$ (Proposition 2.5). Ideals and prime ideals in the localized ring (Proposition 2.2), local rings. R Noetherian implies $R[U^{-1}]$ Noetherian (Corollary 2.3). Checking if a module is 0 "locally" (Lemma 2.8) and the annihilator of $x \in M$.[1]

Lecture 6 (Sep 23). Motivating examples for the primary decomposition. Statement of the primary decomposition for ideals. Definition of the set of associated primes to a module. Starting the proof of their most important properties, Proposition 3.4.

Lecture 7 (Sep 25). Finishing the proof of Theorem 3.1 in Eisenbud's book and many other auxiliary results concerting associated primes (Corollary 3.5, Lemma 3.6, Proposition 3.7).

Lecture 8 (Sep 30). The notion of P-primary and P-coprimary modules. Proof of Proposition 3.9, which in particular connects the new definition to the one we gave in the case of ideals in Lecture 6. Statement and start of the proof of the primary decomposition theorem. Irreducible ideals and proof that every ideal is an intersection of irreducibles.

[1]Section 2.3 is essentially the content of Problem 3 in PSet 1. We will skip Section 2.4 for now, and come back to it when we discuss dimension.

Lecture 9 (Oct 2). Proof of the primary decomposition theorem. Statement of the compatibility with localization. Relation of the primary decomposition with unique factorization domains.

Lecture 10 (Oct 7). Start of Chapter 4 in [2]. The definitions of integral elements, integral extensions and normalization. Examples. Test 1.

Lecture 11 (Oct 9). The Cayley–Hamilton theorem (Theorem 4.3). As a consequence, the 3 versions of Nakayama's lemma (Corollary 4.7, Corollary 4.8 a) and b)); introduction of the Jacobson radical. Finite and finite type extensions and proof that "finite" is equivalent to "finite type and integral" (Proposition 4.1, Corollary 4.5). Proof that the integral closure is a subalgebra (Theorem 4.2).

Lecture 12 (Oct 16). Examples of how integral maps look on Spec: $A(X) \subseteq A(X)[t]/(p(t))$, $\mathbb{Z} \subseteq \mathbb{Z}[i]$ (see the picture of the course on canvas!) and $\mathbb{Z} \to \mathbb{Z}/2$. The lying over and going up theorems (Theorem 4.15 [2] or Theorems 5.10/5.11 in [1]); I followed the proof in Atiyah–Macdonald which uses Problem 4 in PSet 5 instead of Nakayama. The incomparability of primes on the fibers (Corollary 4.18 in [2]). Definition of the residue field.

Lecture 13 (Oct 21). The fibers of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ as corresponding to primes on $S \otimes_R \kappa(P)$. Fibers of a finite map are finite; the proof uses something about Artinian rings that we'll see only later. Jacobson rings, definition and examples. The characterization of Jacobson rings in Lemma 4.20 of [2]. Statement of the "fancy" Nullstellensatz (Theorem 4.19).

Lecture 14 (Oct 23). Proof of the "fancy" Nullstellensatz. Zariski's lemma as an immediate corollary. Proof of the "standard" Nullstellensatz for finitely generated algebras over algebraically closed fields (Corollary 1.9 and Theorem 1.6). Beginning of Chapter 6 on flatness: an introduction to Tor functors.

Lecture 15 (Oct 28). Flat modules preserve kernels, images and arbitrary exact sequences. The characterization of flatness in terms of Tor (Proposition 6.1). Examples of flat algebras and some geometric intuition. Statement of the going down for flat algebras (Lemma 10.11). Definition of faithfully flat morphisms.[2]

Lecture 16 (Oct 30). The characterization of faithfully flat morphisms [4, Lemma 00HP]; in particular, faithfully flat morphisms induce surjective maps on Spec. Proof of going down for flat morphisms. Beginning of Chapter 5 in Eisenbud. Definitions of graded rings, graded modules and filtrations. The I-adic filtration, I-filtrations and I-stable filtrarions. Statement of the Artin-Rees lemma (Lemma 5.1). Definition of the associated graded algebra or module and statement of Proposition 5.3.

Lecture 17 (Nov 4). Definition of the blow-up algebra and blow-up module. Proof that a filtration is I-stable is equivalent to the blow-up module being finitely generated over the blow-up algebra. Proof of Artin-Rees. Statement and proof of Krull intersection theorem (Corollary 5.4). Test 2.

Lecture 18 (Nov 6). Motivating examples where "Zariski local" and "analytic local" disagree. Definition of completion of a ring. The examples of power series and p-adic numbers. Cauchy sequences and convergence. Proof that completion at a maximal ideal is a local ring. Statement of Hensel's lemma.

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[2]Faithfully flat morphisms are not covered in Eisenbud, and the proof he gives of going down is different. I mostly followed [4, Section 00H9] here.

Lecture 19 (Nov 13). Completion of modules and the universal property of completion/inverse limits. Proof of Hensel's lemma. An arithmetic application to lifting cubic roots mod p to cubic roots mod p^k. Important properties of the completion of Noetherian rings (Theorems 7.1 and 7.2): the completion is complete, flat and Noetherian, and completing a module is the same as tensoring by the completion. We showed 7.1 b), 7.2 a), 7.2 b) and Lemma 7.15 modulo a few details, but 7.1 a) (Noetherian) was only sketched.

Lecture 20 (Nov 18). Start of dimension theory (Chapters 8 and 9 in Eisenbud). Definition of dimension of a ring, dimension and codimension of ideals. Examples. Quotienting by the nilradical does not change dimension. Integral extensions do not change dimension. Examples (without proof) of a Noetherian ring with infinite dimension and a non-Noetherian ring with finite dimension.

Lecture 21 (Nov 20). Artinian rings (Section 2.4 or [1, Chapters 6 and 8]). Definition and examples of Artinian rings and modules. Composition series and the length of a module. Proof that all composition series have the same length [1, Proposition 6.8] and that a module has finite length iff it is Artinian and Noetherian [1, Proposition 6.9] (you can also see [2, Theorem 2.13], but I followed Atiyah-Macdonald). Proofs of Proposition 8.1-8.4 in [1] (if R is Artinian then $\mathrm{Spec}(R) = \mathrm{Spec}_m(R)$ is finite and $J(R)^N = 0$ for some N).

Lecture 22 (Nov 25). Proof that if R is Noetherian of dimension 0 then $\mathrm{Spec}(R) = \mathrm{Spec}_m(R)$ is finite and $J(R)^N = 0$ for some N. Artinian rings are product of local Artinian rings [1, Theorem 8.7]. Proof that Artinian is equivalent to Noetherian of dimension 0 [1, Corollary 6.11, Theorem 8.5]. Statement of fundamental theorem of dimension theory $\dim(R) = \delta(R) = d(R)$ [1, Theorem 11.14] or [**?**, Theorem 12.1] Examples: $k[x_1, \ldots, x_n]$, R Artinian and $R = k[x, y]/(y^2 - x^3)$. Definition of $\delta(R)$. Statement of the polynomiality of Hilbert-Samuel functions [1, Corollary 11.2] or [2, Proposition 12.2] (see also [2, Theorem 1.11] for a baby case).

Lecture 23 (Nov 27). Proof of the polynomiality of Hilbert-Samuel functions. Definition of $d(R)$. Start of the proof of the fundamental theorem of dimension theory (Propositions 11.6, 11.7, 11.8, 11.10, 11.13 in [1]).

Lecture 24 (Dec 2). Continuation of the proof of the fundamental theorem of dimension theory. The Zariski tangent space. Definition of regular rings and some basic properties.

Lecture 25 (Dec 4). Consequences of the fundamental theorem of dimension theory: Krull principal ideal and height theorems; the equality $\dim R[x] = \dim R + 1$; the dimension of fibers.

Lecture 26 (Dec 9). The (co)dimension 1 case: DVRs and Dedekind domains. Serre criterion for normality. Factorization into prime ideals in Dedekind domains. (in these last two lectures most proofs were skipped/very roughly sketched)

Lecture 27 (Dec 11). Noether normalization and some consequences. A sketch of the proof of Noether's normalization. Going-down for integral normal extensions.

References

[1] Michael Atiyah and Ian G. Macdonald, Introduction To Commutative Algebra (1994). Westview Press.

[2] David Eisenbud, Commutative algebra with a view toward algebraic geometry (1995). Grad. Texts in Math. 150, Springer-Verlag, New York.

[3] Miles Reid, Undergraduate commutative algebra (1995). Cambridge University Press, Cambridge.

[4] The stacks project authors, The Stacks project, available at https://stacks.math.columbia.edu, 2024.