COMMUTATIVE ALGEBRA - PSET 9

Problem 1. (Chinese remainder theorem[1]) (5 points) We say that two ideals I, J are coprime if $I + J = R$.

(1) Show that if I and J are coprime then
$$R/(I \cap J) \simeq R/I \oplus R/J \,.$$
(2) Show that if I and J are coprime then $I \cap J = IJ$.
(3) Show that if I is coprime to J, J' then I is also coprime to JJ'.
(4) Use the previous parts to show by induction that if $I_1, \ldots, I_n$ are pairwise coprime ideals then
$$R/\left(I_1 \cap \ldots \cap I_n\right) \simeq R/I_1 \times \ldots \times R/I_n \,.$$
and
$$I_1 \cap \ldots \cap I_n = I_1 \ldots I_n \,.$$

Problem 2. (prime avoidance) (5 points) Let $n \geq 2$ and $I, P_1, \ldots, P_n \subseteq R$ be ideals such that $P_3, P_4, \ldots, P_n$ are prime. Show that if I is contained in the union of $P_1 \cup \ldots \cup P_n$ then it is contained in at least one of the P_i.[2]

Problem 3. (5 points)

(1) Let $x \in R$ and $P \in \operatorname{Spec}(R)$ be a finitely generated prime such that $P \subsetneq (x)$. Show that $xP = P$ and use Nakayama to conclude that P is a minimal prime.
(2) Show that a principal ideal domain has dimension 0 or 1.

Problem 4. (5 points) Let k be an algebraically closed field and $R = k[x, y]$. Let $P \in \operatorname{Spec}(R)$.

(1) Show that if $P \cap k[x] \neq 0$ then either $P = (x - a)$ or $P = (x - a, y - b)$ for some $a, b \in k$.
(2) Suppose now that $P \cap k[x] = 0$. Let $U = k[x] \setminus \{0\} \subseteq k[x, y]$ and $B = k(x)$. Then
$$P[U^{-1}] \subseteq B[y] \,.$$
Use the fact that $B[y]$ is a principal ideal domain (and R is a UFD) to conclude that P is principal.
(3) Show that the dimension of R is 2 and that
$$\dim(P) + \operatorname{codim}(P) = 2$$
for every prime P. Give examples of primes with dimensions 0,1 and 2.

[1] Do you see why this is called the Chinese remainder theorem?

[2] Try to prove the contrapositive by induction on n. This problem looks easier than it is!