COMMUTATIVE ALGEBRA - PSET 7

Problem 1. (5 points) Given a topological space X, let X_0 be the set of closed points, i.e.

$$X_0 = \{x \in X : \{x\} \text{ is closed}\}.$$

The space X is said to be Jacobson if for every closed subset C in X we have[1]

$$\overline{C \cap X_0} = C\,.$$

(1) Show that if $X = \operatorname{Spec}(R)$ then $X_0 = \operatorname{Spec}_{\mathfrak{m}}(R)$. Moreover, show that the closure of $Z(I) \cap \operatorname{Spec}_{\mathfrak{m}}(R)$ is $Z(J)$ where $J = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$.
(2) Show that R is Jacobson if and only if $\operatorname{Spec}(R)$ is Jacobson.

Problem 2. (5+1 points)[2] In this exercise we will show that maximal ideals in $\mathbb{R}[x_1, \dots x_n]$ are in bijection with pairs of complex conjugate points on $\mathbb{A}^n_{\mathbb{C}}$ using the general version of Nullstellensatz for Jacobson rings. Given $a = (a_1, \dots, a_n) \in \mathbb{A}^n_{\mathbb{C}}$ we denote $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n) \in \mathbb{A}^n_{\mathbb{C}}$. Let $\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n)$ be the corresponding maximal ideal in $\mathbb{C}[x_1, \dots, x_n]$. Let $R = \mathbb{R}[x_1, \dots, x_n]$ and $S = \mathbb{C}[x_1, \dots, x_n]$.

(1) Explain why the map

$$\mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec}_{\mathfrak{m}}(S) \to \operatorname{Spec}_{\mathfrak{m}}(R) \tag{1}$$

is well-defined.
(2) Show that if $\mathfrak{m}$ is a maximal ideal in R then $R/\mathfrak{m}$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. Use that to conclude that (1) is surjective.
(3) Show that

$$\mathfrak{m}_a \cap R = \mathfrak{m}_{\bar{a}} \cap R.$$

(4) Conversely, suppose that $\mathfrak{m}_a \cap R = \mathfrak{m}_b \cap R$. Show that, for each i, b_i is either a_i or $\bar{a}_i$. Show a similar statement for $a_i + a_j$, $b_i + b_j$, and use that to conclude that $b = a$ or $b = \bar{a}$.

Problem 3. (5 points) Let $R = k[x, y, z]$, $M = R/(x, y)$ and $N = R/(x, z)$. Find a free resolution of N as a R-module and use it to compute $\operatorname{Tor}_i^R(M, N)$ for every $i \geq 0$.

Problem 4. (5 points) Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finitely generated R-module. Show that if $\operatorname{Tor}_1^R(R/\mathfrak{m}, M) = 0$ then M is free.[3]

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[1] The bar means closure. The closure $\overline{A}$ of a subset A of X is the smallest closed subset containing A.

[2] You can get one extra point by proving the following more general statement instead: if k is a field such that its algebraic closure $\bar{k}$ is a Galois extension, then maximal ideals in $k[x_1, \dots, x_n]$ are in 1:1 correspondence with orbits of the Galois action of $\operatorname{Gal}(\bar{k}/k)$ on $\mathbb{A}^n_{\bar{k}}$. If you show this general version you get an extra point. You'll need some Galois theory for this general version.

[3] Hint: Nakayama III. A consequence of this problem is that if M is finitely generated over a Noetherian ring, then M being flat is equivalent to M being locally free (i.e. M_P free for all P). Locally free modules are important in algebraic geometry: they correspond to vector bundles!