COMMUTATIVE ALGEBRA - PSET 6

Problem 1. (8 points) Recall the definition of the Zariski topology in Problem 4 of PSet 1. Recall also that a continuous map $f: X \to Y$ between topological spaces is called closed if for every closed set $C \subseteq X$ its image $f(C) \subseteq Y$ is also closed. The goal of this exercise is to show that a ring homomorphism $\varphi: R \to S$ satisfies the going-up property if and only if the induced map $\varphi^{\#}: \operatorname{Spec} S \to \operatorname{Spec} R$ is closed.

(1) Start by proving the easy implication "closed implies going up".

(2) Show that if $\varphi: R \to S$ is any injective ring homomorphism then all the minimal primes of R are in the image of $\varphi^{\#}$.[1] Conclude that if φ is injective and satisfies going up then $\varphi^{\#}$ is surjective.[2]

(3) Let $K = \ker \varphi$. Show that if φ satisfies going up then the induced map $R/K \to S$ also satisfies going up. Conclude that if φ satisfies going up then the image of $\varphi^{\#}$ is

$$\varphi^{\#}\big(\operatorname{Spec}(S)\big) = Z(K)\,.$$

(4) More generally, show that if φ satisfies going up then

$$\varphi^{\#}\big(Z(I)\big) = Z\big(\ker(R \to S/I)\big)\,,$$

for any ideal $I \subseteq S$, and hence $\varphi^{\#}$ is closed.

(5) Show that the projection $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ sending $(x, y) \mapsto x$ is not closed (in the Zariski topology) directly from the definition.

(6) Let k be a field. Show that the inclusion $k[x] \subseteq k[x, y]$ does not satisfy going up directly from the definition.

Problem 2. (6 points) Let R be a ring and $P \in \operatorname{Spec}(R)$. The residue field of P is defined to be

$$\kappa(P) = \operatorname{Frac}(R/P)\,.$$

There is a canonical map

$$R \to R/P \to \kappa(P)\,.$$

(1) Show that[3]

$$\kappa(P) \simeq R_P/P_P\,.$$

(2) Use right exactness of the tensor product and (1) to prove that if S is a R-algebra then

$$S \otimes_R \kappa(P) \simeq S_P/PS_P\,.$$

(3) Show that the image the canonical map $\operatorname{Spec}(\kappa(P)) \to \operatorname{Spec}(R)$ is the singleton $\{P\}$.

(4) Show that if $\varphi: R \to S$ is a ring homomorphism and $Q \in \operatorname{Spec}(S)$ lies over $P \in \operatorname{Spec}(R)$ then $\kappa(Q)$ is a field extension of $\kappa(P)$.

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[1] Use the idea in the proof of "lying over".

[2] In class, we showed going up from lying over. This exercise is saying that going up implies lying over.

[3] I am being purposely vague about the category in which these objects are isomorphic because I don't want you to worry too much about writing justifications for boring things like "this map respects products". The same applies to 2.2 and 3.2. In this case it is an isomorphism of R-algebras, and in 2.2 and 3.2 it's an isomorphism of S-algebras.

Problem 3. (6 points) Let $\varphi\colon R \to S$ be a ring homomorphism, making S a R-algebra, and let P be a prime ideal of R.

(1) Show that the map $\mathrm{Spec}(S/PS) \to \mathrm{Spec}(S)$ is an injection whose image is the set of primes Q such that $\varphi^{-1}(Q) \supseteq P$.

(2) Show that
$$S_P \simeq S[\varphi(R \setminus P)^{-1}]\,,$$
and use it to prove that the map $\mathrm{Spec}(S_P) \to \mathrm{Spec}(S)$ is an injection whose image is the set of primes Q such that $\varphi^{-1}(Q) \subseteq P$.

(3) Combine (1) and (2) to show that primes lying over P are in bijection with
$$\mathrm{Spec}(S \otimes_R \kappa(P))\,.$$