COMMUTATIVE ALGEBRA - PSET 5

Problem 1. (5 points) Let $R \subseteq S \subseteq S'$ be subrings.

(1) Show that if R is integrally closed in S and S is integrally closed in S' then R is integrally closed in S'.
(2) Show that if S is integral over R and S' is integral over S then S' is integral over R.
(3) Show that if S is integral over R and S is integrally closed in S' then S is the integral closure of R in S'.

Problem 2. (4 points) Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with integer coefficients which factors as $f(x) = g(x)h(x)$ with $g(x), h(x) \in \mathbb{Q}[x]$ both monic. Show that $g(x), h(x) \in \mathbb{Z}[x]$ by considering the algebra $\mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ generated by the roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ of f and arguing that the coefficients of g, h are integral over $\mathbb{Z}$.[1]

Problem 3. (5 points) Let $p \in \mathbb{Z}_+$ be a prime number (or more generally a square-free integer $\neq -1, 0, 1$). Recall that the ring of fractions of $\mathbb{Z}[\sqrt{p}]$ is $\mathbb{Q}[\sqrt{p}]$ (why?). Show that the normalization of $\mathbb{Z}[\sqrt{p}]$ is

$$\begin{cases} \mathbb{Z}[\sqrt{p}] & \text{if } p \not\equiv 1 \mod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{p}}{2}\right] & \text{if } p \equiv 1 \mod 4\,. \end{cases}$$

Problem 4. (4 points) Let $R \subseteq S$ be domains such that S is integral over R. Show that S is a field if and only if R is a field.

Problem 5. (2 points) Let $R = k[x]_{(x)}$ and $\mathfrak{m}$ be the maximal ideal of R. Give an example of a module $M \neq 0$ (that must be infinitely generated) such that $\mathfrak{m}M = M$.

[1] This fact is also a consequence of Gauss' lemma, but you are not allowed to use it here!