COMMUTATIVE ALGEBRA - PSET 4

Problem 1. (4 points) Show that if $I \subseteq R$ is a radical ideal then every prime associated to R/I is a minimal prime over I (in other words, R/I has no embedded primes). Give an example of an ideal that is not radical and has no embedded primes.

Problem 2. (7 points) This exercise will explain how the primary decomposition works when there is factorization in primes. It also proves that on a unique factorization domains the minimal primes over principal ideals are principal, which we skipped in class. Throughout, let R be a domain.

(1) Suppose that $p \in R$ is prime (i.e. the principal ideal (p) is prime). Show that (p^e) is (p)-primary by induction on e.
(2) Suppose that $r = p^e \cdot g$ for $g \notin (p)$. Prove that $(r) = (p^e) \cap (g)$.
(3) Suppose that $r \in R$ can be written as $r = p_1^{e_1} \dots p_k^{e_k}$ where $p_1, \dots, p_k$ are all prime. Show that the prime ideals minimal over (r) are precisely $(p_1), \dots, (p_k)$.
(4) Let r be as above. Show that
$$(r) = (p_1^{e_1}) \cap \dots (p_k^{e_k})$$
is a minimal primary decomposition (which is actually unique, since the (p_i) are all minimal).

Problem 3. (9 points) Let $R = k[x, y, z]$ and $I = (x^2, xy^2, y^2z, xz)$.[1]

(1) What are the (two) minimal primes containing I? Write both as the annihilator of some element of R/I.
(2) Find an embedded prime of R/I and write it as the annihilator of some element of R/I.
(3) Describe the kernel of the R-module homomorphisms
$$R/I \to (R/I)_P$$
where P are the minimal primes that you found in (1). (You don't need to write down the details for why your answer is the correct one)
(4) The previous part gives you two primary ideals Q_1, Q_2 that appear in the primary decomposition of I. Find a third ideal Q_3 such that
$$I = Q_1 \cap Q_2 \cap Q_3$$
and whose radical is the embedded prime you found in (2).[2]

Explain why Q_3 is primary and why this is a minimal primary decomposition.
(5) Find a different primary decomposition, i.e. a different Q_3' such that we still have
$$I = Q_1 \cap Q_2 \cap Q_3'.$$

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[1] Can you draw a picture of $\mathrm{Spec}(R/I)$ that includes the "fuziness"? That might help you guessing the answers, but it is not required.

[2] Here's a fact that you can use freely and that might help: if I_1 is an ideal generated by monomials $x^{a_i}y^{b_i}z^{c_i}$ and I_2 is generated by $x^{a_i'}y^{b_i'}z^{c_i'}$ then $I_1 \cap I_2$ is generated by $x^{\max(a_i,a_j')}y^{\max(b_i,b_j')}z^{\max(c_i,c_j')}$ for all possible i, j.