COMMUTATIVE ALGEBRA - PROBLEM SET 2

Problem 1. (4 points) A topological space X is said to be disconnected if we can write $X = X_1 \cup X_2$ with X_1, X_2 disjoint closed sets (in particular, an irreducible set is always connected). An element e of a ring R is called idempotent if $e^2 = e$. Show that R has a non-trivial ($e \neq 0, 1$) idempotent if and only if $\operatorname{Spec} R$ is disconnected. You might want to try the following steps:

(1) Start by noting that having a non-trivial idempotent is the same as having $e_1, e_2 \in R$ such that $e_1 + e_2 = 1$, $e_1 e_2 = 0$ and $\{e_1, e_2\} \neq \{0, 1\}$.
(2) To prove the "only if" part, take $X_1 = Z(\{e_1\})$ and $X_2 = Z(\{e_2\})$ and show that $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$.
(3) For the other implication, assume that $X = X_1 \cup X_2$ with $X_j = Z(I_j)$ disjoint closed subsets. Prove that, for any $r_1 \in I_1$ and $r_2 \in I_2$, the element $r_1 r_2$ is in every prime ideal of R, and hence it is nilpotent.
(4) Show that there are $r_1 \in I_1$, $r_2 \in I_2$ such that $r_1 + r_2 = 1$. Use that $(r_1 + r_2)^N = 1$ for every $N > 0$ and part (2) to construct e_1, e_2.

Problem 2. (4 points) Let $n, m \geq 1$ and k be a field. Describe as explicitly as possible the following modules:

(1) $k[x] \otimes_k k[x]$
(2) $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}/(m))$;
(3) $\mathbb{Z}/(n) \otimes_{\mathbb{Z}} \mathbb{Z}/(m)$;
(4) $k[x]/(x^n) \otimes_{k[x]} k[x]/(x^m)$.

You might use the exactness properties from the next problem.

Problem 3. (4 points) In this exercise we will prove the exactness properties of Hom and $\otimes$.

(1) Let
$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{1}$$
be an exact sequence of R-modules, and let M be another R-module. Show that
$$0 \to \operatorname{Hom}_R(C, M) \xrightarrow{\beta^*} \operatorname{Hom}_R(B, M) \xrightarrow{\alpha^*} \operatorname{Hom}_R(A, M) \tag{2}$$
is exact.
(2) Conversely, show that if (2) is exact for every M then (1) is exact.
(3) Let M, N, Q be R-modules. Show that there is an isomorphism
$$\operatorname{Hom}_R(N \otimes_R Q, M) \simeq \operatorname{Hom}_R(N, \operatorname{Hom}_R(Q, M))$$
of R-modules.[1]
(4) Using all the previous parts, and also the other exact sequence for Hom which we have seen in class (its proof is very similar to (1)), conclude that if (1) is exact and N is a R-module then
$$N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$$

[1] In a categorical language, this means that $N \otimes -$ and $\operatorname{Hom}_R(N, -)$ form a pair of adjoint functors. Adjoint functors appear all the time in mathematics!

is exact.[2]

Problem 4. (4 points) Our goal in this problem is to characterize the rings R between $\mathbb{Z}$ and $\mathbb{Q}$.

(1) Let $\mathcal{P}$ be a set of prime numbers. Let $U_{\mathcal{P}} \subseteq \mathbb{Z}$ be the subset of integers whose prime factors are all in $\mathcal{P}$ (for example, $U_{\{2\}} = \{\pm 1, \pm 2, \pm 4, \ldots\}$). Show that $U_{\mathcal{P}}$ is multiplicative and that $\mathbb{Z} \subseteq \mathbb{Z}[U_{\mathcal{P}}^{-1}] \subseteq \mathbb{Q}$.

(2) Give an example of a multiplicative subset of $\mathbb{Z}$ which is not of the form $U_{\mathcal{P}}$.

(3) Let R be a subring of $\mathbb{Q}$ that contains $\mathbb{Z}$. Show that if $x \in R$ and p divides the denominator of x (in its reduced form) then $1/p \in R$. Conclude that

$$R = \mathbb{Z}[U_{\mathcal{P}}^{-1}]$$

where

$$\mathcal{P} = \{p \colon 1/p \in R\}\,.$$

Problem 5. (4 points) Let R be a ring, M a R-module, and $f_1, \ldots, f_n \in R$ such that $f_1 + \ldots + f_n = 1$. Show the following:[3]

(1) If $m \in M$ goes to 0 via the map $M \to M[f_i^{-1}]$ for $i = 1, \ldots, n$, then $m = 0$.

(2) Suppose we have elements $m_i \in M[f_i^{-1}]$. Assume that for every i, j the images of m_i and m_j via the maps $M[f_i^{-1}] \to M[f_i^{-1} f_j^{-1}]$ are the same. Show that there is an element $m \in M$ that goes to m_i in $M[f_i^{-1}]$.

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[2] In fact, this is a general property of adjoint functors. If (F, G) is a pair of adjoint functions, then F is right exact and G is left exact! Even fancier: F commutes with colimits and G commutes with limits.

[3] Here's a geometric analog of this statement: suppose that X is a topological space which admits an open covering $X = U_1 \cup \ldots \cup U_n$; if $g \in C(X)$ is such that $g_{|U_i} = 0$ for every i, then $g = 0$. If we have functions g_i defined on each U_i that agree on the intersections $U_i \cap U_j$, then we can glue them to a function on X.