COMMUTATIVE ALGEBRA - PSET 10

Problem 1. (3 points) Let R be a k-algebra. Show that if R is Artinian and finite type over k then R is finite over k.[1]

Problem 2. (5 points) Let R be a Noetherian local ring.

(1) Let $x \in R$ be a non unit. Prove that[2]

$$\dim R/(x) \geq \dim R - 1\,. \tag{1}$$

(2) Let $x \in R$ be a non zero divisor and non unit. Conclude that

$$\dim R/(x) = \dim R - 1\,.$$

(3) Find an example of a non-local Noetherian ring where the inequality (1) fails.

Problem 3. (6 points) Let R be a Noetherian local ring. A regular sequence in R is a sequence $x_1, \ldots, x_n \in R$ such that $(x_1, \ldots, x_n) \neq R$ and x_i is not a zero-divisor in $R/(x_1, \ldots, x_{i-1})$ for $i = 1, 2, \ldots, n$.

(1) Show that if $x_1, \ldots, x_n$ is a regular sequence in R then $\dim R \geq n$.

The ring R is Cohen-Macaulay if there is some regular sequence in R for which equality in (1) holds.

(2) Suppose that R is of dimension 1. Show that R is Cohen-Macaulay if and only if R has no embedded primes.[3] Give an example of a ring that is not Cohen-Macaulay.

(3) Prove that if R is regular and $\mathfrak{m} = (x_1, \ldots, x_n)$ with $n = \dim R$ then $R/(x_1, \ldots, x_i)$ is regular of dimension $n - i$. Using the fact that a regular ring is a domain (Corollary 10.14 in Eisenbud) conclude that $x_1, \ldots, x_n$ is a regular sequence, and hence regular implies Cohen-Macaulay. Give an example of a ring that is Cohen-Macaulay but not regular.

Problem 4. (6 points) Let k be an algebraically closed field and $f \in k[x_1, \ldots, x_n]$, $f \neq 0$, be such that $f(0, \ldots, 0) = 0$. Let $R = k[x_1, \ldots, x_n]/(f)$ and $\mathfrak{m} = (\overline{x}_1, \ldots, \overline{x}_n)$.

(1) Show that $\dim R_{\mathfrak{m}} = n - 1$.

(2) Show that $R_{\mathfrak{m}}$ is regular if and only if not all of the derivatives $\partial f / \partial x_i$ vanish as $(0, \ldots, 0)$ (i.e. if $(0, \ldots, 0)$ is a non-singular point of $X = Z(f)$).

(3) Use (2) to show that if $R_{\mathfrak{m}}$ is regular then

$$\hat{R}_{\mathfrak{m}} \simeq k[\![y_1, \ldots, y_{n-1}]\!]\,.$$

---

[1] Hint: Start by reducing to the local case. Note that this is an extension of Zariski's lemma, so you might want to use it.

[2] Try to start with a $\mathfrak{m}/(x)$-primary ideal of $R/(x)$ generated by $\dim R/(x)$ elements.

[3] By PSet 4.1, it follows that if R is local, 1 dimensional and reduced then it is Cohen-Macaulay.