COMMUTATIVE ALGEBRA - PROBLEM SET 1

Problem 1. (5 points) Recall that a R-module M is finitely generated if there are finitely many elements $m_1, \ldots, m_N \in M$ such that

$$M = \{r_1 m_1 + \ldots r_N m_N \colon r_1, \ldots r_N \in R\}\,.$$

The R-module M is Noetherian if every submodule of M is finitely generated.

(1) Show that R is Noetherian as a ring if and only if R is Noetherian as an R-module.

(2) Suppose that we have a short exact sequence of R-modules

$$0 \to L \to M \to N \to 0\,.$$

Show that M is Noetherian if and only if L and N are.

(3) Suppose that R is a Noetherian ring. Use (2) to show that M is Noetherian if and only if it is finitely generated.

Problem 2. (5 points) Let R be a ring.

(1) Show that if R is Noetherian then every surjective homomorphism $\varphi\colon R \to R$ is also injective (and hence an isomorphism).[1]

(2) Give an example of a non Noetherian ring R and a surjective but non injective ring homomorphism $R \to R$.

(3) Give an example of a ring homomorphism $k[x] \to k[x]$ which is injective but not surjective.

Problem 3. (5 points) Let R be a ring. The nilradical of R is defined to be

$$\mathfrak{N} = \sqrt{(0)} = \{f \in R\colon f^n = 0 \text{ for some } n > 0\}\,.$$

(1) Suppose that $f \notin \mathfrak{N}$ and consider the (non-empty) collection of ideals

$$\Sigma = \{I \subseteq R\colon I \text{ is an ideal such that } f^n \notin I \text{ for all } n > 0\}\,.$$

Show that if P is maximal among the ideals in Σ (i.e. $P \in \Sigma$ and there is no $I \in \Sigma$ strictly containing P) then P is a prime ideal.

(2) Use the previous part of the problem to show that $\mathfrak{N}$ is the intersection of all the prime ideals of R.

(3) Deduce that for any ideal $J \subseteq R$ we have

$$\sqrt{J} = \bigcap_{\substack{P \supseteq J \\ P \text{ prime}}} P\,.$$

(4) If $R = k[x_1, \ldots, x_n]$ and k is algebraically closed, show that

$$\sqrt{J} = \bigcap_{\substack{\mathfrak{m} \supseteq J \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}\,.$$

[1]Hint: consider the ideals $\ker(\varphi^n)$. This is analogous to the well known statement in set theory that a surjection from a finite set to itself is automatically a bijection. Being Noetherian is a kind of finiteness condition.

Problem 4. (5 points) Let R be a ring and recall that $\mathsf{Spec}(R)$ is the set of prime ideals of R. Given a subset $I \subseteq R$, let

$$Z(I) = \{P \text{ prime} \colon P \supseteq I\} \subseteq \mathsf{Spec}(R)\,.$$

We say that any subset of the form $Z(I)$ is Zariski closed. A subset is called Zariski open if its complement is Zariski closed.[2]

(1) Show that the Zariski open/closed sets form a topology on $\mathsf{Spec}(R)$. I.e., show that $\emptyset, \mathsf{Spec}(R)$ are closed; a union of two closed subsets is closed; and an arbitrary intersection of closed subsets is closed.

(2) If $\varphi\colon S \to R$ is a homomorphism, show that $\varphi^{\#}\colon \mathsf{Spec}(R) \to \mathsf{Spec}(S)$ is a continuous function with respect to the Zariski topologies. I.e., show that the preimage of a closed subset of $\mathsf{Spec}(S)$ under $\varphi^{\#}$ is closed in $\mathsf{Spec}(R)$.

Bonus problem. (1 point)[3] Let X be a compact Hausdorff topological space. We will consider the $\mathbb{R}$-algebra $C(X)$ of continuous functions $f : X \to \mathbb{R}$ with the point-wise sum and point-wise product.

(1) Given $x \in X$, show that

$$\mathfrak{m}_x = \{f \in C(X) : f(x) = 0\}$$

is a maximal ideal of $C(X)$.

(2) Conversely, show that every maximal ideal of $C(X)$ is of the form $\mathfrak{m}_x$ for some $x \in X$. Conclude that $x \mapsto \mathfrak{m}_x$ is a bijection between points of X and the set $\mathrm{Spec}_m(C(X))$ of maximal ideals of $C(X)$.

(3) We consider the Zariski topology in $\mathsf{Spec}_m(C(X))$, which is generated by the base of open sets $\{U_f\}_{f \in C(X)}$ where

$$U_f = \{\mathfrak{m} \in \mathsf{Spec}_m(C(X)) : f \notin \mathfrak{m}\}.$$

Show that the bijection previously defined is a homeomorphism.

[2]Note that if we consider $R = A(X)$ and replace $\mathsf{Spec}(R)$ by $\mathsf{Spec}_m(R) = X$ then the definition of $Z(I)$ matches the one given in lecture. So there is a Zariski topology on the algebraic set X; the closed subsets are precisely the algebraic subsets of X.

[3]Note that the grade of this problem set is capped at 20, so you can get all the points without solving this bonus problem. This is more of a topology problem then an algebra problem (if you didn't take a course on topology, feel free to skip it). The point of this exercise is to illustrate that the principle "ring of functions" determines "geometric space" is literally true for topological spaces. You might want to recall Urysohn's lemma for the last part of the problem.