# Relations in degree d

```
In[ ]:= (*The symbol c[k,j] stands for c_k(j)*)
(*Properties of generators*)
c[k_, j_] := 0 /; k < 0
c[0, 1] = d;
c[0, 0] = 0;
```

```
In[ ]:= (*Normalization*)
c[1, 0] = 0;
c[1, 1] = 0;
```

## Truncated relations in C[T^+] with degree d - 1, d

```
In[ ]:= (*A[s,n] and B[s,n] are A_s, B_s defined in Proposition 2.6*)
A[s_, n_] := (-1) ^ (s + 2) c[s + 1, 0] + (-1) ^ (s + 1) (3 - n - chi / d) c[s, 1] +
    (-1) ^ s / (2 d ^ 2) ((n - 7 / 2) d + chi) ((n - 5 / 2) d + chi) c[s - 1, 2];
B[s_, n_] := (-1) ^ (s + 2) c[s + 1, 0] + (-1) ^ (s + 1) (2 - n - chi / d) c[s, 1] +
    (-1) ^ s / (2 d ^ 2) ((n - 5 / 2) d + chi) ((n - 3 / 2) d + chi) c[s - 1, 2];
```

```
In[ ]:= (*Given a partition m of l,
term9[m,n] is the corresponding term in the sum (9)*)
term9[m_, n_] :=
 (Times @@ Function[c, 1 / c[[2]] !] /@ Tally[m]) * Times @@ (Function[s,
      (s - 1) ! (A[s, n] - Sum[(-1) ^ i beta ^ i / i ! B[s - i, n], {i, 0, 2}])] /@ m)
```

```
In[ ]:= (*rule for truncating equations*)
trunc = c[j_, k_] /; (j + k - d < -1) → 0;
```

```
In[ ]:= (*rule to simplify factorials,
which mathematica does not do automatically*)
simpfactorials = {(d + 1) ! → (d + 1) d !, d ! → d * (d - 1) !,
    (d - 1) ! → (d - 2) ! (d - 1), (d - 3) ! → (d - 2) ! / (d - 2) };
```

```
In[ ]:= (*relsproduct1 are the (truncated) relations in degree l=
 d+1 in the product M x P^2 produced by Proposition 2.6*)
relsproduct1[n_] =
  Simplify[1 / (d - 3) ! Plus @@ Function[m, term9[m, n]] /@ {{d + 1},
        {d, 1}, {d - 1, 1, 1}, {d - 1, 2}, {d - 2, 2, 1},
        {d - 2, 3}, {d - 2, 1, 1, 1}} /. trunc //. simpfactorials];
```

In[∘]:=
```
(*relsproduct2 are the (truncated) relations in degree l=
 d+2 in the product M x P^2 produced by Proposition 2.6*)
relsproduct2[n_] =
  Simplify[1 / (d - 3) ! Plus @@ Function[m, term9[m, n]] /@ {{d + 2},
        {d + 1, 1}, {d, 1, 1}, {d, 2}, {d - 1, 2, 1}, {d - 1, 3},
        {d - 1, 1, 1, 1}, {d - 2, 4}, {d - 2, 3, 1}, {d - 2, 2, 2},
        {d - 2, 2, 1, 1}, {d - 2, 1, 1, 1, 1}} /. trunc //. simpfactorials];
```

In[∘]:=
```
(*relations (a) are relations in degree d-
  1 obtained by pushing forward relations in M x P^2 in degree l=d+1*)
relationa[n_] := Coefficient[relsproduct1[n], beta, 2];
```

In[∘]:=
```
(*relations (b) are relations in degree d obtained from relations in M x
  P^2 in degree l=d+1 by multiplying by beta and pushing forward*)
relationb[n_] := Coefficient[relsproduct1[n], beta, 1];
```

In[∘]:=
```
(*relations (c) produces relations in degree d by
  pushing forward relations in M x P^2 in degree l=d+2*)
relationc[n_] := Coefficient[relsproduct2[n], beta, 2];
```

## Verifying linear independence

In[∘]:=
```
(*Determinant appearing when expressing generators in degree d-
 1 in terms of lower generators*)
det1 = Factor[
  Det[Table[Coefficient[relationa[n], c[d - j, j]], {n, 1, 3}, {j, 0, 2}]]]
```

Out[∘]=

$$\frac{1}{4} (-1)^{3 d} \text{chi} (-2 + d)^4 (-1 + d) (-2 \text{chi} + d) (-\text{chi} + d)$$

In[∘]:=
```
(*Set of monomials Mon_2*)
monomials2 =
 Join[{c[d + 1, 0], c[d, 1], c[d - 1, 2]}, Table[c[d - 1, 0] × c[3 - i, i], {i, 0, 2}]]
```

Out[∘]=
```
{c[1 + d, 0], c[d, 1], c[-1 + d, 2],
 c[3, 0] × c[-1 + d, 0], c[2, 1] × c[-1 + d, 0], c[1, 2] × c[-1 + d, 0]}
```

In[ ]:=
```
(*Determinant of 6x6 matrix of coefficients
 of monomials2 in relations R_(b), R_(c)*)
eqs = Join @@ Table[{relationb[n], relationc[n]}, {n, 1, 3}];
det2 = Factor[
  Det[Table[Coefficient[eqs⟦i⟧, monomials2⟦j⟧], {i, 1, 6}, {j, 1, 6}]] //.
   {d! → d * (d - 1)!, (d - 1)! → (d - 2)! (d - 2), (d - 3)! → (d - 2)! / (d - 2)}]
```

Out[ ]=

$$4 (-1)^{6 d} (-2+d)^6 (-1+d)^3 d^4$$

## Truncated relations in C[T] with degree d - 1, d

In[ ]:=
```
(*Writes c[d,0],c[d-1,1], c[d-2,2] in terms of lower degree generators*)
sol1 = Simplify[Solve[relationa[1] == 0 && relationa[2] == 0 && relationa[3] == 0,
      {c[d, 0], c[d - 1, 1], c[d - 2, 2]}] //.
     {d! → d * (d - 1)!, (d - 2)! → (d - 1)! / (d - 1)}][[1]];
```

In[ ]:=
```
(*Writes c[d+1,0],c[d,1], c[d-1,2] in terms of lower degree generators*)
sol2 = Simplify[Solve[relationb[1] == 0 && relationc[1] == 0 && relationc[2] == 0,
     {c[d + 1, 0], c[d, 1], c[d - 1, 2]}]][[1]];
```

In[ ]:=
```
(*Basis of C[T]^2 x T^(d-2) ordered by <*)
basis = Join[Join @@ Table[c[d - i, i - 1] × c[4 - j, j - 1], {i, 1, 3}, {j, 1, 3}],
  Join @@ Table[{c[d - i, i - 1] × c[2, 0]^2,
     c[d - i, i - 1] × c[2, 0] × c[0, 2], c[d - i, i - 1] × c[0, 2]^2}, {i, 1, 3}]]
```

Out[ ]=

```
{c[3, 0] × c[-1 + d, 0], c[2, 1] × c[-1 + d, 0], c[1, 2] × c[-1 + d, 0],
 c[3, 0] × c[-2 + d, 1], c[2, 1] × c[-2 + d, 1], c[1, 2] × c[-2 + d, 1],
 c[3, 0] × c[-3 + d, 2], c[2, 1] × c[-3 + d, 2], c[1, 2] × c[-3 + d, 2],
 c[2, 0]^2 c[-1 + d, 0], c[0, 2] × c[2, 0] × c[-1 + d, 0], c[0, 2]^2 c[-1 + d, 0],
 c[2, 0]^2 c[-2 + d, 1], c[0, 2] × c[2, 0] × c[-2 + d, 1], c[0, 2]^2 c[-2 + d, 1],
 c[2, 0]^2 c[-3 + d, 2], c[0, 2] × c[2, 0] × c[-3 + d, 2], c[0, 2]^2 c[-3 + d, 2]}
```

In[ ]:=
```
(*Matrix in row-echelon form with the coefficients of the 3 relations R_1,
R_2, R_3. Obtained by plugging sol1, sol2 into the remaining 3 equations*)
TruncRelations = Together[Simplify[
    Factor[RowReduce[Outer[Function[{rel, mon}, Coefficient[rel, mon]],
       {relationb[2], relationb[3], relationc[3]} /. sol2 /. sol1, basis]]]]];
```

In[∘]:=
```
MatrixForm[TruncRelations]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{chi\,(chi-d)\,(2\,chi-d)\,(-2+d)}{8\,d^3} & \frac{-4+d}{8\,(-2+d)} & -\frac{chi\,(chi-d)\,(2\,chi-d)}{2\,(-2+d)\,d^3} & \frac{-48\,chi^4+96\,chi^3\,d+24\,chi^4\,d-\cdots}{} \\ 0 & 1 & 0 & 1 & 0 & \frac{-12\,chi^2+12\,chi\,d+6\,chi^2\,d-d^2-6\,chi\,d^2}{8\,d^2} & 0 & \frac{24\,chi^2-24\,chi\,d+2\,d^2-d^3}{8\,(-2+d)\,d^2} & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{2}{-2+d} & 0 & \end{pmatrix}$$

```
(* Relations[d,chi] is same as the above TruncRelations
 but also keeps track of the dependence on d and chi. *)
```

In[∘]:=
```
Relations[d_, chi_] := {{1, 0, 0, 0, 0, -1/(8 d^3) chi (chi - d) (2 chi - d) (-2 + d),
    (-4 + d)/(8 (-2 + d)), -((chi (chi - d) (2 chi - d)) / (2 (-2 + d) d^3)),
    1/(32 (-2 + d) d^4) (-48 chi^4 + 96 chi^3 d + 24 chi^4 d - 48 chi^2 d^2 -
       48 chi^3 d^2 + 18 chi^2 d^3 + 3 d^4 + 6 chi d^4 + 6 chi^2 d^4 - d^5 - 6 chi d^5),
    (-6 chi^2 + 6 chi d - 2 d^2 + d^3) / (2 chi (2 chi^2 - 3 chi d + d^2)),
    (12 chi^4 - 6 chi^2 d - 24 chi^3 d - 12 chi^4 d + 6 chi d^2 + 24 chi^2 d^2 + 24 chi^3 d^2 -
       12 chi d^3 - 30 chi^2 d^3 - 4 d^4 + 18 chi d^4 + 6 chi^2 d^4 + 4 d^5 - 6 chi d^5 - d^6) /
     (8 chi (chi - d) (2 chi - d) (-2 + d) d), (64 chi^6 - 192 chi^5 d - 128 chi^6 d +
       208 chi^4 d^2 + 384 chi^5 d^2 + 128 chi^6 d^2 - 96 chi^3 d^3 - 392 chi^4 d^3 - 384 chi^5 d^3 -
       64 chi^6 d^3 - 8 chi^2 d^4 + 144 chi^3 d^4 + 344 chi^4 d^4 + 192 chi^5 d^4 + 24 chi d^5 +
       94 chi^2 d^5 - 48 chi^3 d^5 - 160 chi^4 d^5 - 4 d^6 - 102 chi d^6 - 196 chi^2 d^6 + 12 d^7 +
       156 chi d^7 + 140 chi^2 d^7 - 9 d^8 - 108 chi d^8 - 24 chi^2 d^8 + 2 d^9 + 24 chi d^9) /
     (128 chi (chi - d) (2 chi - d) (-2 + d) d^4), (4 - d)/(4 (-2 + d) d), (4 - 15 d + 10 d^2 - 2 d^3)/(16 (-2 + d) d),
    1/(256 (-2 + d) d^5) (192 chi^4 - 384 chi^3 d - 240 chi^4 d + 192 chi^2 d^2 + 480 chi^3 d^2 +
       96 chi^4 d^2 - 96 chi^2 d^3 - 192 chi^3 d^3 - 12 d^4 - 144 chi d^4 - 240 chi^2 d^4 + 31 d^5 +
       336 chi d^5 + 240 chi^2 d^5 - 20 d^6 - 240 chi d^6 - 48 chi^2 d^6 + 4 d^7 + 48 chi d^7),
    ((2 + d) (6 chi^2 - 6 chi d + 2 d^2 - d^3)) / (16 chi (chi - d) (2 chi - d) (-2 + d)),
    (-128 chi^6 + 384 chi^5 d + 160 chi^6 d - 464 chi^4 d^2 - 480 chi^5 d^2 - 80 chi^6 d^2 +
       288 chi^3 d^3 + 688 chi^4 d^3 + 240 chi^5 d^3 - 20 chi^2 d^4 - 576 chi^3 d^4 - 476 chi^4 d^4 -
       60 chi d^5 + 118 chi^2 d^5 + 552 chi^3 d^5 + 84 chi^4 d^5 + 16 d^6 + 90 chi d^6 -
       188 chi^2 d^6 - 168 chi^3 d^6 - 24 d^7 - 48 chi d^7 + 90 chi^2 d^7 + 16 d^8 - 6 chi d^8 -
       6 chi^2 d^8 - 6 d^9 + 6 chi d^9 + d^10) / (64 chi (chi - d) (2 chi - d) (-2 + d)^2 d^4),
    (-3072 chi^8 + 12 288 chi^7 d + 3840 chi^8 d - 18 688 chi^6 d^2 - 15 360 chi^7 d^2 -
       1152 chi^8 d^2 + 13 056 chi^5 d^3 + 21 312 chi^6 d^3 + 4608 chi^7 d^3 - 384 chi^8 d^3 -
       3904 chi^4 d^4 - 10 176 chi^5 d^4 - 1952 chi^6 d^4 + 1536 chi^7 d^4 + 384 chi^3 d^5 -
       816 chi^4 d^5 - 10 272 chi^5 d^5 - 6656 chi^6 d^5 - 208 chi^2 d^6 + 672 chi^3 d^6 +
       12 712 chi^4 d^6 + 14 592 chi^5 d^6 + 1280 chi^6 d^6 + 144 chi d^7 + 972 chi^2 d^7 -
       2928 chi^3 d^7 - 12 680 chi^4 d^7 - 3840 chi^5 d^7 - 24 d^8 - 444 chi d^8 - 1622 chi^2 d^8 +
       2832 chi^3 d^8 + 3680 chi^4 d^8 + 44 d^9 + 606 chi d^9 + 1180 chi^2 d^9 - 960 chi^3 d^9 - 34 d^10 -
```

```
    420 chi d^10 - 316 chi^2 d^10 + 13 d^11 + 156 chi d^11 + 24 chi^2 d^11 - 2 d^12 - 24 chi d^12) /
   (1024 chi (chi - d) (2 chi - d) (-2 + d)^2 d^6)},
{0, 1, 0, 1, 0, 1/(8 d^2) (-12 chi^2 + 12 chi d + 6 chi^2 d - d^2 - 6 chi d^2),
 0, (24 chi^2 - 24 chi d + 2 d^2 - d^3) / (8 (-2 + d) d^2),
 -((chi (chi - d) (2 chi - d) (-16 + 8 d + d^2)) / (8 (-2 + d) d^3)),
 (3 (-1 + d) d)/(chi (chi - d) (-2 + d)),
 (4 chi^2 - 4 chi d - 14 chi^2 d + 14 chi d^2 + 16 chi^2 d^2 + 3 d^3 -
    16 chi d^3 - 2 chi^2 d^3 - 3 d^4 + 2 chi d^4) / (4 chi (chi - d) (-2 + d) d),
 ((-1 + d) (48 chi^4 - 96 chi^3 d - 48 chi^4 d + 48 chi^2 d^2 + 96 chi^3 d^2 + 48 chi^4 d^2 -
       36 chi^2 d^3 - 96 chi^3 d^3 - 3 d^4 - 12 chi d^4 - 4 chi^2 d^4 + 6 d^5 + 52 chi d^5 +
       8 chi^2 d^5 - 8 chi d^6)) / (64 chi (chi - d) (-2 + d) d^3), 0, 0,
 (chi (chi - d) (2 chi - d) (-8 + 10 d - d^2 - 4 d^3 + d^4)) / (8 (-2 + d) d^4),
 (-8 chi^2 + 8 chi d + 8 chi^2 d + 6 d^2 - 8 chi d^2 + 2 chi^2 d^2 - 3 d^3 - 2 chi d^3 - 3 d^4) /
   (8 chi (chi - d) (-2 + d)^2 d),
 (96 chi^4 - 192 chi^3 d - 144 chi^4 d + 72 chi^2 d^2 + 288 chi^3 d^2 + 96 chi^4 d^2 + 24 chi d^3 -
    60 chi^2 d^3 - 192 chi^3 d^3 - 12 d^4 - 84 chi d^4 + 44 chi^2 d^4 + 18 d^5 + 52 chi d^5 - 16 chi^2
     d^5 - 9 d^6 + 16 chi d^6 + 2 chi^2 d^6 + 3 d^7 - 2 chi d^7) / (32 chi (chi - d) (-2 + d)^2 d^3),
 (1920 chi^6 - 5760 chi^5 d - 2304 chi^6 d + 6240 chi^4 d^2 + 6912 chi^5 d^2 - 288 chi^6 d^2 -
    2880 chi^3 d^3 - 8016 chi^4 d^3 + 864 chi^5 d^3 + 1536 chi^6 d^3 + 552 chi^2 d^4 +
    4512 chi^3 d^4 + 576 chi^4 d^4 - 4608 chi^5 d^4 - 384 chi^6 d^4 - 72 chi d^5 - 1368 chi^2 d^5 -
    2592 chi^3 d^5 + 3936 chi^4 d^5 + 1152 chi^5 d^5 + 18 d^6 + 264 chi d^6 + 1774 chi^2 d^6 -
    192 chi^3 d^6 - 1200 chi^4 d^6 - 33 d^7 - 334 chi d^7 - 864 chi^2 d^7 + 480 chi^3 d^7 +
    21 d^8 + 192 chi d^8 + 28 chi^2 d^8 - 6 d^9 - 76 chi d^9 - 8 chi^2 d^9 + 8 chi d^10) /
   (512 chi (chi - d) (-2 + d)^2 d^5)}, {0, 0, 1, 0, 0, 0, 2/(-2 + d), 0,
 (12 chi^2 - 12 chi d - d^2)/(8 (-2 + d) d), (4 d^2) / (chi (2 chi^2 - 3 chi d + d^2)),
 (-6 chi^2 + 6 chi d + 6 chi^2 d - 6 chi d^2 - d^3) / (chi (chi - d) (2 chi - d)),
 -(((-1 + 2 d) (-12 chi^2 + 12 chi d + 12 chi^2 d - d^2 - 12 chi d^2)) /
    (16 chi (chi - d) (2 chi - d))), 0, (3 - d)/(-2 + d),
 ((-3 + d) (12 chi^2 - 12 chi d - 12 chi^2 d + d^2 + 12 chi d^2)) / (8 (-2 + d) d^2),
 -((d^2 (2 + d)) / (2 chi (chi - d) (2 chi - d) (-2 + d))),
 (-48 chi^4 + 96 chi^3 d + 72 chi^4 d - 36 chi^2 d^2 - 144 chi^3 d^2 - 12 chi d^3 +
    66 chi^2 d^3 + 4 d^4 + 6 chi d^4 - 6 chi^2 d^4 - 2 d^5 + 6 chi d^5 + d^6) /
   (8 chi (chi - d) (2 chi - d) (-2 + d) d^2), (128 chi^6 - 384 chi^5 d - 1344 chi^6 d +
    224 chi^4 d^2 + 4032 chi^5 d^2 + 1216 chi^6 d^2 + 192 chi^3 d^3 - 3792 chi^4 d^3 - 3648 chi^5 d^3 -
    232 chi^2 d^4 + 864 chi^3 d^4 + 3520 chi^4 d^4 + 72 chi d^5 + 372 chi^2 d^5 - 960 chi^3 d^5 -
    6 d^6 - 132 chi d^6 - 212 chi^2 d^6 + 5 d^7 + 84 chi d^7 + 24 chi^2 d^7 - 2 d^8 - 24 chi d^8) /
   (128 chi (chi - d) (2 chi - d) (-2 + d) d^4)}}
```

```
(* Put the Relations[d,chi] into the matrix form. TopRelations
 for T^(d-2)xT^(2) and ExtRelations for T^(d-2)xSym^2(T^1). *)
```

In[ ]:=
```
TopRelations[d_, chi_] := {{Relations[d, chi][[1, 1 ;; 3]],
    Relations[d, chi][[1, 4 ;; 6]], Relations[d, chi][[1, 7 ;; 9]]},
   {Relations[d, chi][[2, 1 ;; 3]],
    Relations[d, chi][[2, 4 ;; 6]], Relations[d, chi][[2, 7 ;; 9]]},
   {Relations[d, chi][[3, 1 ;; 3]],
    Relations[d, chi][[3, 4 ;; 6]], Relations[d, chi][[3, 7 ;; 9]]}}
```

In[ ]:=
```
ExtRelations[d_, chi_] := {{Relations[d, chi][[1, 10 ;; 12]],
    Relations[d, chi][[1, 13 ;; 15]], Relations[d, chi][[1, 16 ;; 18]]},
   {Relations[d, chi][[2, 10 ;; 12]],
    Relations[d, chi][[2, 13 ;; 15]], Relations[d, chi][[2, 16 ;; 18]]},
   {Relations[d, chi][[3, 10 ;; 12]],
    Relations[d, chi][[3, 13 ;; 15]], Relations[d, chi][[3, 16 ;; 18]]}}
```

# Proof of main theorem

In[ ]:=
```
(* Automorphism matrices; AutA for SL(T^{d-2}), AutB for SL(T^2),
AutU for Hom(T^2, Sym^2(T^1)) and AutV for GL(Sym^2(T^1)). *)
AutA := Table[a[i, j], {i, 1, 3}, {j, 1, 3}]
AutB := Table[b[i, j], {i, 1, 3}, {j, 1, 3}]
AutU := Table[u[i, j], {i, 1, 3}, {j, 1, 3}]
AutV := Table[v[i, j], {i, 1, 3}, {j, 1, 3}]
```

In[ ]:=
```
(* Shuffle matrix S which is obtained by AutA^tr(-)AutB. *)
S := Table[s[i, j], {i, 1, 3}, {j, 1, 3}]
```

## Find the shuffle matrix

In[ ]:=
```
(* Determinant map defined by three
 dimensional span(TopRelations[d,chi]). *)
DET[d_, chi_] := Simplify[Det[Sum[TopRelations[d, chi][[i]] * x[i], {i, 1, 3}]]]
```

In[∘]:=
```
DET[d, chi]
```
Out[∘]=

$$-\frac{1}{32\,(-2+d)\,d^6}\,\mathrm{chi}\,\left(2\,\mathrm{chi}^2-3\,\mathrm{chi}\,d+d^2\right)\left(4\,\mathrm{chi}^3\,(-2+d)\,x[1]^3-6\,\mathrm{chi}^2\,(-2+d)\,d\,x[1]^2\,(x[1]+4\,x[2])+2\,\mathrm{chi}\,(-2+d)\,d^2\,x[1]\left(x[1]^2+12\,x[1]\times x[2]+24\,x[2]^2\right)+d^3\,x[2]\left(\left(8-5\,d+d^2\right)x[1]^2-4\left(-16+8\,d+d^2\right)x[2]^2+8\,x[1]\,(-3\,(-2+d)\,x[2]+d\,x[3])\right)\right)$$

In[∘]:=
```
(* Cubic curve defined by DET[d,chi]
 has a unique nodal singularity at [0:0:1]. *)
Solve[DET[d, chi] == 0 && D[DET[d, chi], x[1]] == 0 &&
  D[DET[d, chi], x[2]] == 0 && D[DET[d, chi], x[3]] == 0 &&
  (x[1] == 1 ∨ x[2] == 1 ∨ x[3] == 1), {x[1], x[2], x[3]}, Complexes]
```
Out[∘]=
```
{{x[1] → 0, x[2] → 0, x[3] → 1}}
```

In[∘]:=
```
(* Solve the shuffle matrix S using the fact that it sends
 the determinant curve of (d,chi1) to that of (d,chi2). *)
f := CoefficientList[DET[d, chi1], {x[1], x[2], x[3]}, {4, 4, 4}]
g := CoefficientList[(DET[d, chi2] /.
     {x[1] → Sum[s[j, 1] * x[j], {j, 1, 3}], x[2] → Sum[s[j, 2] * x[j], {j, 1, 3}],
      x[3] → Sum[s[j, 3] * x[j], {j, 1, 3}]}), {x[1], x[2], x[3]}, {4, 4, 4}]
coeff[u_, v_, w_] := Simplify[(f - g)[[u + 1, v + 1, w + 1]]]
```

In[∘]:=
```
(* By the singularity consideration, we have the following. *)
s[3, 1] := 0
s[3, 2] := 0
```

In[∘]:=
```
(* The following equations allow us
 to divide the solutions into two cases. *)
coeff[0, 2, 1]
coeff[2, 0, 1]
coeff[1, 1, 1]
```
Out[∘]=

$$\frac{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)s[2,1]\times s[2,2]\times s[3,3]}{4\,(-2+d)\,d^2}$$

Out[∘]=

$$\frac{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)s[1,1]\times s[1,2]\times s[3,3]}{4\,(-2+d)\,d^2}$$

Out[∘]=

$$\frac{1}{4\,(-2+d)\,d^2}\left(-2\,\mathrm{chi1}^3+3\,\mathrm{chi1}^2\,d-\mathrm{chi1}\,d^2+\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)(s[1,2]\times s[2,1]+s[1,1]\times s[2,2])\,s[3,3]\right)$$

In[∘]:=
```
Solve[s[2, 1] × s[2, 2] × s[3, 3] == 0 && s[1, 1] × s[1, 2] × s[3, 3] == 0 &&
   (s[1, 2] × s[2, 1] + s[1, 1] × s[2, 2]) s[3, 3] ≠ 0,
  {s[1, 1], s[2, 2], s[3, 3], s[1, 2], s[2, 1]}]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=
```
{{s[1, 1] → 0, s[2, 2] → 0}, {s[1, 2] → 0, s[2, 1] → 0}}
```

## Type I solutions

In[∘]:=
```
(* We start with the first case,
namely s[1,1]=s[2,2]=0, which we call Type I. *)
s[1, 1] := 0
s[2, 2] := 0
```

In[∘]:=
```
coeff[0, 3, 0]
coeff[3, 0, 0]
coeff[1, 1, 1]
```

Out[∘]=

$$\frac{1}{16\,(-2+d)\,d^6}\left(4\,\mathrm{chi1}^3\,d^3\,(-16+8\,d+d^2)-6\,\mathrm{chi1}^2\,d^4\,(-16+8\,d+d^2)+2\,\mathrm{chi1}\,d^5\,(-16+8\,d+d^2)+\mathrm{chi2}^2\,(-2+d)\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)^2\,s[2,1]^3\right)$$

Out[∘]=

$$-\frac{1}{16\,d^6}\left(4\,\mathrm{chi1}^6-12\,\mathrm{chi1}^5\,d+13\,\mathrm{chi1}^4\,d^2-6\,\mathrm{chi1}^3\,d^3+\mathrm{chi1}^2\,d^4+\frac{2\,\mathrm{chi2}\,d^3\,(-16+8\,d+d^2)\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)\,s[1,2]^3}{-2+d}\right)$$

Out[∘]=

$$\frac{-2\,\mathrm{chi1}^3+3\,\mathrm{chi1}^2\,d-\mathrm{chi1}\,d^2+\mathrm{chi2}\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)\,s[1,2]\times s[2,1]\times s[3,3]}{4\,(-2+d)\,d^2}$$

In[∘]:=
```
(* The above three equations imply that s[3,3]^3=
 1. By scaling the matrices using a cubic root of unity,
if necessary, we may assume that s[3,3]=1. *)
s[3, 3] := 1
```

In[∘]:=
```
(* From coeff[0,3,0]=0,
s[2,1] is given by a cubic root. From coeff[1,1,1]=0 and s[3,3]=1,
we also solve s[1,2]. *)
s[2, 1] := CubeRoot[-((2 chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2)) /
      (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))] * d
```

In[•]:=
```
Solve[coeff[1, 1, 1] == 0, s[1, 2]]
```

In[•]:=

$$s[1,2] := -\left(\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Big/\left(2^{1/3}\,\text{chi2}\,d\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\right)\right)$$

```
(* Solve the rest of S. We carefully check
 that the coefficients are nonzero before solving it. *)
```

In[•]:=
```
Coefficient[coeff[1, 2, 0], s[2, 3], 1]
Coefficient[coeff[2, 1, 0], s[1, 3], 1]
```

Out[•]=

$$\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{4\,(-2+d)\,d^2}$$

Out[•]=

$$\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{4\,(-2+d)\,d^2}$$

In[•]:=
```
Simplify[Solve[coeff[1, 2, 0] == 0 && coeff[2, 1, 0] == 0, {s[2, 3], s[1, 3]}]]
```

In[•]:=

$$s[2,3] := \frac{1}{8\,(2-d)\,d^2}\,(-2+d)\left(-48\,\text{chi1}\,(-2+d)-48\,d+24\,d^2+2^{1/3}\left(24\,\text{chi2}^2\,(-2+d)-24\,\text{chi2}\,(-2+d)\,d-d^2\left(8-5\,d+d^2\right)\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\right)$$

$$s[1,3] := \frac{1}{8\,d^3}\,(-2+d)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+\frac{d^2\left(8-5\,d+d^2\right)}{-2+d}+\left(24\times 2^{2/3}\,\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Big/\left(\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\right)-\left(12\times 2^{2/3}\,\text{chi1}\,d\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Big/\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\right)\right)$$

In[ ]:= MatrixForm[S]

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & -\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{2^{1/3}\,\text{chi2}\,d\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,\sqrt[3]{\frac{\text{chi1}\,(-16+8\,d+d^2)\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}^2\,(-2+d)\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)^2}}} \\ -2^{1/3}\,d\,\sqrt[3]{\frac{\text{chi1}\,(-16+8\,d+d^2)\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}^2\,(-2+d)\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)^2}} & 0 \\ 0 & 0 \end{pmatrix}$$

## Type I - Solve for A, B

```
(* We have solved the shuffle matrix S uniquely
  up to a choice of a cubic root in the case of s[1,1]=
 s[2,2]=0. We claim that there is no compatible pair (AutA,AutB) such that
   AutA^t(-)AutB corresponds to this shuffle matrix. To linearize the problem,
we use the inverse matrix AutB^(-1). *)
```

In[ ]:=
```
At := Transpose[AutA]
Binv := Table[bb[i, j], {i, 1, 3}, {j, 1, 3}]
Diff[d_, chi1_, chi2_] := Table[At.TopRelations[d, chi1]⟦i⟧ -
    Sum[S⟦i, j⟧ * TopRelations[d, chi2]⟦j⟧.Binv, {j, 1, 3}], {i, 1, 3}]
```

```
(* Use Diff⟦3⟧ first. *)
(* From now on, we skip checking that we are solving
 linear equations with nonzero leading coefficients,
unless it is nontrivial, for simplicity of the code. The
 solutions have denominators which are evidently nonzero. *)
```

In[ ]:=
```
Solve[Diff[d, chi1, chi2]⟦3⟧ == 0,
 {bb[3, 1], bb[3, 2], bb[3, 3], bb[1, 1], bb[1, 2], bb[1, 3], a[3, 2], a[1, 2]}]
```

In[∘]:=
```
bb[3, 1] := 2 a[3, 1] / (-2 + d)
bb[3, 2] := 0
bb[3, 3] := - 1/(8 (-2 + d) d)
    (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1])
bb[1, 1] := -(((12 chi2^2 - 12 chi2 d - d^2) a[3, 1]) / (8 (-2 + d) d)) + a[3, 3]
bb[1, 2] := 0
bb[1, 3] := 1/(128 (-2 + d) d^2) (12 chi2^2 - 12 chi2 d - d^2)
     (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1]) -
   1/(16 d) (16 d a[1, 3] - 8 d^2 a[1, 3] - 12 chi1^2 a[3, 3] + 12 chi1 d a[3, 3] + d^2 a[3, 3])
a[3, 2] := 0
a[1, 2] := 0
```

In[∘]:=
```
Simplify[Diff[d, chi1, chi2][[3]]]
```

Out[∘]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
(* Solve b[2,j] using Diff[[2]][3,j]. *)
```

In[∘]:=
```
Simplify[
 Solve[Diff[d, chi1, chi2][[2]][[3, 1]] == 0 && Diff[d, chi1, chi2][[2]][[3, 2]] == 0 &&
   Diff[d, chi1, chi2][[2]][[3, 3]] == 0, {bb[2, 1], bb[2, 2], bb[2, 3]}]]
```

In[∘]:=
```
bb[2, 1] :=
 ((-2 + d) d^2 (-16 a[2, 3] + 1/((-2 + d)^2 d^3) 2^(1/3) (-24 chi2^4 (-2 + d) + 48 chi2^3 (-2 + d) d +
          (-3 + d) d^4 + 6 chi2 (-1 + d) d^4 - 6 chi2^2 d^2 (-8 + 3 d + d^2))
       a[3, 1] ((chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2)) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3) -
      1/(2 × 2^(2/3) (-2 + d)^2) (-4 + d) (-12 chi2^2 a[3, 1] + 12 chi2 d a[3, 1] +
          d (-16 a[3, 3] + d (a[3, 1] + 8 a[3, 3])))
       ((chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2)) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3) -
      1/((2 - d) d^2) 4 a[3, 3] (48 chi1 (-2 + d) + 48 d - 24 d^2 -
          2^(1/3) (24 chi2^2 (-2 + d) - 24 chi2 (-2 + d) d - d^2 (8 - 5 d + d^2))
```

```
(chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3) )))) /
  (8 × 2^(1/3) chi2
   (chi2 - d) (-2 chi2 + d) (chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3) )
bb[2, 2] := (d^3 (8 a[1, 3] - a[3, 3]) +
     24 chi1^2 a[3, 3] - 24 chi1 d a[3, 3] + 2 d^2 (-8 a[1, 3] + a[3, 3])) /
   (4 × 2^(1/3) chi2 (chi2 - d) (2 chi2 - d) (chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3) )
bb[2, 3] := ((-2 + d) (32 d^2 (-6 chi1^2 (-2 + d) + 6 chi1 (-2 + d) d + d^2) a[2, 3] +
        1/(-2 + d) 32 chi1 (chi1 - d) (2 chi1 - d) d (-16 + 8 d + d^2) a[3, 3] +
        1/(-2 + d)^2 2^(1/3) (-24 chi2^4 (-2 + d) + 48 chi2^3 (-2 + d) d +
           (-3 + d) d^4 + 6 chi2 (-1 + d) d^4 - 6 chi2^2 d^2 (-8 + 3 d + d^2))
         (d^2 (8 a[1, 1] - a[3, 1]) + 12 chi1^2 a[3, 1] - 4 d (4 a[1, 1] + 3 chi1 a[3, 1]))
         (chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3) + 1/(2 × 2^(2/3) (-2 + d)^2) (4 - d) d^3
         (((12 chi2^2 - 12 chi2 d - d^2) (-12 chi1^2 a[3, 1] + d^2 (-8 a[1, 1] + a[3, 1]) +
               4 d (4 a[1, 1] + 3 chi1 a[3, 1])) + 8 (-2 + d) d (d^2 (8 a[1, 3] - a[3, 3]) +
               12 chi1^2 a[3, 3] - 4 d (4 a[1, 3] + 3 chi1 a[3, 3])))
         (chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3) +
        1/(2 - d) 4 d (-12 chi1^2 a[3, 3] + d^2 (-8 a[1, 3] + a[3, 3]) +
           4 d (4 a[1, 3] + 3 chi1 a[3, 3])) (48 chi1 (-2 + d) + 48 d - 24 d^2 -
           2^(1/3) (24 chi2^2 (-2 + d) - 24 chi2 (-2 + d) d - d^2 (8 - 5 d + d^2))
            (chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3) )))) /
  (128 × 2^(1/3) chi2 (chi2 - d) d^2 (-2 chi2 + d)
```

$$\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\Bigg)$$

```
(* Use the rest of Diff⟦2⟧. *)
```

In[ ]:=
```
Simplify[
 Solve[Diff[d, chi1, chi2]⟦2⟧⟦1, 1⟧ == 0 && Diff[d, chi1, chi2]⟦2⟧⟦2, 1⟧ == 0 &&
   Diff[d, chi1, chi2]⟦2⟧⟦1, 2⟧ == 0, {a[2, 1], a[2, 2], a[1, 1]}]]
```

In[ ]:=

$$a[2,1] := -2^{1/3}\,d\left(\left(\left(-12\,\text{chi2}^2+12\,\text{chi2}\,d+d^2\right)a[3,1]\right)\Big/\left(8\,(-2+d)\,d\right)+a[3,3]\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}+$$

$$\frac{1}{4\,(2-d)\,d^2}\,a[3,1]\Bigg(-48\,\text{chi1}\,(-2+d)-48\,d+24\,d^2+2^{1/3}\left(24\,\text{chi2}^2\,(-2+d)-24\,\text{chi2}\,(-2+d)\,d-d^2\left(8-5\,d+d^2\right)\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\Bigg)$$

$$a[2,2] := \frac{1}{2\times 2^{2/3}\,d^2}\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\,a[3,1]\,\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}$$

$$a[1,1] := \left(\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)a[3,1]\right)\Big/\left(8\,(-2+d)\,d^2\right)$$

```
Simplify[Solve[Diff[d, chi1, chi2]⟦2⟧⟦1, 3⟧ == 0, a[3, 1]]]
```

In[ ]:=

$$a[3,1] := \Bigg((-2+d)\,d^2\left(d^3\,(8\,a[1,3]-a[3,3])+24\,\text{chi1}^2\,a[3,3]-24\,\text{chi1}\,d\,a[3,3]+2\,d^2\,(-8\,a[1,3]+a[3,3])\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\Bigg)\Big/\left(2^{2/3}\,\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)$$

In[ ]:=
```
Simplify[Diff[d, chi1, chi2]⟦2⟧]
```

Out[ ]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
(* We would like to use Diff⟦1⟧ to solve a[1,3] and a[2,3]. We first check
 that the leading coefficients of the linear equations are nonzero. *)
```

In[ ]:=
```
Simplify[Coefficient[Diff[d, chi1, chi2][[1]][[2, 1]], a[1, 3], 1]]
Simplify[Coefficient[Diff[d, chi1, chi2][[1]][[1, 1]], a[2, 3], 1]]
```

Out[ ]=

$$\frac{(-2+d)\, d \left(-24\, \text{chi2}^2 + 24\, \text{chi2}\, d + (-2+d)\, d^2\right)}{2\, \text{chi2} \left(-16 + 8\, d + d^2\right) \left(2\, \text{chi2}^2 - 3\, \text{chi2}\, d + d^2\right)}$$

Out[ ]=

$$\frac{2^{1/3}\, \text{chi1}\, (-2+d)\, d \left(2\, \text{chi1}^2 - 3\, \text{chi1}\, d + d^2\right)}{\text{chi2}^2\, (\text{chi2} - d)^2\, (-2\, \text{chi2} + d)^2 \left(\sqrt[3]{\frac{\text{chi1} \left(-16+8\, d+d^2\right) \left(2\, \text{chi1}^2-3\, \text{chi1}\, d+d^2\right)}{\text{chi2}^2\, (-2+d) \left(2\, \text{chi2}^2-3\, \text{chi2}\, d+d^2\right)^2}}\right)^2}$$

In[ ]:=
```
(* Check that (-24 chi2^2+24 chi2 d+(-2+d) d^2) is nonzero
 if d is coprime to chi1 and chi2. Note that coprimality
 forces d to divide 24 for such solutions to exist. *)
Divlist := Divisors[24]

For[i = 1, i < Length[Divlist] + 1, i++,
 For[chi2 = 1, chi2 < Divlist[[i]] / 2 , chi2++,
  If[(
     (-24 chi2^2 + 24 chi2 d + (-2 + d) d^2) /. d → Divlist[[i]]) == 0,
   Print[{Divlist[[i]], chi1, chi2}]]
 ]
]

Clear[d, chi1, chi2]
```

(* Therefore we solve a[1,3] and a[2,3] using Diff[[1]][[2,1]] and Diff[[1]][[1,1]]. *)

In[ ]:=
```
Simplify[Solve[Diff[d, chi1, chi2][[1]][[2, 1]] == 0 &&
   Diff[d, chi1, chi2][[1]][[1, 1]] == 0, {a[1, 3], a[2, 3]}]]
```

In[∘]:=
```
a[1, 3] :=
 1/(8 (-2 + d) d^2) a[3, 3] (-24 chi1^2 + 24 chi1 d + (-2 + d) d^2 - (8 × 2^(2/3) chi1 (-16 + 8 d + d^2)
      (2 chi1^2 - 3 chi1 d + d^2)) / ((-24 chi2^2 + 24 chi2 d + (-2 + d) d^2)
      ((chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2))/(chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3)))
a[2, 3] := (a[3, 3]
   (24 chi1 (chi1 - d) (-2 + d) (-2 chi1 + d)^2 (-24 chi2^2 + 24 chi2 d + (-2 + d) d^2) +
     2^(1/3) chi1 (-2 + d) (2 chi1^2 - 3 chi1 d + d^2)
      (48 chi2^4 (4 + d) - 96 chi2^3 d (4 + d) + d^4 (8 - 6 d + d^2) - 6 chi2 d^3 (8 - 2 d + d^2) +
        6 chi2^2 d^2 (40 + 6 d + d^2)) ((chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2))/(chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(1/3) +
     8 × 2^(2/3) chi2^2 (-4 + d) (6 chi1^2 - 6 chi1 d + d^2) (2 chi2^2 - 3 chi2 d + d^2)^2
      ((chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2))/(chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))^(2/3))) /
  (4 chi1 (chi1 - d) (2 chi1 - d) (-2 + d) d^2 (-24 chi2^2 + 24 chi2 d + (-2 + d) d^2))
```

```
(* Conclude a[3,3]=0 from Diff[[1]][1,2]. *)
```

In[∘]:=
```
Simplify[Diff[d, chi1, chi2][[1]][[1, 2]]]
```

Out[∘]=

$$-\frac{2\,\text{chi1}\,(-4+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\,a[3,3]}{d\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)}$$

In[∘]:=
```
a[3, 3] := 0
```

```
(* This forces AutA and Binv to be zero,
hence a contradiction with Det[AutA]=Det[AutB]=1. *)
```

In[∘]:=
```
AutA
Binv
```

Out[∘]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Out[∘]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

## Type II

In[∘]:=
```
Clear[s, a, bb]
```

In[∘]:=
```
(* Now we consider the type II case,
namely s[1,2]=s[2,1]=0. Recall that we also had s[3,1]=
   s[3,2]=0 from the singulariy consideration. *)
s[3, 1] := 0
s[3, 2] := 0
s[1, 2] := 0
s[2, 1] := 0
```

```
(* We solve the matrix S similar to the previous case. Consider
 the following three equations involving s[1,1],s[2,2],s[3,3].*)
```

In[∘]:=
```
coeff[3, 0, 0]
coeff[0, 3, 0]
coeff[1, 1, 1]
```

Out[∘]=

$$\frac{1}{16\,d^6}\left(-4\,\text{chi1}^6+12\,\text{chi1}^5\,d-13\,\text{chi1}^4\,d^2+6\,\text{chi1}^3\,d^3-\text{chi1}^2\,d^4+\text{chi2}^2\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2 s[1,1]^3\right)$$

Out[∘]=

$$\frac{\left(-16+8\,d+d^2\right)\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)-\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)s[2,2]^3\right)}{8\,(-2+d)\,d^3}$$

Out[∘]=

$$\frac{-2\,\text{chi1}^3+3\,\text{chi1}^2\,d-\text{chi1}\,d^2+\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)s[1,1]\times s[2,2]\times s[3,3]}{4\,(-2+d)\,d^2}$$

```
(* The above three equations imply that s[3,3]^3=
 1. By scaling the matrices using a cubic root of unity,
if necessary, we may assume that s[3,3]=1. *)
```

In[∘]:=
```
s[3, 3] := 1
```

```
(* From coeff[0,3,0]=0,
s[2,2] is given by a cubic root. From coeff[1,1,1]=0 and s[3,3]=1,
we also solve s[1,1]=s[2,2]^2. *)
```

In[ ]:=
```
s[2, 2] := CubeRoot[(chi1 (2 chi1^2 - 3 chi1 d + d^2)) / (chi2 (2 chi2^2 - 3 chi2 d + d^2))]
s[1, 1] := s[2, 2]^2
```

```
(* Solve the rest of S. *)
```

In[ ]:=
```
Simplify[Solve[coeff[2, 1, 0] == 0 && coeff[1, 2, 0] == 0, {s[1, 3], s[2, 3]}]]
```

In[ ]:=

$$s[1,3] := \frac{1}{8 d^3}\left(-24\,\mathrm{chi1}^2(-2+d) + 24\,\mathrm{chi1}\,(-2+d)\,d + d^2\left(8-5d+d^2\right) + \left(24\,\mathrm{chi2}^2(-2+d) - 24\,\mathrm{chi2}\,(-2+d)\,d - d^2\left(8-5d+d^2\right)\right)\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}^{\,2}\right)$$

$$s[2,3] := \frac{1}{d^2}\,3\,(-2+d)\left(2\,\mathrm{chi1} - d + (-2\,\mathrm{chi2}+d)\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\right)$$

In[ ]:=
```
MatrixForm[S]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}^{\,2} & 0 & \frac{-24\,\mathrm{chi1}^2(-2+d)+24\,\mathrm{chi1}\,(-2+d)\,d+d^2\left(8-5d+d^2\right)+(24\,\mathrm{chi2}}{} \\ 0 & \sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}} & \frac{3\,(-2+d)\,\left(2\,\mathrm{chi1}-d+(-2\right.}{} \\ 0 & 0 & \end{pmatrix}$$

## Type II - Solve for A, B

```
(* We have solved the shuffle matrix S uniquely up to a choice
 of a cubic root for s[2,2]. We claim that for each S there is
 a unique pair (AutA,AutB) up to scaling such that AutA^t(-)AutB
 corresponds to this shuffle matrix. To linearize the problem,
we again use the inverse matrix AutB^(-1). *)
```

In[ ]:=
```
At := Transpose[AutA]
Binv := Table[bb[i, j], {i, 1, 3}, {j, 1, 3}]
Diff'[d_, chi1_, chi2_] := Table[At.TopRelations[d, chi1][[i]] -
    Sum[S[[i, j]] * TopRelations[d, chi2][[j]].Binv, {j, 1, 3}], {i, 1, 3}]
```

In[ ]:=
```
(* Use Diff'[[3]] first. *)
```

```
In[•]:= Solve[Diff'[d, chi1, chi2][[3]] == 0,
          {bb[1, 1], bb[1, 2], bb[1, 3], bb[3, 1], bb[3, 2], bb[3, 3], a[3, 2], a[1, 2]}]
```

```
In[•]:= bb[1, 1] := -(((12 chi2^2 - 12 chi2 d - d^2) a[3, 1]) / (8 (-2 + d) d)) + a[3, 3]
        bb[1, 2] := 0
        bb[1, 3] := 1/(128 (-2 + d) d^2) (12 chi2^2 - 12 chi2 d - d^2)
              (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1]) -
            1/(16 d) (16 d a[1, 3] - 8 d^2 a[1, 3] - 12 chi1^2 a[3, 3] + 12 chi1 d a[3, 3] + d^2 a[3, 3])
        bb[3, 1] := (2 a[3, 1])/(-2 + d)
        bb[3, 2] := 0
        bb[3, 3] := -1/(8 (-2 + d) d)
            (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1])
        a[3, 2] := 0
        a[1, 2] := 0
```

```
In[•]:= Simplify[Diff'[d, chi1, chi2][[3]]]
Out[•]=
        {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

        (* Solve b[2,j] using Diff'[[2]][1,j]. *)

In[•]:= Solve[Diff'[d, chi1, chi2][[2]][[1, 1]] == 0 && Diff'[d, chi1, chi2][[2]][[1, 2]] == 0 &&
          Diff'[d, chi1, chi2][[2]][[1, 3]] == 0, {bb[2, 1], bb[2, 2], bb[2, 3]}]
```

In[∘]:=
```
bb[2, 1] := (d^2 a[2, 1] - 12 chi1 a[3, 1] + 6 d a[3, 1] +
    12 chi2 a[3, 1] ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3) - 6 d a[3, 1]
     ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3)) / (d^2 ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3))
bb[2, 2] :=
  (-16 d^2 a[1, 1] + 8 d^3 a[1, 1] + 24 chi1^2 a[3, 1] - 24 chi1 d a[3, 1] + 2 d^2 a[3, 1] -
    d^3 a[3, 1]) / (8 (-2 + d) d^2 ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3))
bb[2, 3] := -((-(1/(8 d^2)) (-12 chi1^2 + (12 chi1 + 6 chi1^2) d + (-1 - 6 chi1) d^2) a[2, 1] +
      (chi1 (-2 chi1 + d) (-chi1 + d) (-16 + 8 d + d^2) a[3, 1]) / (8 (-2 + d) d^3) - 1/(8 d^3)
       3 (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1])
        (2 chi1 - d + (-2 chi2 + d) ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3))) /
    (((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3)))
```

```
(* Use the rest of Diff'[[2]]. *)
```

In[∘]:=
```
Simplify[
 Solve[Diff'[d, chi1, chi2][[2]][[2, 1]] == 0 && Diff'[d, chi1, chi2][[2]][[3, 2]] == 0 &&
   Diff'[d, chi1, chi2][[2]][[3, 1]] == 0, {a[2, 2], a[1, 3], a[2, 3]}]]
```

In[∘]:=

$$\text{a}[2,2] := \frac{1}{8\,(-2+d)\,d^2}\left(-24\,\text{chi2}^2\,\text{a}[3,1]+24\,\text{chi2}\,d\,\text{a}[3,1]+(-2+d)\,d^2\,(\text{a}[3,1]+8\,\text{a}[3,3])\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\text{a}[1,3] := \frac{1}{64\,(-2+d)^2\,d^4}\Big(\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2\right)\left(d^3\,(8\,\text{a}[1,1]-\text{a}[3,1])+24\,\text{chi1}^2\,\text{a}[3,1]-24\,\text{chi1}\,d\,\text{a}[3,1]+2\,d^2\,(-8\,\text{a}[1,1]+\text{a}[3,1])\right)+8\,(-2+d)\,d^2\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\text{a}[3,3]\Big)$$

$$\text{a}[2,3] := \frac{1}{d^2}\,6\,\text{a}[3,3]\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+\left(1\middle/\left(8\,(-2+d)^2\,d^4\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)$$
$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg(-2\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\,d\left(-16+8\,d+d^2\right)\text{a}[3,1]\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}-(-2+d)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\left(d^2\,\text{a}[2,1]-12\,\text{chi1}\,\text{a}[3,1]+6\,d\,\text{a}[3,1]+6\,(2\,\text{chi2}-d)\,\text{a}[3,1]\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Bigg)$$

In[∘]:=

```
Simplify[Solve[Diff'[d, chi1, chi2][[2]][[3, 3]] == 0, {a[3, 3]}]]
```

In[∘]:=

$$\text{a}[3,3] := -\Bigg(\Bigg(6\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\left(d^3\,(8\,\text{a}[1,1]-\text{a}[3,1])+24\,\text{chi1}^2\,\text{a}[3,1]-24\,\text{chi1}\,d\,\text{a}[3,1]+2\,d^2\,(-8\,\text{a}[1,1]+\text{a}[3,1])\right)\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)-3\,d\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\left(d^3\,(8\,\text{a}[1,1]-\text{a}[3,1])+24\,\text{chi1}^2\,\text{a}[3,1]-24\,\text{chi1}\,d\,\text{a}[3,1]+2\,d^2\,(-8\,\text{a}[1,1]+\text{a}[3,1])\right)$$

$$\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\right)-$$
$$\left(6\,\text{chi1}^2\,(-2+d)-6\,\text{chi1}\,(-2+d)\,d-d^2\right)\Bigg(-2\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)$$
$$d\,(-16+8\,d+d^2)\,a[3,1]\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}-$$
$$(-2+d)\,(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2)\Bigg(d^2\,a[2,1]-12\,\text{chi1}\,a[3,1]+$$
$$6\,d\,a[3,1]+6\,(2\,\text{chi2}-d)\,a[3,1]\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\Bigg)\Bigg)+$$
$$d^2\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\Bigg(\text{chi2}\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)$$
$$(-16+8\,d+d^2)\,\big(-12\,\text{chi1}^2\,a[3,1]+$$
$$d^2\,(-8\,a[1,1]+a[3,1])+4\,d\,(4\,a[1,1]+3\,\text{chi1}\,a[3,1])\big)-$$
$$\Bigg((-2+d)\,(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2)\Bigg(d\,\big(-6\,\text{chi1}^2\,(-2+d)+$$
$$6\,\text{chi1}\,(-2+d)\,d+d^2\big)\,a[2,1]+\frac{1}{-2+d}\,\text{chi1}\,(-2\,\text{chi1}+d)$$
$$(-\text{chi1}+d)\,(-16+8\,d+d^2)\,a[3,1]-3\,\big(-12\,\text{chi1}^2\,a[3,1]+$$
$$d^2\,(-8\,a[1,1]+a[3,1])+4\,d\,(4\,a[1,1]+3\,\text{chi1}\,a[3,1])\big)$$
$$\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\right)\Bigg)\Bigg)\Bigg/$$
$$\left(d\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\right)\Bigg)\Bigg)\Bigg/$$
$$\left(8\,\text{chi1}\,(-2+d)\,d^3\,(-16+8\,d+d^2)\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)\right)\Bigg)$$

In[ ]:=
```
Simplify[Diff'[d, chi1, chi2]⟦2⟧]
```

Out[ ]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
(* Use Diff'⟦1⟧ to solve the rest of AutA. *)
```

In[ ]:=
```
Solve[Diff'[d, chi1, chi2]⟦1⟧⟦1, 2⟧ == 0, a[3, 1]]
```

In[∘]:=
```
a[3, 1] := 0
```

In[∘]:=
```
Simplify[Solve[Diff'[d, chi1, chi2][[1]][[3, 1]] == 0, a[2, 1]]]
```

In[∘]:=
```
a[2, 1] :=
```

$$-\left(\left(\left((-4+d)\, a[1,1]\, \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)\Big/\left(4\,\text{chi1}\,\text{chi2}\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)$$

In[∘]:=
```
Simplify[Diff'[d, chi1, chi2][[1]]]
```

Out[∘]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
(* By rescaling AutA by a cubic root of unity and AutB by the inverse
 of it so that S stays unchanged, we may assume that a[1,1]=s[2,2]. *)
```

In[∘]:=
```
Simplify[Det[AutA]]
```

Out[∘]=

$$\frac{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right) a[1,1]^3}{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}$$

In[∘]:=
```
a[1, 1] := s[2, 2]
```

```
(* We record the unique solutions AutA,
AutB, S up to a choice of a cubic root. *)
```

In[∘]:=
```
Simplify[AutA]
Simplify[Adjugate[Binv]]
Simplify[S]
```

In[∘]:=
```
solA[d_, chi1_, chi2_] := {{
```

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\;,\; 0,$$

$$\frac{1}{8\,(-2+d)\,d^2}\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\,\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\left.\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)\Big\},$$

$$\Big\{-\left(\left((-4+d)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right.\right.$$

$$\left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left(4\,\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\left(4\,\text{chi1}\,\text{chi2}\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right.$$

$$\left.\left.\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right),\ 1,$$

$$\left(\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\,\left(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.\right.$$

$$\left(32-32\,d+6\,d^2+d^3\right)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}$$

$$\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+$$

$$\left((-4+d)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\right) \Big/$$

$$\left.\left.\left((-2\,\text{chi2}+d)\,(-\text{chi2}+d)\right)\right)\right) \Big/ \left(32\,\text{chi1}\,\text{chi2}\,(-2+d)\,d^2\right.$$

$$\left.\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big\},$$

$$\Big\{0,\ 0,\ \left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Big\}\Big\}$$

In[ ]:=

```
solB[d_, chi1_, chi2_] :=
```

$$\Big\{\Big\{\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Big/\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right),$$

$$0,\ -\left(\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right.\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-6\,\text{chi2}^2\,(-2+d)+6\,\text{chi2}\,(-2+d)\,d+d^2\right)-\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-6\,\text{chi1}^2\,(-2+d)+6\,\text{chi1}\,(-2+d)\,d+d^2\right)$$

$$\left.\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)\Big/$$

$$\left(8\,\text{chi1}\,(\text{chi1}-\text{d})\,(2\,\text{chi1}-\text{d})\,\text{d}^2\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}\,\right)\Bigg)\Bigg\},$$

$$\Bigg\{\left((-4+\text{d})\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\right.$$

$$\left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,\text{d}+(-4+\text{d})\,\text{d}^2\right)+\right.$$

$$\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}$$

$$\left.\left.\left(4\,\left(6\,\text{chi1}^2-6\,\text{chi1}\,\text{d}+\text{d}^2\right)-\text{d}^3\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\left(4\,\text{chi1}\,\text{chi2}\,(-2\,\text{chi2}+\text{d})\,(-\text{chi2}+\text{d})\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\right),\ 1,$$

$$\left(\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\,\left(-\left(\left((-4+\text{d})\,\left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)\right.\right.\right.\right.\right.$$

$$\left(-6\,\text{chi2}^2\,(-2+\text{d})+6\,\text{chi2}\,(-2+\text{d})\,\text{d}+\text{d}^2\right)-\text{chi2}$$

$$\left.\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)\,\left(-6\,\text{chi1}^2\,(-2+\text{d})+6\,\text{chi1}\,(-2+\text{d})\,\text{d}+\text{d}^2\right)\right)$$

$$\left.\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}\,\right)\left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)\right.$$

$$\left(-24\,\text{chi2}^2+24\,\text{chi2}\,\text{d}+(-4+\text{d})\,\text{d}^2\right)+\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}\,\left(4\,\left(6\,\text{chi1}^2-6\,\text{chi1}\,\text{d}+\text{d}^2\right)-\right.$$

$$\left.\left.\left.\text{d}^3\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\left.(\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d))\right)+$$

$$\Bigg(\text{chi2}^2\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(32-32\,d+6\,d^2+d^3\right)$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}$$

$$\left(96\,(-2+d)\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+\right.$$

$$\Bigg((-4+d)\left(6\,\text{chi1}^2\,(-2+d)-6\,\text{chi1}\,(-2+d)\,d-d^2\right)$$

$$\Bigg(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+$$

$$\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\,\Bigg(4\,\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-$$

$$\left.\left.\left.d^3\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\Bigg(\text{chi1}\,\text{chi2}\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)$$

$$\left.\left.\left.\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)\right)\Big/$$

$$\left.\left.\left((-2+d)\left(-16+8\,d+d^2\right)\right)\right)\right)\Big/\Bigg(32\,\text{chi1}\,\text{chi2}\,d^2$$

$$\left.\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Big\},$$

$$\left\{0, 0, \left(\text{chi2}\left(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2\right)\left(\frac{\text{chi1}\left(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2\right)}\right)^{2/3}\right) \Big/ \left(\text{chi1}\left(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2\right)\right)\right\}\right\}$$

In[∘]:=

```
solS[d_, chi1_, chi2_] :=
```

$$\left\{\left\{\left(\frac{\text{chi1}\left(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2\right)}\right)^{2/3}, 0, \frac{1}{8\,d^3}\left(-24\,\text{chi1}^2\,(-2+d) + 24\,\text{chi1}\,(-2+d)\,d + d^2\left(8 - 5\,d + d^2\right) + \left(24\,\text{chi2}^2\,(-2+d) - 24\,\text{chi2}\,(-2+d)\,d - d^2\left(8 - 5\,d + d^2\right)\right)\left(\frac{\text{chi1}\left(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2\right)}\right)^{2/3}\right)\right\}, \left\{0, \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2\right)}}, \frac{1}{d^2}\,3\,(-2+d)\left(2\,\text{chi1} - d + (-2\,\text{chi2} + d)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2\right)}}\right)\right\}, \{0, 0, 1\}\right\}$$

```
(* Solutions solA, solB,
solS are just sign matrices when chi1=chi2 and chi1+chi2=d. *)
```

In[∘]:=

```
Simplify[MatrixForm[solA[d, chi, chi]]]
Simplify[MatrixForm[solB[d, chi, chi]]]
Simplify[MatrixForm[solS[d, chi, chi]]]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
Simplify[MatrixForm[solA[d, chi, d - chi]]]
Simplify[MatrixForm[solB[d, chi, d - chi]]]
Simplify[MatrixForm[solS[d, chi, d - chi]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Type II - Solve for U, V

```
(* Using the solutions above,
we find AutU and AutV. Existence of the solution,
which we assumed, will lead to a contradiction later. *)
```

In[ ]:=
```
ExtDiff[d_, chi1_, chi2_] :=
 Table[Transpose[solA[d, chi1, chi2]].TopRelations[d, chi1]〚i〛.AutU +
   Transpose[solA[d, chi1, chi2]].ExtRelations[d, chi1]〚i〛.AutV - Sum[
    solS[d, chi1, chi2]〚i, j〛 * ExtRelations[d, chi2]〚j〛, {j, 1, 3}], {i, 1, 3}]
```

```
(* We solve the linear equation for variables u[i,1] and v[i,1] by
 looking at the first columns of ExtDiff〚1〛,ExtDiff〚2〛,ExtDiff〚3〛. *)
```

In[ ]:=
```
MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2]〚3〛〚1 ;; 3, 1〛, u[1, 1], 1]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ \dfrac{2\,\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{2/3}}{\text{chi1}\,(-2+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)} \end{pmatrix}$$

In[ ]:=
```
Solve[ExtDiff[d, chi1, chi2]〚3〛〚3, 1〛 == 0, {u[1, 1]}]
```

In[ ]:=

$$u[1,1] := \Bigg(\text{chi1}\,(-2+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)$$

$$\Bigg(-\left(\left(d^2\,(2+d)\right)/\left(2\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\,(-2+d)\right)\right)-$$

$$\Bigg(\Bigg(\mathrm{chi2}\,\left(12\,\mathrm{chi1}^2-12\,\mathrm{chi1}\,d-d^2\right)\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Big/$$

$$\left(8\,\mathrm{chi1}\,(-2+d)\,d\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)\right)+\frac{1}{8\,(-2+d)\,d^2}$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\;\Bigg(24\,\mathrm{chi2}^2-24\,\mathrm{chi2}\,d-(-2+d)\,d^2+$$

$$\Bigg(\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)\,\left(-24\,\mathrm{chi1}^2+24\,\mathrm{chi1}\,d+(-2+d)\,d^2\right)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\Bigg)\Big/$$

$$\left(\mathrm{chi1}\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)\right)\Bigg)\Bigg)\;u[3,\,1]\;-$$

$$\Bigg(-\Bigg(\Bigg(\mathrm{chi2}\,d^2\,(2+d)\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)\,\sqrt[3]{\frac{\mathrm{chi1}\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Big/$$

$$\left(2\,\mathrm{chi1}^2\,(\mathrm{chi1}-d)\,(2\,\mathrm{chi1}-d)\,(-2+d)\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)\right)\Bigg)+$$

$$\Bigg(\sqrt[3]{\frac{\mathrm{chi1}\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\;\Bigg(24\,\mathrm{chi2}^2-24\,\mathrm{chi2}\,d-(-2+d)\,d^2+$$

$$\Bigg(\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)\,\left(-24\,\mathrm{chi1}^2+24\,\mathrm{chi1}\,d+(-2+d)\,d^2\right)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\Bigg)\Big/\;\Big(\mathrm{chi1}\,\Big(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+$$

$$d^2\Big)\Big)\Bigg)\Bigg)\Big/\;\left(2\,\mathrm{chi1}\,(\mathrm{chi1}-d)\,(2\,\mathrm{chi1}-d)\,(-2+d)\right)\Bigg)\;v[1,\,1]\;-$$

$$
\Bigg(\Bigg(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-48\,\text{chi1}^4+\left(96\,\text{chi1}^3+72\,\text{chi1}^4\right)d+\left(-36\,\text{chi1}^2-144\,\text{chi1}^3\right)d^2+\left(-12\,\text{chi1}+66\,\text{chi1}^2\right)d^3+\left(4+6\,\text{chi1}-6\,\text{chi1}^2\right)d^4+(-2+6\,\text{chi1})\,d^5+d^6\right)\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)^2\Bigg)\Big/
\left(8\,\text{chi1}^2\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+
\Bigg(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\left(6\,d-6\,d^2\right)\right)
\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+
\Bigg(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)
\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Big/\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)\Bigg)\Big/\left(8\,\text{chi1}\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\right)+
\Bigg((3-d)\Bigg(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)
\left(32-32\,d+6\,d^2+d^3\right)\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)^2
\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+
\Bigg((-4+d)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)
\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg(4\left(6\,\text{chi1}^2-\right.
$$

$$6\,\text{chi1}\,d + d^2\Big) - d^3 \left(\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\right)^2\Bigg)\Bigg)\Bigg)\Bigg/$$

$$((-2\,\text{chi2} + d)\,(-\text{chi2} + d))\Bigg)\Bigg)\Bigg/\Big(32\,\text{chi1}\,\text{chi2}$$

$$(-2 + d)^2\,d^2\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2))\Bigg)\,v[2, 1] -$$

$$\Bigg(\Bigg(\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)\,\big(128\,\text{chi1}^6 + (-384\,\text{chi1}^5 - 1344\,\text{chi1}^6)\,d +$$

$$(224\,\text{chi1}^4 + 4032\,\text{chi1}^5 + 1216\,\text{chi1}^6)\,d^2 +$$
$$(192\,\text{chi1}^3 - 3792\,\text{chi1}^4 - 3648\,\text{chi1}^5)\,d^3 + (-232\,\text{chi1}^2 + 864\,\text{chi1}^3 +$$
$$3520\,\text{chi1}^4)\,d^4 + (72\,\text{chi1} + 372\,\text{chi1}^2 - 960\,\text{chi1}^3)\,d^5 +$$
$$(-6 - 132\,\text{chi1} - 212\,\text{chi1}^2)\,d^6 + (5 + 84\,\text{chi1} + 24\,\text{chi1}^2)\,d^7 +$$

$$(-2 - 24\,\text{chi1})\,d^8\big) \left(\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\right)^2\Bigg)\Bigg/$$

$$\big(128\,\text{chi1}^2\,(-2 + d)\,d^4\,(-2\,\text{chi1} + d)\,(-\text{chi1} + d)\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)\big) +$$

$$\Bigg(3\,\left(-\frac{1}{2} + d\right)\left(\text{chi1}^2\,(1 - d) + \frac{d^2}{12} + \text{chi1}\,(-d + d^2)\right)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\,\Bigg(24\,\text{chi2}^2 - 24\,\text{chi2}\,d - (-2 + d)\,d^2 +$$

$$\Bigg(\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)\,(-24\,\text{chi1}^2 + 24\,\text{chi1}\,d + (-2 + d)\,d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\Bigg)\Bigg/\Big(\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d +$$

$$d^2))\Bigg)\Bigg)\Bigg/\big(16\,\text{chi1}\,(-2 + d)\,d^2\,(-2\,\text{chi1} + d)\,(-\text{chi1} + d)\big) +$$

$$\Bigg(3\,(-3 + d)\,\left(\text{chi1}^2 + (-\text{chi1} - \text{chi1}^2)\,d + \left(\frac{1}{12} + \text{chi1}\right)d^2\right)$$

$$\left(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.$$

$$\left(32-32\,d+6\,d^2+d^3\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}$$

$$\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+$$

$$\left((-4+d)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\right.\right.\right.$$

$$\left.d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\text{chi2}\left(2\,\text{chi2}^2-\right.$$

$$\left.3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(4\left(6\,\text{chi1}^2-\right.\right.$$

$$\left.\left.\left.\left.6\,\text{chi1}\,d+d^2\right)-d^3\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\left.\left.((-2\,\text{chi2}+d)\,(-\text{chi2}+d))\right)\right)\Big/\left(64\,\text{chi1}\,\text{chi2}\,(-2+d)^2\right.$$

$$\left.\left.d^4\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\,\text{v}[3,1]\right)\right)\Big/$$

$$\left(2\,\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)$$

In[ ]:=
```
MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2]⟦3⟧⟦1 ;; 3, 1⟧, u[3, 1], 1]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix}\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\\ 0\\ 0\end{pmatrix}$$

In[ ]:=
```
Solve[ExtDiff[d, chi1, chi2]⟦3⟧⟦1, 1⟧ == 0, {u[3, 1]}]
```

In[ ]:=
```
u[3, 1] := ((4 d^2) / (chi2 (chi2 - d) (2 chi2 - d)) -
```

$$\left(4\, d^2 \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\ v[1,1]\right) \Big/ \left(\text{chi1}\left(\text{chi1}-d\right)\left(2\,\text{chi1}-d\right)\right) -$$

$$\left(\left(\left(-d^3+\text{chi1}^2\left(-6+6\,d\right)+\text{chi1}\left(6\,d-6\,d^2\right)\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\right.$$

$$\left(\text{chi1}\left(-2\,\text{chi1}+d\right)\left(-\text{chi1}+d\right)\right) - \left((3-d)\left(-4+d\right)\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+\left(-4+d\right)d^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\Big/\right.$$

$$\left.\left(4\,\text{chi1}\,\text{chi2}\left(-2+d\right)\left(-2\,\text{chi2}+d\right)\left(-\text{chi2}+d\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)$$

$$v[2,1] - \left(\left(3\left(-\frac{1}{2}+d\right)\left(\text{chi1}^2\left(1-d\right)+\frac{d^2}{12}+\text{chi1}\left(-d+d^2\right)\right)\right.\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\left(2\,\text{chi1}\left(-2\,\text{chi1}+d\right)\left(-\text{chi1}+d\right)\right) -$$

$$\left(3\left(-4+d\right)\left(-3+d\right)\left(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right)d+\left(\frac{1}{12}+\text{chi1}\right)d^2\right)\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+\left(-4+d\right)d^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\Big/\right.$$

```
(8 chi1 chi2 (-2 + d) d^2 (-2 chi2 + d) (-chi2 + d) (2 chi1^2 - 3 chi1 d + d^2)))

v[3, 1]) / (CubeRoot[(chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2))])
```

In[∘]:= 
```
MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2]⟦2⟧⟦1 ;; 3, 1⟧, u[2, 1], 1]]]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}} \\ 0 \\ -\frac{(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2)\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}}{8\,(-2+d)\,d^2} \end{pmatrix}$$

In[∘]:= 
```
Solve[ExtDiff[d, chi1, chi2]⟦2⟧⟦1, 1⟧ == 0, {u[2, 1]}]
```

In[∘]:= 
```
u[2, 1] :=
 ((3 (-1 + d) d CubeRoot[(chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2))]) / (chi2 (chi2 - d) (-2 + d)) +

   (12 (-2 + d) (2 chi1 - d + (-2 chi2 + d) CubeRoot[(chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2))])) /
   (chi2 (chi2 - d) (2 chi2 - d)) -
   (3 (-1 + d) d CubeRoot[(chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2))] v[1, 1]) /

   (chi1 (chi1 - d) (-2 + d)) - ((-4 chi1^2 + (4 chi1 + 14 chi1^2) d +
       (-14 chi1 - 16 chi1^2) d^2 + (-3 + 16 chi1 + 2 chi1^2) d^3 + (3 - 2 chi1) d^4)
     CubeRoot[(chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2))] v[2, 1]) / (4 chi1 (-2 + d) d (-chi1 + d)) -

   (((-1 + d) (-6 chi1^4 + (12 chi1^3 + 6 chi1^4) d - 6 chi1^2 (1 + chi1)^2 d^2 +

        (9 chi1^2/2 + 12 chi1^3) d^3 + (3/8 + 3 chi1/2 + chi1^2/2) d^4 +
        (-3/4 - 13 chi1/2 - chi1^2) d^5 + chi1 d^6)
```

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Bigg/\left(8\,\text{chi1}\,(-2+d)\,d^3\,(-\text{chi1}+d)\right)-$$

$$\Bigg((-4+d)\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\left(-8+10\,d-d^2-4\,d^3+d^4\right)$$

$$\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Bigg)\Bigg)\Bigg/$$

$$\left(32\,\text{chi2}\,(-2+d)\,d^4\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)$$

$$v[3,1]+\Bigg((-4+d)\left(-12\,\text{chi1}^2+\left(12\,\text{chi1}+6\,\text{chi1}^2\right)d+(-1-6\,\text{chi1})\,d^2\right)$$

$$\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Bigg)$$

$$\Bigg(\left(4\,d^2\right)\Big/\left(\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\right)-$$

$$\left(4\,d^2\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\;v[1,1]\right)\Bigg/$$

$$\left(\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\right)-\Bigg(\Bigg(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\left(6\,d-6\,d^2\right)\right)$$

$$\left.\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\right)\Big/\ (\mathrm{chi1}\ (-2\,\mathrm{chi1}+d)$$

$$(-\mathrm{chi1}+d))-\Big((3-d)\ (-4+d)\ \Big(\mathrm{chi1}\ \left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)$$

$$\left(-24\,\mathrm{chi2}^2+24\,\mathrm{chi2}\,d+(-4+d)\,d^2\right)+\mathrm{chi2}\ \left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\ \Big(4\ \left(6\,\mathrm{chi1}^2-6\,\mathrm{chi1}\,d+d^2\right)-$$

$$\left.\left.\left.d^3\ \sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\Big/\ \Big(4\,\mathrm{chi1}\,\mathrm{chi2}$$

$$(-2+d)\ (-2\,\mathrm{chi2}+d)\ (-\mathrm{chi2}+d)\ \left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)\Big)\Big)$$

$$v[2,1]-\left(\left(3\ \left(-\frac{1}{2}+d\right)\ \left(\mathrm{chi1}^2\ (1-d)+\frac{d^2}{12}+\mathrm{chi1}\ \left(-d+d^2\right)\right)\right.\right.$$

$$\left.\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\right)\Big/$$

$$(2\,\mathrm{chi1}\ (-2\,\mathrm{chi1}+d)\ (-\mathrm{chi1}+d))-\Big(3\ (-4+d)\ (-3+d)\ \Big(\mathrm{chi1}^2+$$

$$\left(-\mathrm{chi1}-\mathrm{chi1}^2\right)\,d+\left(\frac{1}{12}+\mathrm{chi1}\right)\,d^2\Big)\ \Big(\mathrm{chi1}\ \left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)$$

$$\left(-24\,\mathrm{chi2}^2+24\,\mathrm{chi2}\,d+(-4+d)\,d^2\right)+\mathrm{chi2}\ \left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\ \Big(4\ \left(6\,\mathrm{chi1}^2-6\,\mathrm{chi1}\,d+d^2\right)-$$

$$\left.\left.\left.d^3\ \sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\Big/\ \Big(8\,\mathrm{chi1}\,\mathrm{chi2}\ (-2+d)$$

$$d^2\ (-2\,\mathrm{chi2}+d)\ (-\mathrm{chi2}+d)\ \left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)\Big)\Big)\ v[3,1]\Big)\Big)\Big/$$

$$\Big(32\,\text{chi1}\,\text{chi2}\,d^2\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\big)$$

$$\sqrt[3]{\frac{\text{chi1}\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\big)}{\text{chi2}\,\big(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\big)}}\;\Big)+\Big((-4+d)\,(-2+d)$$

$$\Big(\text{chi1}\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\big)\,\big(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\big)+$$

$$\text{chi2}\,\big(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\big)\;\sqrt[3]{\frac{\text{chi1}\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\big)}{\text{chi2}\,\big(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\big)}}$$

$$\Big(4\,\big(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\big)-d^3\;\sqrt[3]{\frac{\text{chi1}\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\big)}{\text{chi2}\,\big(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\big)}}^{\,2}\;\Big)\Big)\Big)$$

$$\Big(-\Big(\big(d^2\,(2+d)\big)\Big/\big(2\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\,(-2+d)\big)\Big)-$$

$$\Big(-\Big(\Big(\text{chi2}\,d^2\,(2+d)\,\big(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\big)$$

$$\sqrt[3]{\frac{\text{chi1}\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\big)}{\text{chi2}\,\big(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\big)}}^{\,2}\;\Big)\Big/\;\big(2\,\text{chi1}^2\,(\text{chi1}-d)$$

$$(2\,\text{chi1}-d)\,(-2+d)\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\big)\big)\Big)+$$

$$\Big(\sqrt[3]{\frac{\text{chi1}\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\big)}{\text{chi2}\,\big(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\big)}}\;\Big(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+$$

$$\Big(\text{chi2}\,\big(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\big)\,\big(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\big)$$

$$\sqrt[3]{\frac{\text{chi1}\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\big)}{\text{chi2}\,\big(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\big)}}\;\Big)\Big/\;\big(\text{chi1}\,\big(2\,\text{chi1}^2-3\,\text{chi1}\,d+$$

$$d^2\big)\big)\Big)\Big)\Big/\;\big(2\,\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\,(-2+d)\big)\Big)\;v[1,1]\;-$$

$$\left(\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-48\,\text{chi1}^4+\left(96\,\text{chi1}^3+72\,\text{chi1}^4\right)d+\left(-36\,\text{chi1}^2-144\,\text{chi1}^3\right)d^2+\left(-12\,\text{chi1}+66\,\text{chi1}^2\right)d^3+\left(4+6\,\text{chi1}-6\,\text{chi1}^2\right)d^4+(-2+6\,\text{chi1})\,d^5+d^6\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Big/$$

$$\left(8\,\text{chi1}^2\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+$$

$$\left(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\left(6\,d-6\,d^2\right)\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)\right)\Big/\left(8\,\text{chi1}\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\right)+$$

$$\left((3-d)\left(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(32-32\,d+6\,d^2+d^3\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+\right.\right.$$

$$\left((-4+d)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.\right.$$

$$\sqrt[3]{\frac{\text{chi1}\ (2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2)}{\text{chi2}\ (2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2)}}\ (4\ (6\ \text{chi1}^2-6\ \text{chi1}\ d+$$

$$d^2)-d^3\ \sqrt[3]{\frac{\text{chi1}\ (2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2)}{\text{chi2}\ (2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2)}}^2\ ))))\Big/$$

$$((-2\ \text{chi2}+d)\ (-\text{chi2}+d))))\Big/\ (32\ \text{chi1}\ \text{chi2}$$

$$(-2+d)^2\ d^2\ (2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2)))\ v[2,1]-$$

$$(((\text{chi2}\ (2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2)\ (128\ \text{chi1}^6+(-384\ \text{chi1}^5-1344\ \text{chi1}^6)\ d+$$

$$(224\ \text{chi1}^4+4032\ \text{chi1}^5+1216\ \text{chi1}^6)\ d^2+(192\ \text{chi1}^3-3792\ \text{chi1}^4-3648\ \text{chi1}^5)\ d^3+(-232\ \text{chi1}^2+864\ \text{chi1}^3+3520\ \text{chi1}^4)\ d^4+(72\ \text{chi1}+372\ \text{chi1}^2-960\ \text{chi1}^3)\ d^5+(-6-132\ \text{chi1}-212\ \text{chi1}^2)\ d^6+(5+84\ \text{chi1}+24\ \text{chi1}^2)\ d^7+(-2-24\ \text{chi1})\ d^8)$$

$$\sqrt[3]{\frac{\text{chi1}\ (2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2)}{\text{chi2}\ (2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2)}}^2\ )\Big/\ (128\ \text{chi1}^2\ (-2+d)$$

$$d^4\ (-2\ \text{chi1}+d)\ (-\text{chi1}+d)\ (2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2))+$$

$$(3\ \left(-\frac{1}{2}+d\right)\ \left(\text{chi1}^2\ (1-d)+\frac{d^2}{12}+\text{chi1}\ (-d+d^2)\right)$$

$$\sqrt[3]{\frac{\text{chi1}\ (2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2)}{\text{chi2}\ (2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2)}}\ (24\ \text{chi2}^2-24\ \text{chi2}\ d-(-2+d)\ d^2+$$

$$(\text{chi2}\ (2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2)\ (-24\ \text{chi1}^2+24\ \text{chi1}\ d+(-2+d)\ d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\ (2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2)}{\text{chi2}\ (2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2)}}\ )\Big/\ (\text{chi1}\ (2\ \text{chi1}^2-3\ \text{chi1}\ d+$$

$$d^2))))\Big/\ (16\ \text{chi1}\ (-2+d)\ d^2\ (-2\ \text{chi1}+d)\ (-\text{chi1}+d))+$$

$$\Bigg(3\,(-3+d)\,\left(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right)d+\left(\frac{1}{12}+\text{chi1}\right)d^2\right)$$

$$\Bigg(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\left(32-32\,d+6\,d^2+d^3\right)\,\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{2/3}$$

$$\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+$$

$$\Bigg((-4+d)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)$$

$$\Bigg(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\,\Big(-24\,\text{chi2}^2+24\,\text{chi2}\,d+$$

$$(-4+d)\,d^2\Big)+\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\,\Bigg(4\,\Big(6\,\text{chi1}^2-6\,\text{chi1}\,d+$$

$$d^2\Big)-d^3\,\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{2/3}\Bigg)\Bigg)\Bigg)\Bigg/$$

$$((-2\,\text{chi2}+d)\,(-\text{chi2}+d))\Bigg)\Bigg)\Bigg/\Big(64\,\text{chi1}\,\text{chi2}$$

$$(-2+d)^2\,d^4\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\Big)\Bigg)\,v[3,1]-$$

$$\Bigg(\Bigg(\Bigg(\text{chi2}\,\left(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2\right)\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{2/3}\Bigg)\Bigg/$$

$$\left(8\,\text{chi1}\,(-2+d)\,d\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+\frac{1}{8\,(-2+d)\,d^2}$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\;\Big(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+$$

$$\Big(\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\Big)\Big/\,(\text{chi1}\,(2\,\text{chi1}^2-$$

$$3\,\text{chi1}\,d+d^2))\Big)\Big)\;\Big((4\,d^2)\,/\,(\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d))-$$

$$\Big(4\,d^2\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\;v[1,1]\Big)\Big/\,(\text{chi1}\,(\text{chi1}-d)$$

$$(2\,\text{chi1}-d))-\Big(\Big((-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\,(6\,d-6\,d^2))$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\Big)\Big/\,(\text{chi1}\,(-2\,\text{chi1}+d)$$

$$(-\text{chi1}+d))-\Big((3-d)\,(-4+d)\,\Big(\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)$$

$$(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2)+\text{chi2}$$

$$(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}$$

$$\Big(4\,(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2)-$$

$$d^3\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}^{\,2}\Big)\Big)\Big)\Big/\,\Big(4\,\text{chi1}\,\text{chi2}$$

$$(-2+d)\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2))\Big)$$

$$v[2,1]-\Bigg(\Bigg(\Bigg(3\left(-\frac{1}{2}+d\right)\left(\text{chi1}^2(1-d)+\frac{d^2}{12}+\text{chi1}\left(-d+d^2\right)\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Big/\ (2\,\text{chi1}\,(-2\,\text{chi1}+d)$$

$$(-\text{chi1}+d))-\Bigg(3\,(-4+d)\,(-3+d)\,\Big(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right)d+$$

$$\left(\frac{1}{12}+\text{chi1}\right)d^2\Big)\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\Big(-24\,\text{chi2}^2+$$

$$24\,\text{chi2}\,d+(-4+d)\,d^2\Big)+\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\ \Bigg(4\,\Big(6\,\text{chi1}^2-6\,\text{chi1}\,d+$$

$$d^2\Big)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Bigg)\Bigg)\Big/$$

$$\Big(8\,\text{chi1}\,\text{chi2}\,(-2+d)\,d^2\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)$$

$$\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\Big)\Bigg)\,v[3,1]\Bigg)\Bigg)\Big/$$

$$\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Bigg)\Bigg)\Big/\ \Bigg(8\,\text{chi2}^2$$

$$(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Bigg)\Big/$$

$$\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)$$

In[•]:=
```
MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2][[3]][[1 ;; 3, 1]], v[2, 1], 1]]]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 \\ \frac{3-d}{-2+d} \\ 0 \end{pmatrix}$$

In[•]:=
```
Solve[ExtDiff[d, chi1, chi2][[3]][[2, 1]] == 0, v[2, 1]]
```

In[∘]:=

$$v[2,1] := -\frac{1}{8\,d^2}\left(-12\,\text{chi1}^2 + 12\,\text{chi1}\,d + 12\,\text{chi1}^2\,d - d^2 - 12\,\text{chi1}\,d^2\right) v[3,1]$$

In[∘]:=

```
MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2]〚2〛〚1 ;; 3, 1〛, v[1, 1], 1]]]
```

Out[∘]//MatrixForm=

$$\left(\frac{8\,\text{chi1}\,\text{chi2}\,(2\,\text{chi1}-d)\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\,d\,\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-3\,d^2\left(-2+d+d^2\right)+2\,\text{chi}\ldots\right.}{\ldots}\right.$$

In[∘]:=

```
Solve[ExtDiff[d, chi1, chi2]〚2〛〚2, 1〛 == 0, v[1, 1]]
```

In[∘]:=

$$v[1,1] := \Bigg(\left(\text{chi1}\,d^2\,(2+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right) \Big/ \Bigg(4\,\text{chi2}^2\,(\text{chi2}-d)$$
$$(2\,\text{chi2}-d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)^2\Bigg) -$$
$$\left(-12\,\text{chi1}^2+\left(12\,\text{chi1}+6\,\text{chi1}^2\right)d+(-1-6\,\text{chi1})\,d^2\right) \Big/$$
$$\left(2\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right) +$$
$$\Bigg(2\,(-2+d)\,d^2\,\Bigg(\Bigg(\text{chi2}\left(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$
$$\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)^2\Bigg) \Big/$$
$$\left(8\,\text{chi1}\,(-2+d)\,d\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right) + \frac{1}{8\,(-2+d)\,d^2}$$
$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\,\Bigg(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+$$
$$\Bigg(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)$$
$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right) \Big/$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)\Bigg)\Bigg)\Bigg/\ (\text{chi2}\ (\text{chi2}-d)\ (2\,\text{chi2}-d))\ -$$

$$\frac{1}{8\ (-2+d)\ d^4}\ \text{chi1}\ (-2\,\text{chi1}+d)\ (-\text{chi1}+d)\ \left(-8+10\,d-d^2-4\,d^3+d^4\right)\ v[3,1]\ -$$

$$\Bigg(\left(-12\,\text{chi1}^2+\left(12\,\text{chi1}+6\,\text{chi1}^2\right)d+(-1-6\,\text{chi1})\,d^2\right)$$

$$\left(-12\,\text{chi1}^2+12\,\text{chi1}\,d+12\,\text{chi1}^2\,d-d^2-12\,\text{chi1}\,d^2\right)$$

$$\Bigg(\Bigg(\left(-d^3+\text{chi1}^2\ (-6+6\,d)+\text{chi1}\left(6\,d-6\,d^2\right)\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Bigg/$$

$$(\text{chi1}\ (-2\,\text{chi1}+d)\ (-\text{chi1}+d))\ -\ \Bigg((3-d)\ (-4+d)$$

$$\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\Bigg(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Bigg)\Bigg)\Bigg/$$

$$\left(4\,\text{chi1}\,\text{chi2}\ (-2+d)\ (-2\,\text{chi2}+d)\ (-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)$$

$$v[3,1]\Bigg)\Bigg/\left(64\,d^4\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+$$

$$\frac{1}{16\,d^2}\ (-2+d)\left(-12\,\text{chi1}^2+12\,\text{chi1}\,d+12\,\text{chi1}^2\,d-d^2-12\,\text{chi1}\,d^2\right)$$

$$\Bigg(\Bigg(\text{chi2}\left(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Bigg/$$

$$\left(8\,\text{chi1}\,(-2+d)\,d\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+\frac{1}{8\,(-2+d)\,d^2}$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\,\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\ \left(\text{chi1}\right.$$

$$\left.\left.\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)$$

$$\left(\left(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\,\left(6\,d-6\,d^2\right)\right)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\right.$$

$$(\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d))-\left((3-d)\,(-4+d)\right.$$

$$\left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left.\left(4\,\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^2\right)\right)\right)\right)\Big/$$

$$\left.\left(4\,\text{chi1}\,\text{chi2}\,(-2+d)\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)$$

$$\text{v}[3,1]+\left(\left(-12\,\text{chi1}^2+\left(12\,\text{chi1}+6\,\text{chi1}^2\right)\,d+(-1-6\,\text{chi1})\,d^2\right)\right.$$

$$\left(\left(3\,\left(-\frac{1}{2}+d\right)\,\left(\text{chi1}^2\,(1-d)+\frac{d^2}{12}+\text{chi1}\,\left(-d+d^2\right)\right)\right.\right.$$

$$
\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Big/\ (2\,\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d))-
$$

$$
\Bigg(3\,(-4+d)\,(-3+d)\,\left(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right)d+\left(\frac{1}{12}+\text{chi1}\right)d^2\right)
$$

$$
\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+
$$

$$
\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}
$$

$$
\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Bigg)\Bigg)\Big/
$$

$$
\left(8\,\text{chi1}\,\text{chi2}\,(-2+d)\,d^2\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)
$$

$$
\text{v[3, 1]}\Bigg)\Big/\left(8\,d^2\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)-
$$

$$
\frac{1}{2}\,(-2+d)\,\Bigg(\Bigg(\text{chi2}\left(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)
$$

$$
\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Big/
$$

$$
\left(8\,\text{chi1}\,(-2+d)\,d\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+\frac{1}{8\,(-2+d)\,d^2}
$$

$$
\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\ \Bigg(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+
$$

$$
\Bigg(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)
$$

$$
\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Big/\ \left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)\Bigg)
$$

$$\left(\left(3\left(-\frac{1}{2}+d\right)\left(\text{chi1}^2\,(1-d)+\frac{d^2}{12}+\text{chi1}\left(-d+d^2\right)\right)\right.\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\;(2\,\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d))\;-$$

$$\left(3\,(-4+d)\,(-3+d)\left(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right)d+\left(\frac{1}{12}+\text{chi1}\right)d^2\right)\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\right)\Big/$$

$$\left.\left(8\,\text{chi1}\,\text{chi2}\,(-2+d)\,d^2\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)$$

$$v[3,1]-\left(\text{chi1}\,(-2+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right.$$

$$\left(-12\,\text{chi1}^2+12\,\text{chi1}\,d+12\,\text{chi1}^2\,d-d^2-12\,\text{chi1}\,d^2\right)$$

$$\left(\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.\right.$$

$$\left(-48\,\text{chi1}^4+\left(96\,\text{chi1}^3+72\,\text{chi1}^4\right)d+\left(-36\,\text{chi1}^2-144\,\text{chi1}^3\right)d^2+\left(-12\,\text{chi1}+66\,\text{chi1}^2\right)d^3+\left(4+6\,\text{chi1}-6\,\text{chi1}^2\right)d^4+\right.$$

$$\left.\left.(-2+6\,\text{chi1})\,d^5+d^6\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Big/$$

$$\left(8\,\text{chi1}^2\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+$$

$$\left(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\left(6\,d-6\,d^2\right)\right)\right.$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+\right.\right.$$

$$\left.\left.\left.d^2\right)\right)\right)\Big/\left(8\,\text{chi1}\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\right)+$$

$$\left((3-d)\left(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.\right.$$

$$\left(32-32\,d+6\,d^2+d^3\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}$$

$$\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+$$

$$\left((-4+d)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\right.\right.\right.$$

$$\left.d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\text{chi2}\left(2\,\text{chi2}^2-\right.$$

$$\left.3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(4\left(6\,\text{chi1}^2-\right.\right.$$

$$\left.\left.\left.\left.6\,\text{chi1}\,d+d^2\right)-d^3\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\left.\left.\left((-2\,\text{chi2}+d)\,(-\text{chi2}+d)\right)\right)\right)\Big/\left(32\,\text{chi1}\,\text{chi2}\,(-2+d)^2\right.$$

$$\left.\left.d^2\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\,v[3,1]\right)\Big/$$

$$\left(16\,\text{chi2}\,d^2\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)+$$

$$\left(\text{chi1}\,(-2+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right.$$

$$\left(\left(\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(128\,\text{chi1}^6+\left(-384\,\text{chi1}^5-1344\,\text{chi1}^6\right)d+\right.\right.\right.\right.$$

$$\left(224\,\text{chi1}^4+4032\,\text{chi1}^5+1216\,\text{chi1}^6\right)d^2+$$
$$\left(192\,\text{chi1}^3-3792\,\text{chi1}^4-3648\,\text{chi1}^5\right)d^3+\left(-232\,\text{chi1}^2+864\,\text{chi1}^3+\right.$$
$$\left.3520\,\text{chi1}^4\right)d^4+\left(72\,\text{chi1}+372\,\text{chi1}^2-960\,\text{chi1}^3\right)d^5+$$
$$\left(-6-132\,\text{chi1}-212\,\text{chi1}^2\right)d^6+\left(5+84\,\text{chi1}+24\,\text{chi1}^2\right)d^7+$$

$$\left.(-2-24\,\text{chi1})\,d^8\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Bigg/$$

$$\left(128\,\text{chi1}^2\,(-2+d)\,d^4\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+$$

$$\left(3\left(-\frac{1}{2}+d\right)\left(\text{chi1}^2\,(1-d)+\frac{d^2}{12}+\text{chi1}\left(-d+d^2\right)\right)\right.$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Bigg/\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+\right.\right.$$

$$\left.\left.\left.\left.d^2\right)\right)\right)\right)\Bigg/\left(16\,\text{chi1}\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\right)+$$

$$\left(3\,(-3+d)\left(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right)d+\left(\frac{1}{12}+\text{chi1}\right)d^2\right)\right.$$

$$\left(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.$$

$$(32 - 32\,d + 6\,d^2 + d^3)\ \left(\sqrt[3]{\frac{chi1\,(2\,chi1^2 - 3\,chi1\,d + d^2)}{chi2\,(2\,chi2^2 - 3\,chi2\,d + d^2)}}\right)^2$$

$$\Bigg(2\,chi1 - d + (-2\,chi2 + d)\ \sqrt[3]{\frac{chi1\,(2\,chi1^2 - 3\,chi1\,d + d^2)}{chi2\,(2\,chi2^2 - 3\,chi2\,d + d^2)}}\Bigg) +$$

$$\Bigg((-4 + d)\ (-24\,chi2^2 + 24\,chi2\,d + (-2 + d)\ d^2)\ \Bigg(chi1\ (2\,chi1^2 - 3\,chi1$$

$$d + d^2)\ (-24\,chi2^2 + 24\,chi2\,d + (-4 + d)\ d^2) + chi2\ (2\,chi2^2 -$$

$$3\,chi2\,d + d^2)\ \sqrt[3]{\frac{chi1\,(2\,chi1^2 - 3\,chi1\,d + d^2)}{chi2\,(2\,chi2^2 - 3\,chi2\,d + d^2)}}\ \Bigg(4\ (6\,chi1^2 -$$

$$6\,chi1\,d + d^2) - d^3\ \left(\sqrt[3]{\frac{chi1\,(2\,chi1^2 - 3\,chi1\,d + d^2)}{chi2\,(2\,chi2^2 - 3\,chi2\,d + d^2)}}\right)^2\Bigg)\Bigg)\Bigg) \Big/$$

$$((-2\,chi2 + d)\ (-chi2 + d))\Bigg)\Bigg) \Big/\ (64\,chi1\,chi2\ (-2 + d)^2$$

$$d^4\ (2\,chi1^2 - 3\,chi1\,d + d^2))\Bigg)\ v[3, 1]\Bigg) \Big/$$

$$\Bigg(2\,chi2\ (2\,chi2^2 - 3\,chi2\,d + d^2)\ \left(\sqrt[3]{\frac{chi1\,(2\,chi1^2 - 3\,chi1\,d + d^2)}{chi2\,(2\,chi2^2 - 3\,chi2\,d + d^2)}}\right)^2\Bigg)\Bigg) \Big/$$

$$\Bigg(-((-12\,chi1^2 + (12\,chi1 + 6\,chi1^2)\ d + (-1 - 6\,chi1)\ d^2) \Big/$$

$$(2\,chi1\ (chi1 - d)\ (2\,chi1 - d))) -$$

$$\Bigg(chi1\ (-2 + d)\ (2\,chi1^2 - 3\,chi1\,d + d^2)\ \Bigg(-\Bigg(\Bigg(chi2\,d^2\ (2 + d)$$

$$(2\,chi2^2 - 3\,chi2\,d + d^2)\ \left(\sqrt[3]{\frac{chi1\,(2\,chi1^2 - 3\,chi1\,d + d^2)}{chi2\,(2\,chi2^2 - 3\,chi2\,d + d^2)}}\right)^2\Bigg) \Big/$$

$$(2\,chi1^2\ (chi1 - d)\ (2\,chi1 - d)\ (-2 + d)\ (2\,chi1^2 - 3\,chi1\,d + d^2))\Bigg) +$$

$$\left(\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\;(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$(\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\;\Bigg)\Big/\;(\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+$$

$$d^2))\Big)\Big)\Big/\;(2\,\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\,(-2+d))\Big)\Big)\Big/$$

$$\left(2\,\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}^{\,2}\right)+$$

$$(2\,(-2+d)\,d^2\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)$$

$$((\text{chi2}\,(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2)\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}^{\,2}\;\Bigg)\Big/$$

$$(8\,\text{chi1}\,(-2+d)\,d\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2))+\frac{1}{8\,(-2+d)\,d^2}$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\;(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+$$

$$(\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\;\Bigg)\Big/\;(\text{chi1}$$

$$(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2))\Big)\Big)\Big)\Big/$$

$$\left(\text{chi2}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)^2\right)\right)$$

```
(* We are left with only v[3,1] variable. We
 use collections of two linear equations of v[3,1]
 (from ExtDiff[d,chi1,chi2]〚1〛〚1,1〛 and ExtDiff[d,chi1,chi2]〚2〛〚1,1〛)
 to obtain constraints for the existence of the solution. *)
```

In[•]:=
```
AA[d_, chi1_, chi2_] := Coefficient[ExtDiff[d, chi1, chi2]〚1〛〚1, 1〛, v[3, 1], 1]
BB[d_, chi1_, chi2_] := Coefficient[ExtDiff[d, chi1, chi2]〚1〛〚1, 1〛, v[3, 1], 0]
CC[d_, chi1_, chi2_] := Coefficient[ExtDiff[d, chi1, chi2]〚1〛〚2, 1〛, v[3, 1], 1]
DD[d_, chi1_, chi2_] := Coefficient[ExtDiff[d, chi1, chi2]〚1〛〚2, 1〛, v[3, 1], 0]
```

In[•]:=
```
Constraint[d_, chi1_, chi2_] := Simplify[Together[
   AA[d, chi1, chi2] * DD[d, chi1, chi2] - BB[d, chi1, chi2] * CC[d, chi1, chi2]]]
```

In[•]:=
```
(* Show that there are no (d,chi1,chi2) with coprime 0<chi1,
chi2<d unless chi2=chi1 or chi2=d-chi1. *)
```

In[∘]:= AA[d, chi1, chi2]

Out[∘]=

```
((64 chi1^6 - 192 chi1^5 d - 128 chi1^6 d + 208 chi1^4 d^2 + 384 chi1^5 d^2 + 128 chi1^6 d^2 - 96 chi1^3 d^3 -
    392 chi1^4 d^3 - 384 chi1^5 d^3 - 64 chi1^6 d^3 - 8 chi1^2 d^4 + 144 chi1^3 d^4 + 344 chi1^4 d^4 + 192 chi1^5 d^4 +
    24 chi1 d^5 + 94 chi1^2 d^5 - 48 chi1^3 d^5 - 160 chi1^4 d^5 - 4 d^6 - 102 chi1 d^6 - 196 chi1^2 d^6 +
    12 d^7 + 156 chi1 d^7 + 140 chi1^2 d^7 - 9 d^8 - 108 chi1 d^8 - 24 chi1^2 d^8 + 2 d^9 + 24 chi1 d^9)
   ((chi1 (2 chi1^2-3 chi1 d+d^2))/(chi2 (2 chi2^2-3 chi2 d+d^2)))^(1/3)) / (128 chi1 (chi1 - d) (2 chi1 - d) (-2 + d) d^4) -
 ((-4 + d) (⋯ 28 ⋯ + 48 chi1 d^7) (chi1 (2 chi1^2 - 3 chi1 d + d^2) (-24 chi2^2 + 24 chi2 d + (-4 + d) d^2) +
    chi2 (2 chi2^2 - 3 chi2 d + d^2) (⋯ 1 ⋯)^(1/3) (4 (6 chi1^2 - 6 chi1 d + d^2) - d^3 (⋯ 1 ⋯)^(1/3)^2))) /
  (1024 chi1 chi2 (-2 + d) d^5 (-2 chi2 + d) (-chi2 + d) (2 chi1^2 - 3 chi1 d + d^2)) -
 ⋯ 1 ⋯/(8 ⋯ 1 ⋯) + ⋯ 1 ⋯ +
 ((chi1-d) ⋯ 4 ⋯ (-⋯ 1 ⋯/⋯ 1 ⋯ + ⋯ 4 ⋯)) / (32 chi2 ⋯ 3 ⋯ (⋯ 1 ⋯) (⋯ 1 ⋯/⋯ 1 ⋯)^(1/3)) +
 (chi1 ⋯ 2 ⋯
   (-(chi2 (2 chi2^2-3 chi2 d+d^2) (128 chi1^6+(-384 chi1^5-1344 chi1^6) d+⋯ 5 ⋯+(5+84 chi1+24 chi1^2) d^7+(-2-24 chi1) d^8)
       ((chi1 ⋯ 1 ⋯)/(chi2 ⋯ 1 ⋯))^(1/3)^2) /
     (128 chi1^2 (-2+d) d^4 (-2 chi1+d) (-chi1+d) (2 chi1^2-3 chi1 d+d^2)) -
    (3 ⋯ 3 ⋯ (⋯ 1 ⋯))/(16 chi1 ⋯ 3 ⋯ (-chi1+d)) - ⋯ 1 ⋯ + ⋯ 1 ⋯ + ⋯ 1 ⋯/⋯ 1 ⋯ -
    ((chi2 ⋯ 2 ⋯ ⋯ 1 ⋯^2/(8 chi1 (⋯ 1 ⋯) d (⋯ 1 ⋯)) + ⋯ 1 ⋯) (⋯ 1 ⋯)) / ((chi1 ⋯ 1 ⋯)/(chi2 ⋯ 1 ⋯))^(1/3))) /
  (2 chi2 (2 chi2^2 - 3 chi2 d + d^2) ((chi1 (2 chi1^2-3 chi1 d+d^2))/(chi2 (2 chi2^2-3 chi2 d+d^2)))^(1/3))
```

Full expression not available (original memory size: 0.7 MB)

In[ ]:=
```
Constraint[d, chi1, chi2]
```

Out[ ]=

$$-\left(\left(\text{chi1}\,(\text{chi1}-d)\,\left(12-9d+2d^2\right)\left(6\,(-1+d)^2 d^2\left(32-26d+5d^2\right)+4\,\text{chi1}^4\left(-120+150d+3d^2-47d^3+11d^4\right)-8\,\text{chi1}^3 d\left(-120+150d+3d^2-47d^3+11d^4\right)+\text{chi1}\,d\left(-768+2256d-2184d^2+822d^3-153d^4+50d^5-11d^6\right)+\text{chi1}^2\left(768-2256d+1704d^2-222d^3+165d^4-238d^5+55d^6\right)\right)\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(6\,\text{chi2}^2-6\,\text{chi2}\,d+d^2\right)-\text{chi2}\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)\Big/\left(128\,\text{chi2}^2\,(\text{chi2}-d)^2\,(-2+d)^3 d^4\,(-2\,\text{chi2}+d)^2\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)$$

## Type II - Analysing constraints to existence of solutions

```
(* We can ignore chi1 (chi1-d) (12-9d+2 d^2) and the
 denominator. The rest is a product of two parts P1 and P2. *)
```

In[ ]:=
```
P1[d_, chi1_] :=
 6 (-1 + d)^2 d^2 (32 - 26 d + 5 d^2) + 4 chi1^4 (-120 + 150 d + 3 d^2 - 47 d^3 + 11 d^4) -
  8 chi1^3 d (-120 + 150 d + 3 d^2 - 47 d^3 + 11 d^4) +
  chi1 d (-768 + 2256 d - 2184 d^2 + 822 d^3 - 153 d^4 + 50 d^5 - 11 d^6) +
  chi1^2 (768 - 2256 d + 1704 d^2 - 222 d^3 + 165 d^4 - 238 d^5 + 55 d^6)
P2[d_, chi1_, chi2_] :=
 (chi1 (2 chi1^2 - 3 chi1 d + d^2) (6 chi2^2 - 6 chi2 d + d^2) - chi2 (6 chi1^2 - 6 chi1 d + d^2)
    (2 chi2^2 - 3 chi2 d + d^2) CubeRoot[(chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2))])
```

```
(* We first show that P2 is nonzero unless chi1=chi2 or chi1+chi2=
  d. Note that P2 vanishing is equivalent to the vanishing of the following
   by taking cubes. By the argument in the paper involving coprimality,
this is nonzero unless chi1=chi2 or chi1+chi2=d. *)
```

In[ ]:=
```
Factor[chi1 ^2 (2 chi1^2 - 3 chi1 d + d^2) ^2 (6 chi2^2 - 6 chi2 d + d^2) ^3 -
  chi2^2 (6 chi1^2 - 6 chi1 d + d^2) ^3 (2 chi2^2 - 3 chi2 d + d^2) ^2 ]
```

Out[ ]=

$$(chi1 - chi2)\ (chi1 + chi2 - d)\ d^2 \left(216\ chi1^4\ chi2^4 - 432\ chi1^4\ chi2^3\ d - 432\ chi1^3\ chi2^4\ d + 288\ chi1^4\ chi2^2\ d^2 + 864\ chi1^3\ chi2^3\ d^2 + 288\ chi1^2\ chi2^4\ d^2 - 72\ chi1^4\ chi2\ d^3 - 576\ chi1^3\ chi2^2\ d^3 - 576\ chi1^2\ chi2^3\ d^3 - 72\ chi1\ chi2^4\ d^3 + 4\ chi1^4\ d^4 + 144\ chi1^3\ chi2\ d^4 + 382\ chi1^2\ chi2^2\ d^4 + 144\ chi1\ chi2^3\ d^4 + 4\ chi2^4\ d^4 - 8\ chi1^3\ d^5 - 94\ chi1^2\ chi2\ d^5 - 94\ chi1\ chi2^2\ d^5 - 8\ chi2^3\ d^5 + 5\ chi1^2\ d^6 + 22\ chi1\ chi2\ d^6 + 5\ chi2^2\ d^6 - chi1\ d^7 - chi2\ d^7\right)$$

```
(* Therefore, the Constraint vanishing is equivalent to P1[d,chi1]=
 0. By the symmetric role of chi1 and chi2, we also have P1[d,chi2]=0. *)
```

In[ ]:=
```
P1[d, chi1]
P1[d, chi2]
```

Out[ ]=

$$6\ (-1 + d)^2\ d^2\ (32 - 26\ d + 5\ d^2) + 4\ chi1^4\ (-120 + 150\ d + 3\ d^2 - 47\ d^3 + 11\ d^4) - 8\ chi1^3\ d\ (-120 + 150\ d + 3\ d^2 - 47\ d^3 + 11\ d^4) + chi1\ d\ (-768 + 2256\ d - 2184\ d^2 + 822\ d^3 - 153\ d^4 + 50\ d^5 - 11\ d^6) + chi1^2\ (768 - 2256\ d + 1704\ d^2 - 222\ d^3 + 165\ d^4 - 238\ d^5 + 55\ d^6)$$

Out[ ]=

$$6\ (-1 + d)^2\ d^2\ (32 - 26\ d + 5\ d^2) + 4\ chi2^4\ (-120 + 150\ d + 3\ d^2 - 47\ d^3 + 11\ d^4) - 8\ chi2^3\ d\ (-120 + 150\ d + 3\ d^2 - 47\ d^3 + 11\ d^4) + chi2\ d\ (-768 + 2256\ d - 2184\ d^2 + 822\ d^3 - 153\ d^4 + 50\ d^5 - 11\ d^6) + chi2^2\ (768 - 2256\ d + 1704\ d^2 - 222\ d^3 + 165\ d^4 - 238\ d^5 + 55\ d^6)$$

```
(* P1[d,x] is a degree four polynomial in x with a symmetry. Therefore
 it has four distinct roots chi1,chi2,d-chi1,d-chi2. *)
```

In[ ]:=
```
Simplify[P1[d, x] - P1[d, d - x]]
```

Out[ ]=

0

```
(* We check that P1[d,0]>0 and P1[d,1]<0. This implies that
  P1[d,x] must have one more additional root between 0 and 1,
contradicting that it is a degree four polynomial.  *)
```

In[∘]:=
```
Factor[P1[d, 0]]
Factor[P1[d, 1]]
```

Out[∘]=

$6\,(-2+d)\,(-1+d)^2\,d^2\,(-16+5\,d)$

Out[∘]=

$-\left((-3+d)\,(-2+d)^2\,(-1+d)\,\left(-24+66\,d-47\,d^2+11\,d^3\right)\right)$

In[∘]:=
```
Solve[P1[d, 0] == 0, d]
Solve[P1[d, 1] == 0, d]
```

Out[∘]=

$\left\{\{d\to 0\},\{d\to 0\},\{d\to 1\},\{d\to 1\},\{d\to 2\},\left\{d\to \frac{16}{5}\right\}\right\}$

Out[∘]=

$\{\{d\to 1\},\{d\to 2\},\{d\to 2\},\{d\to 3\},\{d\to 0.554\ldots\},$
$\{d\to 1.86\ldots-0.695\ldots\,i\},\{d\to 1.86\ldots+0.695\ldots\,i\}\}$

In[∘]:= `(* This proves that there are no ring isomorphisms. *)`