Weyl symmetry for curve counting invariants via spherical twists

Miguel Moreira
ETHZ
Joint with Tim Buelles

YMSC – Tsinghua University
Enumerative Geometry seminar
26 May 2022

Gromov-Witten invariants

Given a smooth projective variety X, Gromov-Witten theory uses the moduli of stable maps and its virtual fundamental class

$$[M_g(X,\beta)]^{\text{vir}} \in A_{\text{virdim}}(M_g(X,\beta)).$$

Gromov-Witten invariants

Given a smooth projective variety X, Gromov-Witten theory uses the moduli of stable maps and its virtual fundamental class

$$[M_g(X,\beta)]^{\text{vir}} \in A_{\text{virdim}}(M_g(X,\beta)).$$

A special case is when X is a Calabi-Yau 3-fold (CY3): the virtual dimension of is 0 for all $g \geq 0$, $\beta \in H_2(X;\mathbb{Z})$ so we get numbers

$$\mathrm{GW}^X_{g,\beta} = \int_{[M_g(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Q}.$$

Gromov-Witten invariants

Given a smooth projective variety X, Gromov-Witten theory uses the moduli of stable maps and its virtual fundamental class

$$[M_g(X,\beta)]^{\mathrm{vir}} \in A_{\mathrm{virdim}}(M_g(X,\beta)).$$

A special case is when X is a Calabi-Yau 3-fold (CY3): the virtual dimension of is 0 for all $g \geq 0$, $\beta \in H_2(X;\mathbb{Z})$ so we get numbers

$$\mathrm{GW}^X_{g,\beta} = \int_{[M_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

Goal:

Compute all numbers $\mathrm{GW}^X_{g,\beta}$. Equivalently, understand the partition function

$$Z_X = \exp\left(\sum_{g,\beta} \mathrm{GW}^X_{g,\beta} u^{2g-2} z^\beta\right).$$

Stable pairs

Stable pairs provide an alternative approach to curve counting on $CY3$.

Stable pairs

Stable pairs provide an alternative approach to curve counting on $CY3$.

Definition (Pandharipande-Thomas '09)

A stable pair on X is an object $\{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$ in the derived category where F is a coherent sheaf and s a section satisfying the following two stability conditions:

1. F is pure of dimension 1: every non-trivial coherent sub-sheaf of F has dimension 1.
2. The cokernel of s has dimension 0.

Stable pairs

Stable pairs provide an alternative approach to curve counting on $CY3$.

Definition (Pandharipande-Thomas '09)

A stable pair on X is an object $\{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$ in the derived category where F is a coherent sheaf and s a section satisfying the following two stability conditions:

1. F is pure of dimension 1: every non-trivial coherent sub-sheaf of F has dimension 1.
2. The cokernel of s has dimension 0.

We associate two discrete invariants:

$$\beta = \mathrm{ch}_2(F) = [\mathrm{supp}(F)] \in H_2(X;\mathbb{Z}) \quad \text{and} \quad n = \chi(X, F).$$

Stable pairs

Stable pairs provide an alternative approach to curve counting on $CY3$.

Definition (Pandharipande-Thomas '09)

A stable pair on X is an object $\{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$ in the derived category where F is a coherent sheaf and s a section satisfying the following two stability conditions:

1. F is pure of dimension 1: every non-trivial coherent sub-sheaf of F has dimension 1.
2. The cokernel of s has dimension 0.

We associate two discrete invariants:

$$\beta = \mathrm{ch}_2(F) = [\mathrm{supp}(F)] \in H_2(X;\mathbb{Z}) \quad \text{and} \quad n = \chi(X,F).$$

The space $P_n(X,\beta)$ parametrizing stable pairs with fixed discrete invariants is a projective fine moduli space.

Pandharipande-Thomas invariants

The moduli of stable pairs $P_n(X,\beta)$ also has a virtual fundamental class, and when X is a CY3 its virtual dimension is 0, producing again numbers

$$\mathrm{PT}^X_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Pandharipande-Thomas invariants

The moduli of stable pairs $P_n(X,\beta)$ also has a virtual fundamental class, and when X is a CY3 its virtual dimension is 0, producing again numbers

$$\mathrm{PT}^X_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande '06)

The Gromov-Witten and Pandharipande-Thomas invariants determine each other:

Pandharipande-Thomas invariants

The moduli of stable pairs $P_n(X, \beta)$ also has a virtual fundamental class, and when X is a CY3 its virtual dimension is 0, producing again numbers

$$\mathrm{PT}^X_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande '06)

The Gromov-Witten and Pandharipande-Thomas invariants determine each other:

$$\exp\left(\sum_{g,\beta} \mathrm{GW}^X_{g,\beta} u^{2g-2} z^\beta\right) = \sum_{n,\beta} \mathrm{PT}^X_{n,\beta} (-q)^n z^\beta$$

after the change of variables $q = e^{iu}$.

Rationality and symmetry

Theorem (Bridgeland, Toda '16)

Rationality and symmetry

Theorem (Bridgeland, Toda '16)

For each β the generating function

$$\sum_{n\in\mathbb{Z}} \mathrm{PT}^X_{n,\beta}(-q)^n$$

is the expansion of a rational function f_β satisfying the symmetry

$$f_\beta(1/q) = f_\beta(q).$$

Think of the theorem as $\mathrm{PT}_{n,\beta} \sim \mathrm{PT}_{-n,\beta}$ after analytic continuation.

Rationality and symmetry

Theorem (Bridgeland, Toda '16)

For each β the generating function

$$\sum_{n\in\mathbb{Z}} \mathrm{PT}^X_{n,\beta}(-q)^n$$

is the expansion of a rational function f_β satisfying the symmetry

$$f_\beta(1/q) = f_\beta(q).$$

Think of the theorem as $\mathrm{PT}_{n,\beta} \sim \mathrm{PT}_{-n,\beta}$ after analytic continuation.

Typical example (contribution of isolated rational curve):

$$f(q) = \frac{q}{(1-q)^2} = q + 2q^2 + 3q^3 + \dots$$
$$= q^{-1} + 2q^{-2} + 3q^{-3} + \dots$$

Proof of rationality

The proof of rationality illustrates a very general principle:

Proof of rationality

The proof of rationality illustrates a very general principle:

$$\text{Symmetry of the derived category } \phi \in \mathrm{Aut}(D^b(X))$$
$$\downarrow$$
$$\text{Constraints on curve counting on } X.$$

Proof of rationality

The proof of rationality illustrates a very general principle:

$$\text{Symmetry of the derived category } \phi \in \mathrm{Aut}(D^b(X))$$
$$\downarrow$$
$$\text{Constraints on curve counting on } X.$$

The proof of rationality uses the derived dual

$$\phi = \mathbb{D} = \mathrm{RHom}(-, \mathcal{O}_X)[2].$$

Proof of rationality

The proof of rationality illustrates a very general principle:

Symmetry of the derived category $\phi \in \mathrm{Aut}(D^b(X))$

$$\downarrow$$

Constraints on curve counting on X.

The proof of rationality uses the derived dual

$$\phi = \mathbb{D} = \mathrm{RHom}(-, \mathcal{O}_X)[2].$$

Note: $\chi(\mathbb{D}(F)) = -\chi(F)$.

Proof of rationality

The proof of rationality illustrates a very general principle:

$$\text{Symmetry of the derived category } \phi \in \mathrm{Aut}(D^b(X))$$
$$\downarrow$$
$$\text{Constraints on curve counting on } X.$$

The proof of rationality uses the derived dual

$$\phi = \mathbb{D} = \mathrm{RHom}(-, \mathcal{O}_X)[2].$$

Note: $\chi(\mathbb{D}(F)) = -\chi(F)$.
Basic idea: use wall-crossing in the derived category to relate

$$P_n(X, \beta) \leftrightsquigarrow \phi(P_n(X, \beta)) \subseteq D^b(X).$$

Geometric setting

Let W be a ruled surface over a genus g curve C, i.e.

$$W = \mathbb{P}_C(\mathcal{E}) \to C.$$

Geometric setting

Let W be a ruled surface over a genus g curve C, i.e.

$$W = \mathbb{P}_C(\mathcal{E}) \to C.$$

Let X be a Calabi-Yau 3-fold containing W as a divisor. Let $B = [\mathbb{P}^1] \in H_2(X)$ be the curve class of the fibers of the ruling.

$$\begin{array}{ccccc} B & \hookrightarrow & W & \overset{\iota}{\hookrightarrow} & X \\ & & \downarrow{\scriptstyle p} & & \\ & & C & & \end{array}$$

Geometric setting

Let W be a ruled surface over a genus g curve C, i.e.

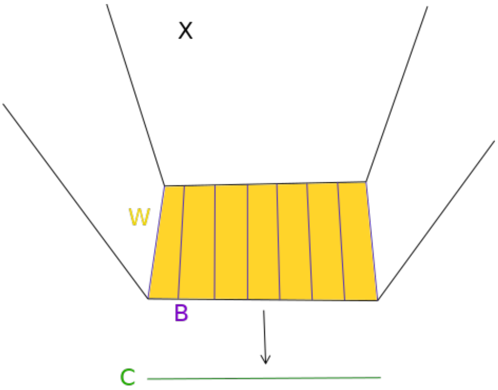
$$W = \mathbb{P}_C(\mathcal{E}) \to C.$$

Let X be a Calabi-Yau 3-fold containing W as a divisor. Let $B = [\mathbb{P}^1] \in H_2(X)$ be the curve class of the fibers of the ruling.

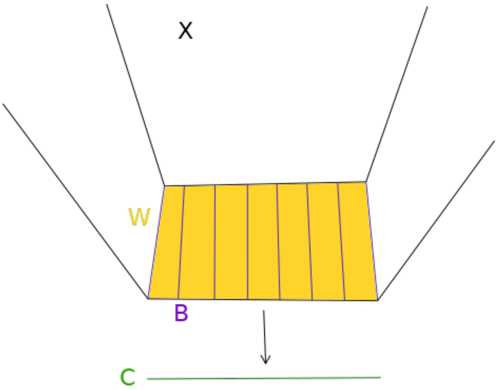
$$\begin{array}{ccccc} B & \hookrightarrow & W & \overset{\iota}{\hookrightarrow} & X \\ & & \downarrow{\scriptstyle p} & & \\ & & C & & \end{array}$$

We also assume that the ray generated by B is extremal in the effective cone of X, i.e. if C_1, C_2 are effective curve classes such that $C_1 + C_2$ is a multiple of B then both C_1, C_2 are multiples of B.

Geometric setting



Geometric setting



Examples

- $X = K_W$
- X elliptic fibration over W
- $X =$ STU model, which is a particular elliptic fibration over $\mathbb{P}^1 \times \mathbb{P}^1$.

Weyl symmetry

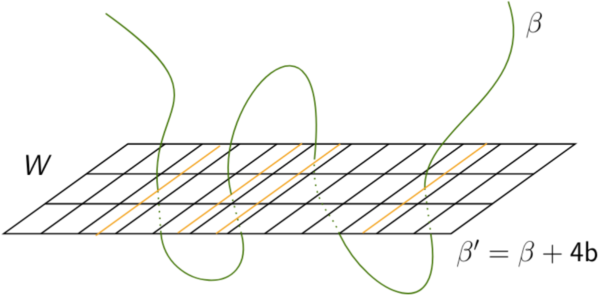
Consider the involution defined on $H_2(X)$ by

$$\beta \mapsto \beta' = \beta + (W \cdot \beta)B.$$

Weyl symmetry

Consider the involution defined on $H_2(X)$ by

$$\beta \mapsto \beta' = \beta + (W \cdot \beta)B.$$



(it's an involution because $W \cdot B = -2$)

Weyl symmetry for PT invariants

Our work is about some symmetry relating curve counting invariants in classes β and β'

$$\mathrm{GW}_{g,\beta} \sim \mathrm{GW}_{g,\beta'}$$
$$\mathrm{PT}_{n,\beta} \sim \mathrm{PT}_{n,\beta'}.$$

Weyl symmetry for PT invariants

Our work is about some symmetry relating curve counting invariants in classes β and β'

$$\mathrm{GW}_{g,\beta} \sim \mathrm{GW}_{g,\beta'}$$
$$\mathrm{PT}_{n,\beta} \sim \mathrm{PT}_{n,\beta'}.$$

Let

$$\mathrm{PT}_\beta(q,Q) = \sum_{n,j\in\mathbb{Z}} P_{n,\beta+jB}\,(-q)^n Q^j\,.$$

Weyl symmetry for PT invariants

Our work is about some symmetry relating curve counting invariants in classes β and β'

$$\mathrm{GW}_{g,\beta} \sim \mathrm{GW}_{g,\beta'}$$
$$\mathrm{PT}_{n,\beta} \sim \mathrm{PT}_{n,\beta'}.$$

Let

$$\mathrm{PT}_\beta(q,Q) = \sum_{n,j\in\mathbb{Z}} P_{n,\beta+jB}\,(-q)^n Q^j\,.$$

The generating series PT_0 of multiples of B can be shown to equal

$$\mathrm{PT}_0(q,Q) = \prod_{j\geq 1}(1-q^j Q)^{(2g-2)j}.$$

Weyl symmetry for PT invariants

Theorem (Buelles-M. '21/22)

Let X be a Calabi-Yau 3-fold containing a smooth, ruled divisor W as described before.

Weyl symmetry for PT invariants

Theorem (Buelles-M. '21/22)

Let X be a Calabi-Yau 3-fold containing a smooth, ruled divisor W as described before. Then

$$\frac{\mathrm{PT}_\beta(q,Q)}{\mathrm{PT}_0(q,Q)} \in \mathbb{Q}(q,Q)$$

is the expansion of a rational function $f_\beta(q,Q)$

Weyl symmetry for PT invariants

Theorem (Buelles-M. '21/22)

Let X be a Calabi-Yau 3-fold containing a smooth, ruled divisor W as described before. Then

$$\frac{\mathrm{PT}_\beta(q,Q)}{\mathrm{PT}_0(q,Q)} \in \mathbb{Q}(q,Q)$$

is the expansion of a rational function $f_\beta(q,Q)$ which satisfies the functional equations

$$f_\beta(q^{-1},Q) = f_\beta(q,Q) \text{ and } f_\beta(q,Q^{-1}) = Q^{-W\cdot\beta} f_\beta(q,Q).$$

Weyl symmetry for GW invariants

Corollary (Assuming GW/PT)

Weyl symmetry for GW invariants

Corollary (Assuming GW/PT)

For all $(g, \beta) \neq (0, mB), (1, mB)$ *the series*

$$\sum_{j \in \mathbb{Z}} \mathrm{GW}_{g, \beta + jB} \, Q^j$$

is the expansion of a rational function $f_\beta(Q)$ *with functional equation*

$$f_\beta(Q^{-1}) = Q^{-W \cdot \beta} f_\beta(Q) \, .$$

Think of the functional equation as equality

$$\mathrm{GW}_{g,\beta} \sim \mathrm{GW}_{g,\beta'}$$

after analytic continuation.

Weyl symmetry for GW invariants

Corollary (Assuming GW/PT)

For all $(g, \beta) \neq (0, mB), (1, mB)$ *the series*

$$\sum_{j \in \mathbb{Z}} \mathrm{GW}_{g,\beta+jB}\, Q^j$$

is the expansion of a rational function $f_\beta(Q)$ *with functional equation*

$$f_\beta(Q^{-1}) = Q^{-W\cdot\beta} f_\beta(Q)\,.$$

Think of the functional equation as equality

$$\mathrm{GW}_{g,\beta} \sim \mathrm{GW}_{g,\beta'}$$

after analytic continuation.

Predicted by physics, at least in the local case K_W (Katz-Klemm-Vafa '97).

Examples

Example

Let $X = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ and let C be the other $\mathbb{P}^1$ in the product. A computation with the topological vertex shows:

$$\frac{\mathrm{PT}_C(q, Q)}{\mathrm{PT}_0(q, Q)} = \frac{2q}{(1-q)^2(1-Q)^2}$$

Examples

Example

Let $X = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ and let C be the other $\mathbb{P}^1$ in the product. A computation with the topological vertex shows:

$$\frac{\mathrm{PT}_C(q,Q)}{\mathrm{PT}_0(q,Q)} = \frac{2q}{(1-q)^2(1-Q)^2}$$

$$\frac{\mathrm{PT}_{2C}(q,Q)}{\mathrm{PT}_0(q,Q)} = \frac{2q^4}{(1-q)^2(1-q^2)^2(1-qQ)^2(1-Q)^2} + \frac{2q^4}{(1-q)^2(1-q^2)^2(q-Q)^2(1-Q)^2} + \frac{2q^4}{(1-q)^4(1-qQ)^2(q-Q)^2}.$$

Spherical twists

The main ingredient of our symmetry is the existence of a certain anti-equivalence $\rho \in \mathrm{Aut}(D^b(X))$ promoting the involution

$$\beta \mapsto \beta' = \beta + (W \cdot \beta)B$$

on $H_2(X)$ to the derived category.

Spherical twists

The main ingredient of our symmetry is the existence of a certain anti-equivalence $\rho \in \mathrm{Aut}(D^b(X))$ promoting the involution

$$\beta \mapsto \beta' = \beta + (W \cdot \beta)B$$

on $H_2(X)$ to the derived category. It's constructed using the spherical functor

$$\begin{aligned} \Phi \colon D^b(C) &\to D^b(X) \\ V &\mapsto \iota_* \left(\mathcal{O}_p(-1) \otimes p^* V\right). \end{aligned}$$

Spherical twists

The main ingredient of our symmetry is the existence of a certain anti-equivalence $\rho \in \mathrm{Aut}(D^b(X))$ promoting the involution

$$\beta \mapsto \beta' = \beta + (W \cdot \beta)B$$

on $H_2(X)$ to the derived category. It's constructed using the spherical functor

$$\begin{aligned}\Phi\colon D^b(C) &\to D^b(X)\\ V &\mapsto \iota_*\left(\mathcal{O}_p(-1) \otimes p^* V\right).\end{aligned}$$

From a spherical functor we associate an automorphism of the derived category, the spherical twist ST defined by

$$\Phi \circ \Phi_R \longrightarrow \mathrm{id} \longrightarrow \mathrm{ST}.$$

Derived equivalence ρ

We already have the derived equivalence $\mathrm{ST} \in \mathsf{Aut}(D^b(X))$. The derived equivalence ρ is then

$$\rho = \mathrm{ST} \circ \mathbb{D}.$$

Derived equivalence ρ

We already have the derived equivalence $\mathrm{ST} \in \mathrm{Aut}(D^b(X))$. The derived equivalence ρ is then

$$\rho = \mathrm{ST} \circ \mathbb{D}.$$

Facts

1. ρ is an involution, i.e. $\rho \circ \rho = \mathrm{id}$.

Derived equivalence ρ

We already have the derived equivalence $\mathrm{ST} \in \mathrm{Aut}(D^b(X))$. The derived equivalence ρ is then

$$\rho = \mathrm{ST} \circ \mathbb{D}.$$

Facts

1. ρ is an involution, i.e. $\rho \circ \rho = \mathrm{id}$.
2. $\rho(\mathcal{O}_X) = \mathcal{O}_X[2]$.

Derived equivalence ρ

We already have the derived equivalence $\mathrm{ST} \in \mathrm{Aut}(D^b(X))$. The derived equivalence ρ is then

$$\rho = \mathrm{ST} \circ \mathbb{D}.$$

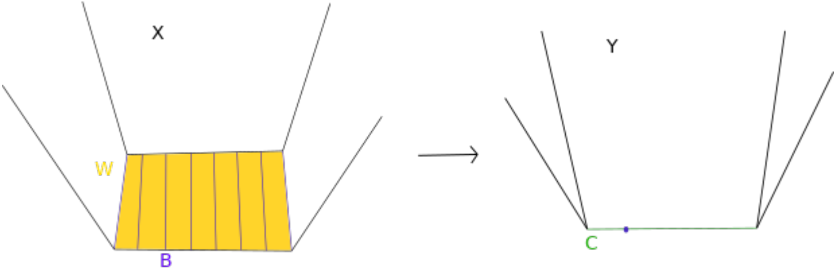
Facts

1. ρ is an involution, i.e. $\rho \circ \rho = \mathrm{id}$.
2. $\rho(\mathcal{O}_X) = \mathcal{O}_X[2]$.
3. If F is a sheaf of dimension 1 and $\mathrm{ch}_2(F) = \beta, \chi(F) = n$ then

$$\mathrm{ch}_2(\rho(F)) = \beta' = \beta + (W \cdot \beta)B$$
$$\chi(\rho(F)) = -n.$$

Orbifold inspiration

When X arises as a crepant resolution $X \to \mathcal{Y}$ of an orbifold with $\mathbb{Z}/2$-singularities along the curve C so that W is the exceptional divisor (and the fibers B are contracted to points), the main result is a consequence of the DT crepant resolution conjecture proven by Beentjes-Calabrese-Rennemo ('18).



Orbifold inspiration

Their proof immitates Bridgeland-Toda proof of rationality using $\mathbb{D}^{\mathcal{Y}}$ to prove the symmetry of PT invariants in $\mathcal{Y}$.

Orbifold inspiration

Their proof immitates Bridgeland-Toda proof of rationality using $\mathbb{D}^{\mathcal{Y}}$ to prove the symmetry of PT invariants in $\mathcal{Y}$.

Proposition

Under the McKay correspondence

$$\Psi : D^b(X) \xrightarrow{\sim} D^b(\mathcal{Y})$$

the derived dual $\mathbb{D}^{\mathcal{Y}}$ corresponds to ρ, i.e.

$$\rho = \Psi^{-1} \circ \mathbb{D}^{\mathcal{Y}} \circ \Psi.$$

Orbifold inspiration

Their proof immitates Bridgeland-Toda proof of rationality using $\mathbb{D}^{\mathcal{Y}}$ to prove the symmetry of PT invariants in $\mathcal{Y}$.

Proposition

Under the McKay correspondence

$$\Psi : D^b(X) \xrightarrow{\sim} D^b(\mathcal{Y})$$

the derived dual $\mathbb{D}^{\mathcal{Y}}$ corresponds to ρ, i.e.

$$\rho = \Psi^{-1} \circ \mathbb{D}^{\mathcal{Y}} \circ \Psi.$$

Important examples (e.g. the STU) don't arise as such crepant resolution.

Homological mirror symmetry?

What can we say about the mirror geometry $\check{X}$? In particular, how to interpret the derived equivalence ST under HMS:

$$\mathrm{ST} \in \mathsf{Aut}(D^b(X)) \cong \mathsf{Aut}(\mathsf{Fuk}(\check{X}))?$$

Homological mirror symmetry?

What can we say about the mirror geometry $\check{X}$? In particular, how to interpret the derived equivalence ST under HMS:

$$\mathrm{ST} \in \mathsf{Aut}(D^b(X)) \cong \mathsf{Aut}(\mathsf{Fuk}(\check{X}))?$$

When the genus of C is $g = 0$ we can write ST as a composition of twists around spherical objects

$$\mathrm{ST} = \mathrm{ST}_{\mathcal{O}_W(-C+B)} \circ \mathrm{ST}_{\mathcal{O}_W(-C)}$$

so (the mirror of) ST should be induced by a symplectomorphism obtained as a composition of two Dehn twists.

Homological mirror symmetry?

How to think about this composition and what about $g > 0$?

Homological mirror symmetry?

How to think about this composition and what about $g > 0$? A typical way in which spherical functors appear in the Fukaya category is through symplectic fibrations: if $w \colon Z \to \mathbb{C}$ is a symplectic fibration with general fiber $\check{X}$ then we get a spherical functor

$$\mathrm{FS}(Z, w) \xrightarrow{\cap} \mathrm{Fuk}(\check{X})$$

Homological mirror symmetry?

How to think about this composition and what about $g > 0$? A typical way in which spherical functors appear in the Fukaya category is through symplectic fibrations: if $w\colon Z \to \mathbb{C}$ is a symplectic fibration with general fiber $\check{X}$ then we get a spherical functor

$$\begin{array}{ccc} \mathrm{FS}(Z, w) & \xrightarrow{\cap} & \mathrm{Fuk}(\check{X}) \\ \text{HMS?}\Big\| & & \Big\|\text{HMS} \\ D^b(C) & \xrightarrow{\Phi} & D^b(X) \end{array}$$

Homological mirror symmetry?

How to think about this composition and what about $g > 0$? A typical way in which spherical functors appear in the Fukaya category is through symplectic fibrations: if $w\colon Z \to \mathbb{C}$ is a symplectic fibration with general fiber $\check{X}$ then we get a spherical functor

$$\begin{array}{ccc} \mathrm{FS}(Z, w) & \xrightarrow{\cap} & \mathrm{Fuk}(\check{X}) \\ \text{HMS?} \Big\| & & \Big\| \text{HMS} \\ D^b(C) & \xrightarrow{\Phi} & D^b(X) \end{array}$$

In such a situation, the derived equivalence corresponding to ST on the symplectic side would be induced by monodromy of $w\colon Z \to \mathbb{C}$ around ∞.

Perverse stable pairs

Recall that stable pairs are of the form $s\colon \mathcal{O}_X \to F$ with $F \in \mathrm{Coh}_1(X), \mathrm{coker}(s) \in \mathrm{Coh}_0(X)$.

Perverse stable pairs

Recall that stable pairs are of the form $s\colon \mathcal{O}_X \to F$ with $F \in \mathrm{Coh}_1(X), \mathrm{coker}(s) \in \mathrm{Coh}_0(X)$.
Bridgeland's proof of rationality with the derived dual uses

$$\mathbb{D}(\mathrm{Coh}_1(X)) = \mathrm{Coh}_1(X) \text{ and } \mathbb{D}(\mathrm{Coh}_0(X)) = \mathrm{Coh}_0(X)[-1]$$

to describe the image of $\mathbb{D}(P_n(X,\beta))$ and to help finding wall-crossing between $\mathbb{D}(P_n(X,\beta))$ and $P_n(X,\beta)$.

Perverse stable pairs

Recall that stable pairs are of the form $s\colon \mathcal{O}_X \to F$ with $F \in \mathrm{Coh}_1(X), \mathrm{coker}(s) \in \mathrm{Coh}_0(X)$.
Bridgeland's proof of rationality with the derived dual uses

$$\mathbb{D}(\mathrm{Coh}_1(X)) = \mathrm{Coh}_1(X) \text{ and } \mathbb{D}(\mathrm{Coh}_0(X)) = \mathrm{Coh}_0(X)[-1]$$

to describe the image of $\mathbb{D}(P_n(X,\beta))$ and to help finding wall-crossing between $\mathbb{D}(P_n(X,\beta))$ and $P_n(X,\beta)$.

Example

If $x \in W$ is a point in the divisor lying in a fiber B then

$$\rho(\mathcal{O}_x) = \{\mathcal{O}_B(-1)[-1] \to \mathcal{O}_B(-2)\}.$$

Perverse sheaves

To study ρ it's more appropriate to use a tilt of $\mathrm{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

Perverse sheaves

To study ρ it's more appropriate to use a tilt of $\mathrm{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$\mathcal{T} = \{T \in \mathrm{Coh}(X) : R^1 p_* T_{|W} = 0\}$$
$$\mathcal{F} = \{F \in \mathrm{Coh}(X) : \mathrm{Hom}(\mathcal{T}, F) = 0\}$$
$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}}.$$

Perverse sheaves

To study ρ it's more appropriate to use a tilt of $\mathrm{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$\mathcal{T} = \{T \in \mathrm{Coh}(X) : R^1 p_* T_{|W} = 0\}$$
$$\mathcal{F} = \{F \in \mathrm{Coh}(X) : \mathrm{Hom}(\mathcal{T}, F) = 0\}$$
$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}}.$$

$\mathcal{A}$ is a heart of $D^b(X)$ and its elements are perverse sheaves. The dimension of a perverse sheaf is the dimension of its support after we contract the fibers B.

Perverse sheaves

To study ρ it's more appropriate to use a tilt of $\mathrm{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$\mathcal{T} = \{T \in \mathrm{Coh}(X) : R^1 p_* T_{|W} = 0\}$$
$$\mathcal{F} = \{F \in \mathrm{Coh}(X) : \mathrm{Hom}(\mathcal{T}, F) = 0\}$$
$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}}.$$

$\mathcal{A}$ is a heart of $D^b(X)$ and its elements are perverse sheaves. The dimension of a perverse sheaf is the dimension of its support after we contract the fibers B.

Example

1. $\mathrm{Coh}_0 \subseteq \mathcal{A}_0$;

Perverse sheaves

To study ρ it's more appropriate to use a tilt of $\mathrm{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$\mathcal{T} = \{T \in \mathrm{Coh}(X) : R^1 p_* T_{|W} = 0\}$$
$$\mathcal{F} = \{F \in \mathrm{Coh}(X) : \mathrm{Hom}(\mathcal{T}, F) = 0\}$$
$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}}.$$

$\mathcal{A}$ is a heart of $D^b(X)$ and its elements are perverse sheaves. The dimension of a perverse sheaf is the dimension of its support after we contract the fibers B.

Example

1. $\mathrm{Coh}_0 \subseteq \mathcal{A}_0$;
2. $\mathcal{O}_B(-1), \mathcal{O}_B(-2)[1] \in \mathcal{A}_0$;

Perverse sheaves

To study ρ it's more appropriate to use a tilt of $\mathrm{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$\mathcal{T} = \{T \in \mathrm{Coh}(X) : R^1 p_* T_{|W} = 0\}$$
$$\mathcal{F} = \{F \in \mathrm{Coh}(X) : \mathrm{Hom}(\mathcal{T}, F) = 0\}$$
$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}}.$$

$\mathcal{A}$ is a heart of $D^b(X)$ and its elements are perverse sheaves. The dimension of a perverse sheaf is the dimension of its support after we contract the fibers B.

Example

1. $\mathrm{Coh}_0 \subseteq \mathcal{A}_0$;
2. $\mathcal{O}_B(-1), \mathcal{O}_B(-2)[1] \in \mathcal{A}_0$;
3. $\mathcal{O}_p(-1), \mathcal{O}_p(-2)[1] \in \mathcal{A}_1$.

Perverse stable pairs

The action of ρ on $\mathcal{A}$ (with perverse dimension) is analogous to the action of $\mathbb{D}$ on Coh(X) (with usual dimension):

$$\rho(\mathcal{A}_1) = \mathcal{A}_1 \text{ and } \rho(\mathcal{A}_0) = \mathcal{A}_0[-1].$$

Perverse stable pairs

The action of ρ on $\mathcal{A}$ (with perverse dimension) is analogous to the action of $\mathbb{D}$ on Coh(X) (with usual dimension):

$$\rho(\mathcal{A}_1) = \mathcal{A}_1 \text{ and } \rho(\mathcal{A}_0) = \mathcal{A}_0[-1].$$

Definition

A perverse stable pair is an object $I \in \langle \mathcal{O}_X[1], \mathcal{A}_{\leq 1} \rangle_{\mathrm{ex}}$ such that $\mathrm{rk}(I) = -1$ and

$$\mathrm{Hom}(\mathcal{A}_0, I) = 0 = \mathrm{Hom}(I, \mathcal{A}_1).$$

Perverse stable pairs

The action of ρ on $\mathcal{A}$ (with perverse dimension) is analogous to the action of $\mathbb{D}$ on Coh(X) (with usual dimension):

$$\rho(\mathcal{A}_1) = \mathcal{A}_1 \text{ and } \rho(\mathcal{A}_0) = \mathcal{A}_0[-1].$$

Definition

A perverse stable pair is an object $I \in \langle \mathcal{O}_X[1], \mathcal{A}_{\leq 1} \rangle_{\mathrm{ex}}$ such that $\mathrm{rk}(I) = -1$ and

$$\mathrm{Hom}(\mathcal{A}_0, I) = 0 = \mathrm{Hom}(I, \mathcal{A}_1).$$

We define the virtual counts of perverse stable pairs:

$${}^p\mathrm{PT}_{n,\beta} \in \mathbb{Z},$$

$${}^p\mathrm{PT}_\beta(q, Q) = \sum_{n,j\in\mathbb{Z}} {}^p\mathrm{PT}_{n,\beta+jB}(-q)^n Q^j.$$

Rationality for ${}^p\mathrm{PT}$

Theorem (Buelles-M)

The series ${}^p\mathrm{PT}_\beta(q, Q)$ *is the expansion of a rational function* $f_\beta \in \mathbb{Q}(q, Q)$ *satisfying the symmetry*

$$f_\beta(q^{-1}, Q^{-1}) = Q^{-W\cdot\beta} f_\beta(q, Q).$$

Rationality for ${}^p\mathrm{PT}$

Theorem (Buelles-M)

The series ${}^p\mathrm{PT}_\beta(q, Q)$ is the expansion of a rational function $f_\beta \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_\beta(q^{-1}, Q^{-1}) = Q^{-W\cdot\beta} f_\beta(q, Q).$$

- Rationality of $\mathrm{PT}_\beta(q)$
- Rationality of ${}^p\mathrm{PT}_\beta(q, Q)$

Rationality for ${}^p\mathrm{PT}$

Theorem (Buelles-M)

The series ${}^p\mathrm{PT}_\beta(q, Q)$ is the expansion of a rational function $f_\beta \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_\beta(q^{-1}, Q^{-1}) = Q^{-W \cdot \beta} f_\beta(q, Q).$$

- Rationality of $\mathrm{PT}_\beta(q)$
- Anti-equivalence $\mathbb{D}$

- Rationality of ${}^p\mathrm{PT}_\beta(q, Q)$
- Anti-equivalence ρ

Rationality for ${}^p\mathrm{PT}$

Theorem (Buelles-M)

The series ${}^p\mathrm{PT}_\beta(q,Q)$ is the expansion of a rational function $f_\beta \in \mathbb{Q}(q,Q)$ satisfying the symmetry

$$f_\beta(q^{-1},Q^{-1}) = Q^{-W\cdot\beta} f_\beta(q,Q).$$

- Rationality of $\mathrm{PT}_\beta(q)$
- Anti-equivalence $\mathbb{D}$
- Torsion pair $\langle \mathrm{Coh}_0, \mathrm{Coh}_1 \rangle$

- Rationality of ${}^p\mathrm{PT}_\beta(q,Q)$
- Anti-equivalence ρ
- Torsion pair $\langle \mathcal{A}_0, \mathcal{A}_1 \rangle$

Rationality for ${}^p\mathrm{PT}$

Theorem (Buelles-M)

The series ${}^p\mathrm{PT}_\beta(q, Q)$ is the expansion of a rational function $f_\beta \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_\beta(q^{-1}, Q^{-1}) = Q^{-W\cdot\beta} f_\beta(q, Q).$$

- Rationality of $\mathrm{PT}_\beta(q)$
- Anti-equivalence $\mathbb{D}$
- Torsion pair $\langle \mathrm{Coh}_0, \mathrm{Coh}_1 \rangle$
- Usual slope stability

- Rationality of ${}^p\mathrm{PT}_\beta(q, Q)$
- Anti-equivalence ρ
- Torsion pair $\langle \mathcal{A}_0, \mathcal{A}_1 \rangle$
- Nironi slope stability

Rationality for ${}^p\mathrm{PT}$

Theorem (Buelles-M)

The series ${}^p\mathrm{PT}_\beta(q, Q)$ is the expansion of a rational function $f_\beta \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_\beta(q^{-1}, Q^{-1}) = Q^{-W\cdot\beta} f_\beta(q, Q).$$

- Rationality of $\mathrm{PT}_\beta(q)$
- Anti-equivalence $\mathbb{D}$
- Torsion pair $\langle \mathrm{Coh}_0, \mathrm{Coh}_1 \rangle$
- Usual slope stability
- Vanishing of Poisson brackets $\{\mathrm{Coh}_{\leq 1}, \mathrm{Coh}_{\leq 1}\} = 0$

- Rationality of ${}^p\mathrm{PT}_\beta(q, Q)$
- Anti-equivalence ρ
- Torsion pair $\langle \mathcal{A}_0, \mathcal{A}_1 \rangle$
- Nironi slope stability
- No vanishing, extra combinatorial difficulty (dealt with in [BCR]).

Wall-crossing

We proved rationality of perverse PT invariants, but now need to relate them to classical stable pairs.

Wall-crossing

We proved rationality of perverse PT invariants, but now need to relate them to classical stable pairs.

Theorem (Buelles-M)

For any $\beta \in H_2(X;\mathbb{Z})$ we have the following identity of rational functions:

$$ {}^{p}\mathrm{PT}_{\beta}(q,Q) = \frac{\mathrm{PT}_{\beta}(q,Q)}{\mathrm{PT}_{0}(q,Q)}. $$

Wall-crossing

We proved rationality of perverse PT invariants, but now need to relate them to classical stable pairs.

Theorem (Buelles-M)

For any $\beta \in H_2(X;\mathbb{Z})$ we have the following identity of rational functions:

$$^{p}\mathrm{PT}_\beta(q,Q) = \frac{\mathrm{PT}_\beta(q,Q)}{\mathrm{PT}_0(q,Q)}.$$

The wall-crossing establishing the equality has two steps and uses the counting of a third type of objects: Bryan-Steinberg invariants.

Wall-crossing PT/BS

When X arises as a crepant resolution $X \to \mathcal{Y}$, Bryan-Steinberg introduced ('12) invariants $\mathrm{BS}_{n,\beta}$. Roughly speaking, they count sheafs+sections $\{\mathcal{O}_X \xrightarrow{s} F\}$ but allowing the cokernel to have support on finitely many fibers B.

Wall-crossing PT/BS

When X arises as a crepant resolution $X \to \mathcal{Y}$, Bryan-Steinberg introduced ('12) invariants $\mathrm{BS}_{n,\beta}$. Roughly speaking, they count sheafs+sections $\{\mathcal{O}_X \xrightarrow{s} F\}$ but allowing the cokernel to have support on finitely many fibers B.
They provide a natural interpretation for the quotient $\mathrm{PT}_\beta/\mathrm{PT}_0$ via a DT/PT type wall-crossing.

Proposition

$$\mathrm{BS}_\beta(q, Q) \equiv \sum_{n,j\in\mathbb{Z}} \mathrm{BS}_{n,\beta+jB}(-q)^n Q^j = \frac{\mathrm{PT}_\beta(q, Q)}{\mathrm{PT}_0(q, Q)}.$$

Wall-crossing PT/BS

When X arises as a crepant resolution $X \to \mathcal{Y}$, Bryan-Steinberg introduced ('12) invariants $\mathrm{BS}_{n,\beta}$. Roughly speaking, they count sheafs+sections $\{\mathcal{O}_X \xrightarrow{s} F\}$ but allowing the cokernel to have support on finitely many fibers B.
They provide a natural interpretation for the quotient $\mathrm{PT}_\beta/\mathrm{PT}_0$ via a DT/PT type wall-crossing.

Proposition

$$\mathrm{BS}_\beta(q, Q) \equiv \sum_{n,j \in \mathbb{Z}} \mathrm{BS}_{n,\beta+jB}(-q)^n Q^j = \frac{\mathrm{PT}_\beta(q, Q)}{\mathrm{PT}_0(q, Q)}.$$

Unlike ${}^p\mathrm{PT}$, BS are defined using the heart Coh(X), no need to tilt.

Wall-crossing ${}^p\mathrm{PT}/\mathrm{BS}$

Final step is comparing ${}^p\mathrm{PT}$ and BS.

Wall-crossing ${}^p\mathrm{PT}/\mathrm{BS}$

Final step is comparing ${}^p\mathrm{PT}$ and BS.

Proposition

We have the following identity of rational functions:

$$\mathrm{BS}_\beta(q, Q) = {}^p\mathrm{PT}_\beta(q, Q).$$

Wall-crossing ${}^p\mathrm{PT}/\mathrm{BS}$

Final step is comparing ${}^p\mathrm{PT}$ and BS.

Proposition

We have the following identity of rational functions:

$$\mathrm{BS}_\beta(q, Q) = {}^p\mathrm{PT}_\beta(q, Q).$$

The identity above is strictly of rational functions, the coefficients are not the same on the nose. When we cross a wall in the path of stability conditions we change the direction in which we expand the same rational function.

Crossing a wall – re-expansion

Example

The rational function $\frac{1}{q-Q}$ can be expanded in two different ways:

Crossing a wall – re-expansion

Example

The rational function $\frac{1}{q-Q}$ can be expanded in two different ways:

$$\frac{1}{q-Q} = \frac{q^{-1}}{1-Qq^{-1}} = \sum_{i\geq 0} Q^i q^{-1-i}$$

.

Crossing a wall – re-expansion

Example

The rational function $\frac{1}{q-Q}$ can be expanded in two different ways:

$$\frac{1}{q-Q} = \frac{q^{-1}}{1-Qq^{-1}} = \sum_{i\geq 0} Q^i q^{-1-i}$$

$$\frac{1}{q-Q} = -\frac{Q^{-1}}{1-Q^{-1}q} = -\sum_{i\geq 0} Q^{-1-i} q^i.$$

Thank you!