Weyl symmetry for curve counting invariants via spherical twists

Miguel Moreira
ETHZ
Joint with Tim Buelles

YMSC – Tsinghua University
Enumerative Geometry seminar
26 May 2022
Given a smooth projective variety $X$, Gromov-Witten theory uses the moduli of stable maps and its virtual fundamental class

$$[M_g(X, \beta)]^{\text{vir}} \in A_{\text{virdim}}(M_g(X, \beta)).$$
Gromov-Witten invariants

Given a smooth projective variety $X$, Gromov-Witten theory uses the moduli of stable maps and its virtual fundamental class

$$[M_g(X, \beta)]^{\text{vir}} \in A_{\text{virdim}}(M_g(X, \beta)).$$

A special case is when $X$ is a Calabi-Yau 3-fold (CY3): the virtual dimension of is 0 for all $g \geq 0$, $\beta \in H_2(X; \mathbb{Z})$ so we get numbers

$$\text{GW}^X_{g, \beta} = \int [M_g(X, \beta)]^{\text{vir}} 1 \in \mathbb{Q}.$$
Gromov-Witten invariants

Given a smooth projective variety $X$, Gromov-Witten theory uses the moduli of stable maps and its virtual fundamental class

$$[M_g(X, \beta)]^{\text{vir}} \in A_{\text{virdim}}(M_g(X, \beta)).$$

A special case is when $X$ is a Calabi-Yau 3-fold (CY3): the virtual dimension of is 0 for all $g \geq 0$, $\beta \in H_2(X; \mathbb{Z})$ so we get numbers

$$GW^X_{g, \beta} = \int_{[M_g(X, \beta)]^{\text{vir}}} 1 \in \mathbb{Q}.$$ 

**Goal:**

Compute all numbers $GW^X_{g, \beta}$. Equivalently, understand the partition function

$$Z_X = \exp \left( \sum_{g, \beta} GW^X_{g, \beta} u^{2g-2} z^\beta \right).$$
Stable pairs

Stable pairs provide an alternative approach to curve counting on CY3.
Stable pairs

Stable pairs provide an alternative approach to curve counting on CY3.

Definition (Pandharipande-Thomas ’09)

A stable pair on $X$ is an object $\{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$ in the derived category where $F$ is a coherent sheaf and $s$ a section satisfying the following two stability conditions:

1. $F$ is pure of dimension 1: every non-trivial coherent sub-sheaf of $F$ has dimension 1.
2. The cokernel of $s$ has dimension 0.
Stable pairs

Stable pairs provide an alternative approach to curve counting on CY3.

Definition (Pandharipande-Thomas '09)

A stable pair on $X$ is an object $\{O_X \xrightarrow{s} F\} \in D^b(X)$ in the derived category where $F$ is a coherent sheaf and $s$ a section satisfying the following two stability conditions:

1. $F$ is pure of dimension 1: every non-trivial coherent sub-sheaf of $F$ has dimension 1.
2. The cokernel of $s$ has dimension 0.

We associate two discrete invariants:

$$\beta = \text{ch}_2(F) = [\text{supp}(F)] \in H_2(X; \mathbb{Z}) \quad \text{and} \quad n = \chi(X, F).$$
Stable pairs

Stable pairs provide an alternative approach to curve counting on CY3.

**Definition (Pandharipande-Thomas ’09)**

A stable pair on $X$ is an object $\{O_X \xrightarrow{s} F\} \in D^b(X)$ in the derived category where $F$ is a coherent sheaf and $s$ a section satisfying the following two stability conditions:

1. $F$ is pure of dimension 1: every non-trivial coherent sub-sheaf of $F$ has dimension 1.
2. The cokernel of $s$ has dimension 0.

We associate two discrete invariants:

$$\beta = \text{ch}_2(F) = [\text{supp}(F)] \in H_2(X; \mathbb{Z}) \quad \text{and} \quad n = \chi(X, F).$$

The space $P_n(X, \beta)$ parametrizing stable pairs with fixed discrete invariants is a projective fine moduli space.
Pandharipande-Thomas invariants

The moduli of stable pairs $P_n(X, \beta)$ also has a virtual fundamental class, and when $X$ is a CY3 its virtual dimension is 0, producing again numbers

$$PT^X_{n, \beta} = \int [P_n(X, \beta)]^{\text{vir}} 1 \in \mathbb{Z}.$$
Pandharipande-Thomas invariants

The moduli of stable pairs $P_n(X, \beta)$ also has a virtual fundamental class, and when $X$ is a CY3 its virtual dimension is 0, producing again numbers

$$PT^X_{n, \beta} = \int [P_n(X, \beta)]_{\text{vir}} 1 \in \mathbb{Z}.$$ 

Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande '06)

*The Gromov-Witten and Pandharipande-Thomas invariants determine each other:*
Pandharipande-Thomas invariants

The moduli of stable pairs $P_n(X, \beta)$ also has a virtual fundamental class, and when $X$ is a CY3 its virtual dimension is 0, producing again numbers

$$PT^X_{n, \beta} = \int [P_n(X, \beta)]^{vir} 1 \in \mathbb{Z}.$$ 

Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande ’06)

The Gromov-Witten and Pandharipande-Thomas invariants determine each other:

$$\exp \left( \sum_{g, \beta} GW^X_{g, \beta} u^{2g-2} z^\beta \right) = \sum_{n, \beta} PT^X_{n, \beta} (-q)^n z^\beta$$

after the change of variables $q = e^{iu}$. 
Rationality and symmetry

Theorem (Bridgeland, Toda ’16)

For each \( \beta \) the generating function

\[
\sum_{n \in \mathbb{Z}} \text{PT}^X_n, \beta(-q^n)
\]

is the expansion of a rational function \( f_{\beta} \) satisfying the symmetry

\[
f_{\beta}(1/q) = f_{\beta}(q)
\]

Think of the theorem as \( \text{PT}^X_n, \beta \sim \text{PT}^{-n}, \beta \) after analytic continuation.

Typical example (contribution of isolated rational curve):

\[
f(q) = q(1 - q) = q + 2q^2 + 3q^3 + \ldots = q - 1 + 2q - 2 + 3q - 3 + \ldots
\]
## Theorem (Bridgeland, Toda ’16)

For each $\beta$ the generating function

$$
\sum_{n \in \mathbb{Z}} \mathrm{PT}^X_{n, \beta}(-q)^n
$$

is the expansion of a rational function $f_\beta$ satisfying the symmetry

$$
f_\beta(1/q) = f_\beta(q).
$$

Think of the theorem as $\mathrm{PT}_{n, \beta} \sim \mathrm{PT}_{-n, \beta}$ after analytic continuation.
Rationality and symmetry

Theorem (Bridgeland, Toda ’16)

For each \( \beta \) the generating function

\[
\sum_{n \in \mathbb{Z}} \text{PT}_{n,\beta}^X (-q)^n
\]

is the expansion of a rational function \( f_\beta \) satisfying the symmetry

\[
f_\beta(1/q) = f_\beta(q).
\]

Think of the theorem as \( \text{PT}_{n,\beta} \sim \text{PT}_{-n,\beta} \) after analytic continuation.

Typical example (contribution of isolated rational curve):

\[
f(q) = \frac{q}{(1 - q)^2} = q + 2q^2 + 3q^3 + \ldots
\]

\[
= q^{-1} + 2q^{-2} + 3q^{-3} + \ldots
\]
Proof of rationality

The proof of rationality illustrates a very general principle:
The proof of rationality illustrates a very general principle:

Symmetry of the derived category $\phi \in \text{Aut}(D^b(X))$

$\Downarrow$

Constraints on curve counting on $X$. 
Proof of rationality

The proof of rationality illustrates a very general principle:

Symmetry of the derived category $\phi \in \text{Aut}(D^b(X))$

↓

Constraints on curve counting on $X$.

The proof of rationality uses the derived dual

$$\phi = \mathbb{D} = \text{RHom}(-, \mathcal{O}_X)[2].$$
Proof of rationality

The proof of rationality illustrates a very general principle:

Symmetry of the derived category $\phi \in \text{Aut}(D^b(X))$

\[
\downarrow
\]

Constraints on curve counting on $X$.

The proof of rationality uses the derived dual

$\phi = \mathbb{D} = \text{RHom}(\mathcal{O}_X, \mathcal{O}_X)[2]$.

Note: $\chi(\mathbb{D}(F)) = -\chi(F)$. 
Proof of rationality

The proof of rationality illustrates a very general principle:

Symmetry of the derived category $\phi \in \text{Aut}(D^b(X))$

$\downarrow$

Constraints on curve counting on $X$.

The proof of rationality uses the derived dual

$\phi = \mathbb{D} = \text{RHom}(-, \mathcal{O}_X)[2]$.

Note: $\chi(\mathbb{D}(F)) = -\chi(F)$.

Basic idea: use wall-crossing in the derived category to relate

$P_n(X, \beta) \leftrightarrow \phi(P_n(X, \beta)) \subseteq D^b(X)$. 
Let $W$ be a ruled surface over a genus $g$ curve $C$, i.e.

$$W = \mathbb{P}_C(E) \to C.$$
Let $W$ be a ruled surface over a genus $g$ curve $C$, i.e.

$$W = \mathbb{P}_{C}(\mathcal{E}) \to C.$$ 

Let $X$ be a Calabi-Yau 3-fold containing $W$ as a divisor. Let $B = [\mathbb{P}^1] \in H_2(X)$ be the curve class of the fibers of the ruling.
Geometric setting

Let $W$ be a ruled surface over a genus $g$ curve $C$, i.e.

$$W = \mathbb{P}_C(\mathcal{E}) \to C.$$  

Let $X$ be a Calabi-Yau 3-fold containing $W$ as a divisor. Let $B = [\mathbb{P}^1] \in H_2(X)$ be the curve class of the fibers of the ruling.

$$\begin{array}{ccc}
B & \hookrightarrow & W \\
\downarrow & & \downarrow \iota \\
C & & X
\end{array}$$

We also assume that the ray generated by $B$ is extremal in the effective cone of $X$, i.e. if $C_1, C_2$ are effective curve classes such that $C_1 + C_2$ is a multiple of $B$ then both $C_1, C_2$ are multiples of $B$. 
Geometric setting

\(X = K_{\text{elliptic fibration}}\) over \(W\), which is a particular elliptic fibration over \(\mathbb{P}^1 \times \mathbb{P}^1\).
Geometric setting

Examples

- $X = K_W$
- $X$ elliptic fibration over $W$
- $X = \text{STU model, which is a particular elliptic fibration over } \mathbb{P}^1 \times \mathbb{P}^1.$
Consider the involution defined on $H_2(X)$ by

$$\beta \mapsto \beta' = \beta + (W \cdot \beta)B.$$
Consider the involution defined on $H_2(X)$ by

$$\beta \mapsto \beta' = \beta + (W \cdot \beta)B.$$
Weyl symmetry for PT invariants

Our work is about some symmetry relating curve counting invariants in classes $\beta$ and $\beta'$:

\[
GW_{g,\beta} \sim GW_{g,\beta'} \\
PT_{n,\beta} \sim PT_{n,\beta'}.
\]
Our work is about some symmetry relating curve counting invariants in classes $\beta$ and $\beta'$

\[
GW_{g,\beta} \sim GW_{g,\beta'} \\
PT_{n,\beta} \sim PT_{n,\beta'}.
\]

Let

\[
PT_{\beta}(q, Q) = \sum_{n,j \in \mathbb{Z}} P_{n,\beta+jB} (-q)^n Q^j.
\]
Our work is about some symmetry relating curve counting invariants in classes $\beta$ and $\beta'$

$$GW_{g,\beta} \sim GW_{g,\beta'}$$
$$PT_{n,\beta} \sim PT_{n,\beta'}.$$ 

Let

$$PT_\beta(q, Q) = \sum_{n,j \in \mathbb{Z}} P_{n,\beta+jB} (-q)^n Q^j.$$ 

The generating series $PT_0$ of multiples of $B$ can be shown to equal

$$PT_0(q, Q) = \prod_{j \geq 1} (1 - q^j Q)^{(2g-2)j}.$$
Weyl symmetry for PT invariants

Theorem (Buelles-M. ’21/22)

Let $X$ be a Calabi-Yau 3-fold containing a smooth, ruled divisor $W$ as described before.
Theorem (Buelles-M. ’21/22)

Let \( X \) be a Calabi-Yau 3-fold containing a smooth, ruled divisor \( W \) as described before. Then

\[
\frac{\text{PT}_\beta(q, Q)}{\text{PT}_0(q, Q)} \in \mathbb{Q}(q, Q)
\]

is the expansion of a rational function \( f_\beta(q, Q) \)
Weyl symmetry for PT invariants

Theorem (Buelles-M. ’21/22)

Let $X$ be a Calabi-Yau 3-fold containing a smooth, ruled divisor $W$ as described before. Then

$$\frac{\text{PT}_\beta(q, Q)}{\text{PT}_0(q, Q)} \in \mathbb{Q}(q, Q)$$

is the expansion of a rational function $f_\beta(q, Q)$ which satisfies the functional equations

$$f_\beta(q^{-1}, Q) = f_\beta(q, Q) \quad \text{and} \quad f_\beta(q, Q^{-1}) = Q^{-W \cdot \beta} f_\beta(q, Q).$$
Weyl symmetry for GW invariants

Corollary (Assuming GW/PT)

For all \((g, \beta) \neq (0, m_B)\), \((1, m_B)\), the series

\[
\sum_{j \in \mathbb{Z}} \text{GW}_{g, \beta} + jB \cdot Q^j
\]

is the expansion of a rational function \(f_{\beta}(Q)\) with functional equation

\[
f_{\beta}(Q - 1) = Q - W \cdot \beta \cdot f_{\beta}(Q).
\]

Think of the functional equation as equality \(\text{GW}_{g, \beta} \sim \text{GW}_{g, \beta}'\) after analytic continuation. Predicted by physics, at least in the local case (Katz-Klemm-Vafa '97).
Weyl symmetry for GW invariants

Corollary (Assuming GW/PT)

For all \((g, \beta) \neq (0, mB), (1, mB)\) the series

\[
\sum_{j \in \mathbb{Z}} GW_{g,\beta + jB} Q^j
\]

is the expansion of a rational function \(f_\beta(Q)\) with functional equation

\[
f_\beta(Q^{-1}) = Q^{-W \cdot \beta} f_\beta(Q).
\]

Think of the functional equation as equality

\[
GW_{g,\beta} \sim GW_{g,\beta'}
\]

after analytic continuation.
Weyl symmetry for GW invariants

**Corollary (Assuming GW/PT)**

For all \((g, \beta) \neq (0, mB), (1, mB)\) the series

\[
\sum_{j \in \mathbb{Z}} GW_{g,\beta+jB} Q^j
\]

is the expansion of a rational function \(f_\beta(Q)\) with functional equation

\[
f_\beta(Q^{-1}) = Q^{-W \cdot \beta} f_\beta(Q).
\]

Think of the functional equation as equality

\[
GW_{g,\beta} \sim GW_{g,\beta'}
\]

after analytic continuation.

Predicted by physics, at least in the local case \(K_W\)
(Katz-Klemm-Vafa '97).
Examples

Example

Let $X = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ and let $C$ be the other $\mathbb{P}^1$ in the product. A computation with the topological vertex shows:

\[
\frac{\text{PT}_C(q, Q)}{\text{PT}_0(q, Q)} = \frac{2q}{(1 - q)^2(1 - Q)^2}
\]
**Example**

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $C$ be the other $\mathbb{P}^1$ in the product. A computation with the topological vertex shows:

$$\frac{\text{PT}_C(q, Q)}{\text{PT}_0(q, Q)} = \frac{2q}{(1 - q)^2(1 - Q)^2}$$

$$\frac{\text{PT}_{2C}(q, Q)}{\text{PT}_0(q, Q)} = \frac{2q^4}{(1 - q)^2(1 - q^2)^2(1 - qQ)^2(1 - Q)^2} + \frac{2q^4}{(1 - q)^2(1 - q^2)^2(q - Q)^2(1 - Q)^2} + \frac{2q^4}{(1 - q)^4(1 - qQ)^2(q - Q)^2}.$$
Spherical twists

The main ingredient of our symmetry is the existence of a certain anti-equivalence \( \rho \in \text{Aut}(D^b(X)) \) promoting the involution

\[
\beta \mapsto \beta' = \beta + (W \cdot \beta)B
\]

on \( H_2(X) \) to the derived category.
Spherical twists

The main ingredient of our symmetry is the existence of a certain anti-equivalence $\rho \in \text{Aut}(D^b(X))$ promoting the involution

$$\beta \mapsto \beta' = \beta + (W \cdot \beta)B$$

on $H_2(X)$ to the derived category. It’s constructed using the spherical functor

$$\Phi: D^b(C) \to D^b(X)$$

$$V \mapsto \iota_*(O_p(-1) \otimes p^* V).$$
The main ingredient of our symmetry is the existence of a certain anti-equivalence \( \rho \in \text{Aut}(D^b(X)) \) promoting the involution

\[
\beta \mapsto \beta' = \beta + (W \cdot \beta)B
\]

on \( H_2(X) \) to the derived category. It’s constructed using the spherical functor

\[
\Phi : D^b(C) \to D^b(X) \\
V \mapsto \iota^* (\mathcal{O}_p(-1) \otimes p^* V).
\]

From a spherical functor we associate an automorphism of the derived category, the spherical twist \( ST \) defined by

\[
\Phi \circ \Phi_R \longrightarrow \text{id} \longrightarrow ST.
\]
Derived equivalence $\rho$

We already have the derived equivalence $ST \in \text{Aut}(D^b(X))$. The derived equivalence $\rho$ is then

$$\rho = ST \circ \mathbb{D}.$$
Derived equivalence $\rho$

We already have the derived equivalence $ST \in \text{Aut}(D^b(X))$. The derived equivalence $\rho$ is then

$$\rho = ST \circ \mathbb{D}.$$ 

Facts

1. $\rho$ is an involution, i.e. $\rho \circ \rho = \text{id}$. 
Derived equivalence $\rho$

We already have the derived equivalence $\text{ST} \in \text{Aut}(D^b(X))$. The derived equivalence $\rho$ is then

$$\rho = \text{ST} \circ \mathbb{D}.$$ 

Facts

1. $\rho$ is an involution, i.e. $\rho \circ \rho = \text{id}$.
2. $\rho(\mathcal{O}_X) = \mathcal{O}_X[2]$. 
Derived equivalence $\rho$

We already have the derived equivalence $ST \in \Aut(D^b(X))$. The derived equivalence $\rho$ is then

$$\rho = ST \circ \mathbb{D}.$$ 

### Facts

1. $\rho$ is an involution, i.e. $\rho \circ \rho = \text{id}$.
2. $\rho(\mathcal{O}_X) = \mathcal{O}_X[2]$.
3. If $F$ is a sheaf of dimension 1 and $\text{ch}_2(F) = \beta$, $\chi(F) = n$ then

$$\text{ch}_2(\rho(F)) = \beta' = \beta + (W \cdot \beta)B$$

$$\chi(\rho(F)) = -n.$$
When $X$ arises as a crepant resolution $X \rightarrow Y$ of an orbifold with $\mathbb{Z}/2$-singularities along the curve $C$ so that $W$ is the exceptional divisor (and the fibers $B$ are contracted to points), the main result is a consequence of the DT crepant resolution conjecture proven by Beentjes-Calabrese-Rennemo ('18).
Their proof immitates Bridgeland-Toda proof of rationality using $\mathbb{D}^\mathcal{Y}$ to prove the symmetry of PT invariants in $\mathcal{Y}$.
Their proof imitates Bridgeland-Toda proof of rationality using \( D^\mathcal{Y} \) to prove the symmetry of PT invariants in \( \mathcal{Y} \).

**Proposition**

*Under the McKay correspondence*

\[
\Psi : D^b(X) \xrightarrow{\sim} D^b(\mathcal{Y})
\]

the derived dual \( D^\mathcal{Y} \) corresponds to \( \rho \), i.e.

\[
\rho = \Psi^{-1} \circ D^\mathcal{Y} \circ \Psi.
\]
Orbifold inspiration

Their proof imitates Bridgeland-Toda proof of rationality using $\mathbb{D}^\mathcal{Y}$ to prove the symmetry of PT invariants in $\mathcal{Y}$.

**Proposition**

*Under the McKay correspondence*

$$\Psi : D^b(X) \sim \rightarrow D^b(\mathcal{Y})$$

the derived dual $\mathbb{D}^\mathcal{Y}$ corresponds to $\rho$, i.e.

$$\rho = \Psi^{-1} \circ \mathbb{D}^\mathcal{Y} \circ \Psi.$$  

Important examples (e.g. the STU) don’t arise as such crepant resolution.
Homological mirror symmetry?

What can we say about the mirror geometry $\check{X}$? In particular, how to interpret the derived equivalence $ST$ under HMS:

$$ST \in \text{Aut}(D^b(X)) \cong \text{Aut}(\text{Fuk}(\check{X}))?$$
What can we say about the mirror geometry $\check{X}$? In particular, how to interpret the derived equivalence $ST$ under HMS:

$$ST \in \text{Aut}(D^b(X)) \cong \text{Aut}(\text{Fuk}(\check{X}))?$$

When the genus of $C$ is $g = 0$ we can write $ST$ as a composition of twists around spherical objects

$$ST = ST_{\mathcal{O}_W(-C+B)} \circ ST_{\mathcal{O}_W(-C)}$$

so (the mirror of) $ST$ should be induced by a symplectomorphism obtained as a composition of two Dehn twists.
Homological mirror symmetry?

How to think about this composition and what about $g > 0$?
How to think about this composition and what about $g > 0$? A typical way in which spherical functors appear in the Fukaya category is through symplectic fibrations: if $w: Z \to \mathbb{C}$ is a symplectic fibration with general fiber $\mathcal{X}$ then we get a spherical functor

\[ \text{FS}(Z, w) \xrightarrow{\cap} \text{Fuk}(\mathcal{X}) \]
How to think about this composition and what about $g > 0$? A typical way in which spherical functors appear in the Fukaya category is through symplectic fibrations: if $w : Z \to \mathbb{C}$ is a symplectic fibration with general fiber $\hat{X}$ then we get a spherical functor

\[
\begin{array}{ccc}
FS(Z, w) & \xrightarrow{\cap} & \text{Fuk}(\hat{X}) \\
\text{HMS?} \downarrow & & \downarrow \text{HMS} \\
D^b(C) & \xrightarrow{\Phi} & D^b(X)
\end{array}
\]
How to think about this composition and what about $g > 0$? A typical way in which spherical functors appear in the Fukaya category is through symplectic fibrations: if $w: Z \to \mathbb{C}$ is a symplectic fibration with general fiber $\mathcal{F}X$ then we get a spherical functor $FS(Z, w) \xrightarrow{\cap} \text{Fuk}(\mathcal{F}X)$.

$$FS(Z, w) \xrightarrow{\cap} \text{Fuk}(\mathcal{F}X)$$

$$\text{HMS}? \quad \text{HMS}$$

$$\Phi$$

$D^b(C) \xrightarrow{\Phi} D^b(X)$

In such a situation, the derived equivalence corresponding to $ST$ on the symplectic side would be induced by monodromy of $w: Z \to \mathbb{C}$ around $\infty$. 
Recall that stable pairs are of the form $s: \mathcal{O}_X \to F$ with $F \in \text{Coh}_1(X)$, $\text{coker}(s) \in \text{Coh}_0(X)$. 
Perverse stable pairs

Recall that stable pairs are of the form $s : \mathcal{O}_X \rightarrow F$ with $F \in \text{Coh}_1(X)$, $\text{coker}(s) \in \text{Coh}_0(X)$. Bridgeland’s proof of rationality with the derived dual uses

$$\mathbb{D}(\text{Coh}_1(X)) = \text{Coh}_1(X) \text{ and } \mathbb{D}(\text{Coh}_0(X)) = \text{Coh}_0(X)[-1]$$

to describe the image of $\mathbb{D}(P_n(X, \beta))$ and to help finding wall-crossing between $\mathbb{D}(P_n(X, \beta))$ and $P_n(X, \beta)$.
Perverse stable pairs

Recall that stable pairs are of the form $s : O_X \to F$ with $F \in \text{Coh}_1(X)$, $\text{coker}(s) \in \text{Coh}_0(X)$.

Bridgeland’s proof of rationality with the derived dual uses

$$\mathbb{D}(\text{Coh}_1(X)) = \text{Coh}_1(X) \text{ and } \mathbb{D}(\text{Coh}_0(X)) = \text{Coh}_0(X)[-1]$$

to describe the image of $\mathbb{D}(P_n(X, \beta))$ and to help finding wall-crossing between $\mathbb{D}(P_n(X, \beta))$ and $P_n(X, \beta)$.

Example

If $x \in W$ is a point in the divisor lying in a fiber $B$ then

$$\rho(O_x) = \{O_B(-1)[-1] \to O_B(-2)\}.$$
Perverse sheaves

To study $\rho$ it’s more appropriate to use a tilt of $\text{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)
Perverse sheaves

To study $\rho$ it’s more appropriate to use a tilt of $\text{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$\mathcal{T} = \{ T \in \text{Coh}(X) : R^1p_*T|_W = 0 \}$$
$$\mathcal{F} = \{ F \in \text{Coh}(X) : \text{Hom}(\mathcal{T}, F) = 0 \}$$
$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\text{ex}}.$$
Perverse sheaves

To study $\rho$ it’s more appropriate to use a tilt of $\text{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$\mathcal{T} = \{ T \in \text{Coh}(X) : R^1 p_* T|_W = 0 \}$$
$$\mathcal{F} = \{ F \in \text{Coh}(X) : \text{Hom}(\mathcal{T}, F) = 0 \}$$
$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\text{ex}}.$$

$\mathcal{A}$ is a heart of $D^b(X)$ and its elements are perverse sheaves. The dimension of a perverse sheaf is the dimension of its support after we contract the fibers $B$. 
Perverse sheaves

To study $\rho$ it’s more appropriate to use a tilt of $\text{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$\mathcal{T} = \{ T \in \text{Coh}(X) : R^1 p_* T|_W = 0 \}$$

$$\mathcal{F} = \{ F \in \text{Coh}(X) : \text{Hom}(\mathcal{T}, F) = 0 \}$$

$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\text{ex}}.$$  

$\mathcal{A}$ is a heart of $D^b(X)$ and its elements are perverse sheaves. The dimension of a perverse sheaf is the dimension of its support after we contract the fibers $B$. 

**Example**

1. $\text{Coh}_0 \subseteq \mathcal{A}_0$;
Perverse sheaves

To study $\rho$ it’s more appropriate to use a tilt of $\text{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$\mathcal{T} = \{ T \in \text{Coh}(X) : R^1 p_* T |_W = 0 \}$$
$$\mathcal{F} = \{ F \in \text{Coh}(X) : \text{Hom}(\mathcal{T}, F) = 0 \}$$
$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\text{ex}}.$$

$\mathcal{A}$ is a heart of $D^b(X)$ and its elements are perverse sheaves. The dimension of a perverse sheaf is the dimension of its support after we contract the fibers $B$.

Example

1. $\text{Coh}_0 \subseteq \mathcal{A}_0$;
2. $\mathcal{O}_B(-1), \mathcal{O}_B(-2)[1] \in \mathcal{A}_0$;
Perverse sheaves

To study $\rho$ it’s more appropriate to use a tilt of $\text{Coh}(X)$ and a different notion of dimension (which corresponds to sheaves on the orbifold)

$$
\mathcal{T} = \{ T \in \text{Coh}(X) : R^1 p_* T|_W = 0 \}
$$

$$
\mathcal{F} = \{ F \in \text{Coh}(X) : \text{Hom}(\mathcal{T}, F) = 0 \}
$$

$$
\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\text{ex}}.
$$

$\mathcal{A}$ is a heart of $D^b(X)$ and its elements are perverse sheaves. The dimension of a perverse sheaf is the dimension of its support after we contract the fibers $B$.

Example

1. $\text{Coh}_0 \subseteq \mathcal{A}_0$;
2. $O_B(-1), O_B(-2)[1] \in \mathcal{A}_0$;
3. $O_p(-1), O_p(-2)[1] \in \mathcal{A}_1$. 
Perverse stable pairs

The action of $\rho$ on $\mathcal{A}$ (with perverse dimension) is analogous to the action of $\mathbb{D}$ on $\text{Coh}(X)$ (with usual dimension):

$$\rho(\mathcal{A}_1) = \mathcal{A}_1 \text{ and } \rho(\mathcal{A}_0) = \mathcal{A}_0[-1].$$
Perverse stable pairs

The action of $\rho$ on $\mathcal{A}$ (with perverse dimension) is analogous to the action of $\mathbb{D}$ on $\text{Coh}(X)$ (with usual dimension):

$$\rho(\mathcal{A}_1) = \mathcal{A}_1 \text{ and } \rho(\mathcal{A}_0) = \mathcal{A}_0[-1].$$

**Definition**

A perverse stable pair is an object $I \in \langle \mathcal{O}_X[1], \mathcal{A}_{\leq 1}\rangle_{\text{ex}}$ such that $\text{rk}(I) = -1$ and

$$\text{Hom}(\mathcal{A}_0, I) = 0 = \text{Hom}(I, \mathcal{A}_1).$$
Perverse stable pairs

The action of $\rho$ on $\mathcal{A}$ (with perverse dimension) is analogous to the action of $\mathbb{D}$ on $\text{Coh}(X)$ (with usual dimension):

$$\rho(\mathcal{A}_1) = \mathcal{A}_1 \text{ and } \rho(\mathcal{A}_0) = \mathcal{A}_0[-1].$$

**Definition**

A perverse stable pair is an object $I \in \langle \mathcal{O}_X[1], \mathcal{A}_{\leq 1} \rangle_{\text{ex}}$ such that $\text{rk}(I) = -1$ and

$$\text{Hom}(\mathcal{A}_0, I) = 0 = \text{Hom}(I, \mathcal{A}_1).$$

We define the virtual counts of perverse stable pairs:

$$p^{PT}_{n,\beta} \in \mathbb{Z},$$

$$p^{PT}_{\beta}(q, Q) = \sum_{n,j \in \mathbb{Z}} p^{PT}_{n,\beta+jB}(-q)^n Q^j.$$
Rationality for $\mathfrak{P}T$

**Theorem (Buelles-M)**

The series $\mathfrak{P}T_\beta(q, Q)$ is the expansion of a rational function $f_\beta \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_\beta(q^{-1}, Q^{-1}) = Q^{-W \cdot \beta} f_\beta(q, Q).$$
Rationality for $p_{PT}$

**Theorem (Buelles-M)**

*The series $p_{PT}^\beta(q, Q)$ is the expansion of a rational function $f^\beta \in \mathbb{Q}(q, Q)$ satisfying the symmetry*

$$f^\beta(q^{-1}, Q^{-1}) = Q^{-W \cdot \beta} f^\beta(q, Q).$$

- Rationality of $PT^\beta(q)$
- Rationality of $p_{PT}^\beta(q, Q)$
The series $pPT_{\beta}(q, Q)$ is the expansion of a rational function $f_{\beta} \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_{\beta}(q^{-1}, Q^{-1}) = Q^{-W \cdot \beta} f_{\beta}(q, Q).$$

**Theorem (Buelles-M)**

- Rationality of $PT_{\beta}(q)$
- Anti-equivalence $\mathbb{D}$
- Rationality of $pPT_{\beta}(q, Q)$
- Anti-equivalence $\rho$
Rationality for $\rho^{PT}$

**Theorem (Buelles-M)**

The series $\rho^{PT}_\beta(q, Q)$ is the expansion of a rational function $f_\beta \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_\beta(q^{-1}, Q^{-1}) = Q^{-W_\beta} f_\beta(q, Q).$$

- Rationality of $PT_\beta(q)$
- Anti-equivalence $\mathbb{D}$
- Torsion pair $\langle \text{Coh}_0, \text{Coh}_1 \rangle$
- Rationality of $\rho^{PT}_\beta(q, Q)$
- Anti-equivalence $\rho$
- Torsion pair $\langle \mathcal{A}_0, \mathcal{A}_1 \rangle$
Rationality for $pPT$

**Theorem (Buelles-M)**

The series $pPT_\beta(q, Q)$ is the expansion of a rational function $f_\beta \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_\beta(q^{-1}, Q^{-1}) = Q^{-W \cdot \beta} f_\beta(q, Q).$$

- Rationality of $PT_\beta(q)$
- Anti-equivalence $\mathbb{D}$
- Torsion pair $\langle \text{Coh}_0, \text{Coh}_1 \rangle$
- Usual slope stability
- Rationality of $pPT_\beta(q, Q)$
- Anti-equivalence $\rho$
- Torsion pair $\langle A_0, A_1 \rangle$
- Nironi slope stability
Rationality for $\mathop{pPT}$

**Theorem (Buelles-M)**

The series $\mathop{pPT}_\beta(q, Q)$ is the expansion of a rational function $f_\beta \in \mathbb{Q}(q, Q)$ satisfying the symmetry

$$f_\beta(q^{-1}, Q^{-1}) = Q^{-W \cdot \beta} f_\beta(q, Q).$$

- Rationality of $\mathop{PT}_\beta(q)$
- Anti-equivalence $\mathcal{D}$
- Torsion pair $\langle \text{Coh}_0, \text{Coh}_1 \rangle$
- Usual slope stability
- Vanishing of Poisson brackets
  $\{ \text{Coh}_{\leq 1}, \text{Coh}_{\leq 1} \} = 0$

- Rationality of $\mathop{pPT}_\beta(q, Q)$
- Anti-equivalence $\rho$
- Torsion pair $\langle \mathcal{A}_0, \mathcal{A}_1 \rangle$
- Nironi slope stability
- No vanishing, extra combinatorial difficulty (dealt with in [BCR]).
We proved rationality of perverse PT invariants, but now need to relate them to classical stable pairs.
We proved rationality of perverse PT invariants, but now need to relate them to classical stable pairs.

**Theorem (Buelles-M)**

For any $\beta \in H_2(X; \mathbb{Z})$ we have the following identity of rational functions:

$$p^{\text{PT}}_\beta(q, Q) = \frac{\text{PT}_\beta(q, Q)}{\text{PT}_0(q, Q)}.$$
Wall-crossing

We proved rationality of perverse PT invariants, but now need to relate them to classical stable pairs.

**Theorem (Buelles-M)**

*For any* $\beta \in H_2(X; \mathbb{Z})$ *we have the following identity of rational functions:*

$$p^{\text{PT}}_\beta(q, Q) = \frac{\text{PT}_\beta(q, Q)}{\text{PT}_0(q, Q)}.$$

The wall-crossing establishing the equality has two steps and uses the counting of a third type of objects: Bryan-Steinberg invariants.
When $X$ arises as a crepant resolution $X \to \mathcal{Y}$, Bryan-Steinberg introduced ('12) invariants $\mathcal{BS}_{n,\beta}$. Roughly speaking, they count sheafs + sections $\{\mathcal{O}_X \to F\}$ but allowing the cokernel to have support on finitely many fibers $B$. 
When $X$ arises as a crepant resolution $X \to Y$, Bryan-Steinberg introduced ('12) invariants $BS_{n,\beta}$. Roughly speaking, they count sheafs + sections $\{\mathcal{O}_X \xrightarrow{s} F\}$ but allowing the cokernel to have support on finitely many fibers $B$. They provide a natural interpretation for the quotient $PT_\beta/PT_0$ via a DT/PT type wall-crossing.

**Proposition**

$$BS_\beta(q, Q) \equiv \sum_{n,j \in \mathbb{Z}} BS_{n,\beta+jB}(-q)^nQ^j = \frac{PT_\beta(q, Q)}{PT_0(q, Q)}.$$
When $X$ arises as a crepant resolution $X \to \mathcal{Y}$, Bryan-Steinberg introduced ('12) invariants $\text{BS}_{n,\beta}$. Roughly speaking, they count sheafs + sections $\{\mathcal{O}_X \to F\}$ but allowing the cokernel to have support on finitely many fibers $B$. They provide a natural interpretation for the quotient $\text{PT}_\beta/\text{PT}_0$ via a DT/PT type wall-crossing.

**Proposition**

$$\text{BS}_{\beta}(q, Q) \equiv \sum_{n,j \in \mathbb{Z}} \text{BS}_{n,\beta+jB}(-q)^n Q^j = \frac{\text{PT}_\beta(q, Q)}{\text{PT}_0(q, Q)}.$$

Unlike $p\text{PT}$, BS are defined using the heart $\text{Coh}(X)$, no need to tilt.
Wall-crossing $^p$PT/BS

Final step is comparing $^p$PT and BS.
Wall-crossing $^p$PT/BS

Final step is comparing $^p$PT and BS.

**Proposition**

We have the following identity of rational functions:

$$BS_\beta(q, Q) = ^pPT_\beta(q, Q).$$
Wall-crossing $^p\text{PT}/\text{BS}$

Final step is comparing $^p\text{PT}$ and $\text{BS}$.

**Proposition**

*We have the following identity of rational functions:*

$$\text{BS}_\beta(q, Q) = ^p\text{PT}_\beta(q, Q).$$

The identity above is strictly of rational functions, the coefficients are not the same on the nose. When we cross a wall in the path of stability conditions we change the direction in which we expand the same rational function.
Crossing a wall – re-expansion

Example

The rational function $\frac{1}{q-Q}$ can be expanded in two different ways:
Crossing a wall – re-expansion

Example

The rational function $\frac{1}{q - Q}$ can be expanded in two different ways:

$$\frac{1}{q - Q} = \frac{q^{-1}}{1 - Qq^{-1}} = \sum_{i \geq 0} Q^i q^{-1-i}.$$
Crossing a wall – re-expansion

Example

The rational function \( \frac{1}{q - Q} \) can be expanded in two different ways:

\[
\frac{1}{q - Q} = \frac{q^{-1}}{1 - Qq^{-1}} = \sum_{i \geq 0} Q^i q^{-1-i}
\]

\[
\frac{1}{q - Q} = -\frac{Q^{-1}}{1 - Q^{-1}q} = -\sum_{i \geq 0} Q^{-1-i} q^i.
\]
Thank you!

\[ \text{PT} \xleftarrow{\text{quotient}} \text{BS} \xrightarrow{\text{re}-\text{expansion}} p\text{PT} \]

\[ \rho(p\text{PT}) \]