# Weyl symmetry for curve counting invariants via spherical twists

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Harvard-MIT
Algebraic Geometry Seminar
28 September 2021

# Gromov-Witten invariants

Given a smooth projective variety $X$, Gromov-Witten theory uses the moduli of stable maps and its virtual fundamental class

$$[M_g(X,\beta)]^{\mathrm{vir}} \in A_{\mathrm{virdim}}(M_g(X,\beta)).$$

A special case is when $X$ is a Calabi-Yau 3-fold (CY3): the virtual dimension of is 0 for all $g \geq 0$, $\beta \in H_2(X;\mathbb{Z})$ so we get numbers

$$\mathrm{GW}^X_{g,\beta} = \int_{[M_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

**Goal:**

Compute all numbers $\mathrm{GW}^X_{g,\beta}$. Equivalently, understand the partition function

$$Z_X = \exp\left(\sum_{g,\beta} \mathrm{GW}^X_{g,\beta} u^{2g-2} z^{\beta}\right).$$

# Stable pairs

Stable pairs provide an alternative approach to curve counting on $CY3$.

**Definition (Pandharipande-Thomas '09)**

A stable pair on $X$ is an object $\{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$ in the derived category where $F$ is a coherent sheaf and $s$ a section satisfying the following two stability conditions:

1. $F$ is pure of dimension 1: every non-trivial coherent sub-sheaf of $F$ has dimension 1.
2. The cokernel of $s$ has dimension 0.

We associate two discrete invariants:

$$\beta = [\mathrm{supp}(F)] \in H_2(X;\mathbb{Z}) \text{ and } n = \chi(X, F).$$

The space $P_n(X, \beta)$ parametrizing stable pairs with fixed discrete invariants is a projective fine moduli space.

# Pandharipande-Thomas invariants

The moduli of stable pairs $P_n(X, \beta)$ also has a virtual fundamental class, and when $X$ is a CY3 its virtual dimension is 0, producing again numbers

$$\mathrm{PT}^X_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

**Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande '06)**

*The Gromov-Witten and Pandharipande-Thomas invariants determine each other:*

$$\exp\left(\sum_{g,\beta} \mathrm{GW}^X_{g,\beta} u^{2g-2} z^\beta\right) = \sum_{n,\beta} \mathrm{PT}^X_{n,\beta} (-q)^n z^\beta$$

*after the change of variables $q = e^{iu}$.*

# Rationality and symmetry

To even make sense of the change of variables $q = e^{iu}$ an important structural result is required:

**Theorem (Bridgeland '16)**

*For each $\beta$ the generating function*

$$\sum_{n \in \mathbb{Z}} \mathrm{PT}^X_{n,\beta}(-q)^n$$

*is the expansion of a rational function $f_\beta$ satisfying the symmetry*

$$f_\beta(1/q) = f_\beta(q).$$

Typical example (contribution of isolated rational curve):

$$f(q) = \frac{q}{(1-q)^2}.$$

# Proof of rationality

The proof of rationality illustrates a very general principle:

Symmetry of the derived category $\phi \in \mathrm{Aut}(D^b(X))$

$$\downarrow$$

Constraints on curve counting on $X$.

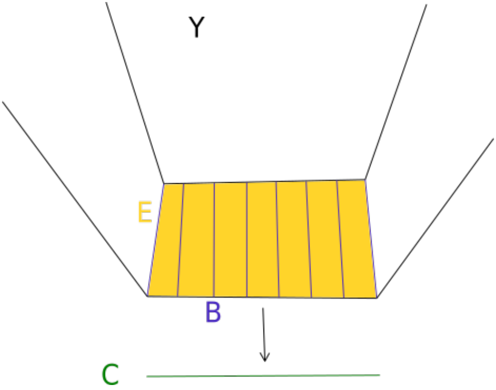
The proof of rationality uses the derived dual

$$\phi = \mathbb{D} = \mathrm{RHom}(-, \mathcal{O}_X)[2].$$

Basic idea: use wall-crossing in the derived category to relate

$$P_n(X, \beta) \leftrightsquigarrow \phi(P_n(X, \beta)) \subseteq D^b(X).$$

# Geometric setting




Let $Y$ be a Calabi-Yau 3-fold containing a smooth divisor $E \subseteq Y$ isomorphic to a Hirzebruch surface (so $E$ is a $\mathbb{P}^1$ bundle $E \to C = \mathbb{P}^1$).
Let $B = [\mathbb{P}^1] \in H_2(Y;\mathbb{Z})$ be the curve class of the fibers of $E \to C$.
(Key examples: $Y = K_E$, $Y$ elliptic fibration over $E$, $Y = STU$)

# $K3$ monodromy

A key source of examples are elliptic fibrations (with section) over Hirzebruch surface $E$. Let $\pi : Y \to E$ be the fibration and $F$ the fiber class. Each fiber $\pi^{-1}(B)$ is a $K3$ surface. The monodromy of $K3$ implies the symmetry

$$\mathrm{GW}^{Y}_{g,hF+iB} = \mathrm{GW}^{Y}_{g,hF+(h-i)B}.$$

For more general $\beta$, our work is about some symmetry relating

$$\mathrm{GW}^{Y}_{g,\beta} \sim \mathrm{GW}^{Y}_{g,\beta'}$$

where $\beta' = \beta + (E \cdot \beta)B$ (note that $\beta \mapsto \beta'$ is an involution since $E \cdot B = -2$).

# Weyl symmetry for PT invariants

Let

$$\mathrm{PT}_\beta(q, Q) = \sum_{n,j\in\mathbb{Z}} P_{n,\beta+jB}\,(-q)^n Q^j\,.$$

The generating series $\mathrm{PT}_0$ of multiples of $B$ is computed (for example via the topological vertex) as

$$\mathrm{PT}_0(q, Q) = \prod_{j\geq 1}(1 - q^j Q)^{-2j}.$$

# Weyl symmetry for PT invariants

**Theorem (Buelles-M. '21)**

*Let $Y$ be a Calabi-Yau 3-fold containing a smooth divisor $E$ isomorphic to a Hirzebruch surface and satisfying a few assumptions (to explain later). Then*

$$\frac{\mathrm{PT}_\beta(q,Q)}{\mathrm{PT}_0(q,Q)} \in \mathbb{Q}(q,Q)$$

*is the expansion of a rational function $f_\beta(q,Q)$ which satisfies the functional equations*

$$f_\beta(q^{-1},Q) = f_\beta(q,Q) \text{ and } f_\beta(q,Q^{-1}) = Q^{-E\cdot\beta} f_\beta(q,Q).$$

# Weyl symmetry for GW invariants

## Corollary

*For all* $(g, \beta) \neq (0, mB), (1, mB)$ *the series*

$$\sum_{j \in \mathbb{Z}} \mathrm{GW}_{g, \beta + jB}\, Q^j$$

*is the expansion of a rational function* $f_\beta(Q)$ *with functional equation*

$$f_\beta(Q^{-1}) = Q^{-E \cdot \beta} f_\beta(Q)\,.$$

Predicted by physics, at least in the local case $K_E$ (Katz-Klemm-Vafa '97).

If $f_\beta$ were a Laurent polynomial (as in the case of $K3$ classes), the functional equation means symmetry holds on the nose

$$\mathrm{GW}^Y_{g,\beta} = \mathrm{GW}^Y_{g,\beta'}.$$

# Assumptions on $Y$

Our proofs at the moment assume the following:

- The curve $B$ generates an extremal ray in the cone of curves of $Y$. I.e. there is a nef divisor $A$ such that

$$\ker\left(A_1(Y) \xrightarrow{A\cdot} \mathbb{Q}\right) = \mathbb{Q}\cdot B.$$

  Holds for any elliptic fibration.
- $-K_E$ is nef, i.e. $E \cong \mathbb{F}_r$ with $r = 0, 1, 2$ (probably not really necessary).
- For the Gromov-Witten corollary we assume the GW/PT correspondence holds.

# Examples

## Example

Let $Y = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ and let $C$ be the other $\mathbb{P}^1$ in the product. A computation with the topological vertex shows:

$$\frac{\mathrm{PT}_C(q,Q)}{\mathrm{PT}_0(q,Q)} = \frac{2q}{(1-q)^2(1-Q)^2}$$

$$\frac{\mathrm{PT}_{2C}(q,Q)}{\mathrm{PT}_0(q,Q)} = \frac{2q^4}{(1-q)^2(1-q^2)^2(1-qQ)^2(1-Q)^2} + \frac{2q^4}{(1-q)^2(1-q^2)^2(q-Q)^2(1-Q)^2} + \frac{2q^4}{(1-q)^4(1-qQ)^2(q-Q)^2}.$$

# Spherical twists

The main ingredient of our symmetry is the existence of a certain anti-equivalence $\rho \in \mathrm{Aut}(D^b(Y))$ promoting the involution

$$\beta \mapsto \beta' = \beta + (E \cdot \beta)B$$

on $H_2(Y;\mathbb{Z})$ to the derived category.Its construction uses spherical twists.

## Definition

An object $G \in D^b(Y)$ is a spherical object if

$$\mathrm{Ext}^i(G,G) = \begin{cases} \mathbb{C} & \text{if } i = 0,3 \\ 0 & \text{otherwise} \end{cases}$$

Given a spherical object $G$, Seidel-Thomas define a spherical twist $\mathrm{ST}_G \in \mathrm{Aut}(D^b(Y))$ by the exact triangle

$$\bigoplus_i \mathrm{Ext}^i(F,G) \otimes G[-i] \to F \to \mathrm{ST}_G(F).$$

# Anti-equivalence $\rho$

Denote by $C \subseteq E \subseteq Y$ the class of one of the sections of the projection $E \to C$. For every $k \in \mathbb{Z}$,

$$\mathcal{O}_E(-C + kB) \in D^b(Y)$$

is a spherical object.

**Definition**

Let

$$\rho = \mathbb{D} \circ \mathrm{ST}_{O_E(-C+kB)} \circ \mathrm{ST}_{O_E(-C+(k+1)B)} \in \mathrm{Aut}(D^b(Y)).$$

(the definition doesn't depend on $k$)

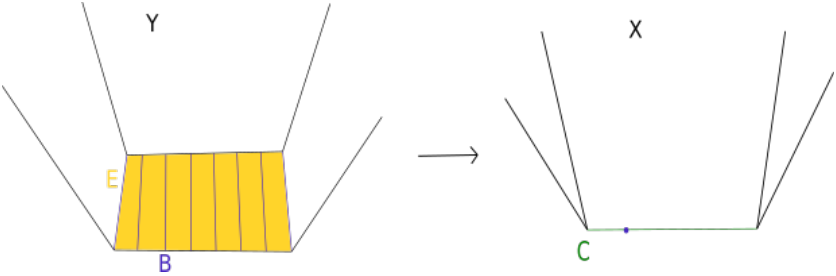
# Properties of $\rho$

1. $\rho$ is an involution, i.e. $\rho \circ \rho = \mathrm{id}$.
2. $\rho(\mathcal{O}_Y) = \mathcal{O}_Y[2]$.
3. If $F$ is supported away from $E$ then $\rho(F) = \mathbb{D}(F)$.
4. $\rho(\mathcal{O}_B(-2)) = \mathcal{O}_B(-2)[1]$ and $\rho(\mathcal{O}_B(-1)) = \mathcal{O}_B(-1)[-1]$.
5. If $F$ is a sheaf of dimension 1 and $\mathrm{ch}_2(F) = \beta, \chi(F) = n$ then

$$\mathrm{ch}_2(\rho(F)) = \beta + (E \cdot \beta)B$$
$$\chi(\rho(F)) = -n.$$

# Orbifold inspiration

When $Y$ arises as a crepant resolution $Y \to \mathcal{X}$ of an orbifold with $\mathbb{Z}/2$-singularities along a $\mathbb{P}^1$ so that $E$ is the exceptional divisor (and the fibers $B$ are contracted to points), the main result is a consequence of the DT crepant resolution conjecture proven by Beentjes-Calabrese-Rennemo ('18).



# Orbifold inspiration

Their proof uses $\mathbb{D}^{\mathcal{X}}$ to prove the symmetry of PT invariants in $\mathcal{X}$.

**Proposition**

*Under the McKay correspondence*

$$\Phi : D^b(Y) \xrightarrow{\sim} D^b(\mathcal{X})$$

*the derived dual* $\mathbb{D}^{\mathcal{X}}$ *corresponds to* $\rho$*, i.e.*

$$\rho = \Phi^{-1} \circ \mathbb{D}^{\mathcal{X}} \circ \Phi.$$

Important examples (e.g. the STU) don't arise as such crepant resolution.

# Perverse stable pairs

Stable pairs are equivalently described as follows:

**Proposition**

*Let $I \in \langle \mathcal{O}_Y[1], \mathsf{Coh}_{\leq 1} \rangle_{\mathsf{ex}}$. Then $I$ is a stable pair if and only if $\mathrm{rk}(I) = -1$ and*

$$\mathsf{Hom}(\mathsf{Coh}_0(Y), I) = 0 = \mathsf{Hom}(I, \mathsf{Coh}_1(Y)).$$

Bridgeland's proof of rationality with the derived dual uses

$$\mathbb{D}(\mathsf{Coh}_1(Y)) = \mathsf{Coh}_1(Y) \text{ and } \mathbb{D}(\mathsf{Coh}_0(Y)) = \mathsf{Coh}_0(Y)[-1].$$

Gives description of the image of $\mathbb{D}(P_n(X, \beta))$ and helps finding wall-crossing back to $P_n(X, \beta)$.

# Perverse sheaves

The derived equivalence $\rho$ doesn't respect $\mathrm{Coh}(Y)$ and the dimension filtration so well.

**Example**

If $x \in E$ is a point in the divisor lying in a fiber $B$ then

$$\rho(\mathcal{O}_x) = \{\mathcal{O}_B(-1)[-1] \to \mathcal{O}_B(-2)\}.$$

We use instead a tilting of $\mathrm{Coh}(Y)$.

$$\mathcal{T} = \{T \in \mathrm{Coh}(Y) : R^1 p_* T_{|E} = 0\}$$
$$\mathcal{F} = \{F \in \mathrm{Coh}(Y) : \mathrm{Hom}(\mathcal{T}, F) = 0\}$$
$$\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}}.$$

$\mathcal{A}$ is a heart of $D^b(Y)$.

# Dimension filtration

Together with $\mathcal{A}$ comes a modified dimension defined by:

$$\dim(F) = \max\{\dim(\mathrm{supp}(F_{|Y\setminus E})), \dim(p(\mathrm{supp}(F_{|E})))\}$$

The modified dimension is used to define $\mathcal{A}_0, \mathcal{A}_1$ which are analogous to $\mathrm{Coh}_0(Y)$, $\mathrm{Coh}_1(Y)$:

$$\rho(\mathcal{A}_1) = \mathcal{A}_1 \text{ and } \rho(\mathcal{A}_0) = \mathcal{A}_0[-1].$$

## Example

1. $\mathrm{Coh}_0 \subseteq \mathcal{A}_0$;
2. $\mathcal{O}_B(-1), \mathcal{O}_B(-2)[1] \in \mathcal{A}_0$;
3. If $F \in \mathrm{Coh}_1(Y)$ and $F_{|E}$ is 0-dimensional then $F \in \mathcal{A}_1$;
4. $\mathcal{O}_E(-C), \mathcal{O}_E(-2C)[1] \in \mathcal{A}_1$.

# Perverse stable pairs

## Definition

A perverse stable pair is an object $I \in \langle \mathcal{O}_Y[1], \mathcal{A}_{\leq 1} \rangle_{\mathrm{ex}}$ such that $\mathrm{rk}(I) = -1$ and

$$\mathrm{Hom}(\mathcal{A}_0, I) = 0 = \mathrm{Hom}(I, \mathcal{A}_1).$$

We define the virtual counts of perverse stable pairs: for

$$\gamma = (\beta, \ell[E]) \in H_2(Y) \oplus \mathbb{Z} \cdot [E]$$

we have

$$ {}^p\mathrm{PT}_{n,\gamma} \in \mathbb{Z},$$

$$ {}^p\mathrm{PT}_{\gamma}(q, Q) = \sum_{n,j \in \mathbb{Z}} {}^p\mathrm{PT}_{n,\gamma+jB}(-q)^n Q^j.$$

# Rationality for ${}^p\mathrm{PT}$

**Theorem**

*The series* ${}^p\mathrm{PT}_\gamma(q, Q)$ *is the expansion of a rational function* $f_\gamma \in \mathbb{Q}(q, Q)$ *satisfying the symmetry*

$$f_\gamma(q^{-1}, Q^{-1}) = Q^{-E\cdot\beta+2\ell} f_\gamma(q, Q).$$

- Rationality of $\mathrm{PT}_\beta(q)$
- Anti-equivalence $\mathbb{D}$
- Torsion pair $\langle \mathrm{Coh}_0, \mathrm{Coh}_1 \rangle$
- Usual slope stability
- Vanishing of Poisson brackets $\{\mathrm{Coh}_{\leq 1}, \mathrm{Coh}_{\leq 1}\} = 0$

- Rationality of ${}^p\mathrm{PT}_\gamma(q, Q)$
- Anti-equivalence $\rho$
- Torsion pair $\langle \mathcal{A}_0, \mathcal{A}_1 \rangle$
- Nironi slope stability
- No vanishing, extra combinatorial difficulty (dealt with in [BCR]).

# Wall-crossing

We proved rationality of perverse PT invariants, but now need to relate them to classical stable pairs.

## Proposition

*For any $\beta \in H_2(Y;\mathbb{Z})$ we have the following identity of rational functions:*

$$^{p}\mathrm{PT}_{\beta}(q,Q) = \frac{\mathrm{PT}_{\beta}(q,Q)}{\mathrm{PT}_{0}(q,Q)}.$$

The wall-crossing establishing the equality has two steps and uses the counting of a third type of objects: Bryan-Steinberg invariants.

# Wall-crossing $\mathrm{PT}/\mathrm{BS}$

When $Y$ arises as a crepant resolution $Y \to \mathcal{X}$, Bryan-Steinberg introduced ('12) invariants $\mathrm{BS}_{n,\beta}$. Roughly speaking, they count sheafs+sections $\{\mathcal{O}_X \xrightarrow{s} F\}$ but allowing the cokernel to have support on fibers of $B$.
They provide a natural interpretation for the quotient $\mathrm{PT}_\beta/\mathrm{PT}_0$ via a DT/PT type wall-crossing.

**Proposition**

$$\mathrm{BS}_\beta(q, Q) \equiv \sum_{n,j\in\mathbb{Z}} \mathrm{BS}_{n,\beta+jB}(-q)^n Q^j = \frac{\mathrm{PT}_\beta(q, Q)}{\mathrm{PT}_0(q, Q)}.$$

Unlike ${}^p\mathrm{PT}$, $\mathrm{BS}$ are defined using the heart Coh$(Y)$, no need to tilt.

# Wall-crossing ${}^p\mathrm{PT}/\mathrm{BS}$

Final step is comparing ${}^p\mathrm{PT}$ and $\mathrm{BS}$.

**Proposition**

*We have the following identity of rational functions:*

$$\mathrm{BS}_\beta(q, Q) = {}^p\mathrm{PT}(q, Q).$$

The identity above is strictly of rational functions, the coefficients are not the same on the nose. When we cross a wall in the path of stability conditions we change the direction in which we expand the same rational function.

# Crossing a wall – re-expansion

## Example

The rational function $\frac{1}{q-Q}$ can be expanded in two different ways:

$$\frac{1}{q-Q} = \frac{q^{-1}}{1-Qq^{-1}} = \sum_{i\geq 0} Q^i q^{-1-i}$$

$$\frac{1}{q-Q} = -\frac{Q^{-1}}{1-Q^{-1}q} = -\sum_{i\geq 0} Q^{-1-i} q^i.$$

# Thank you!

$$\mathrm{PT} \overset{\textit{quotient}}{\longleftrightarrow} \mathrm{BS} \overset{\textit{re-expansion}}{\longleftrightarrow} {}^{p}\mathrm{PT}$$

$${}^{p}\mathrm{PT} \longleftrightarrow \rho({}^{p}\mathrm{PT})$$