VIRASORO CONSTRAINTS FOR MODULI OF SHEAVES

Curves OP '03

Toric
Givental '01

Witten's conjecture $\longrightarrow$ Virasoro GW
Witten '90
Kontsevich '92

EHX '97

MNOP '04
MOOP '20

M'20

Virasoro PT/DT

Virasoro $M_S(r>1,\beta,\chi)$ $\longleftarrow$ Virasoro $\mathrm{Hilb}^n(S)$

van Bree '21

All moduli of sheaves / sheaf-like objects.

Much more general phenomenon?

Moduli of sheaves & pairs

X smooth projective variety. M = moduli of sheaves in X

1) M is projective of finite type

2) M has no strictly semi-stable objects

Can be removed
Arkadij will explain how.

3) There is universal sheaf $\mathcal{G}$ on $X \times M$ which IS NOT necessarily unique. ⟵ Not really necessary.

4) M has 2-term pot $\quad$ Tan $= Ext^1(\mathcal{G},\mathcal{G})$
Obs $= Ext^2(\mathcal{G},\mathcal{G})$
$Ext^{>2}(\mathcal{G},\mathcal{G}) = 0.$

$$[M]^{vir} \in A_{1-\chi(v,v)}(M)$$

Ex: $M_C(r,d) \qquad M_S^H(0,\beta,m) \qquad M_S^H(r,\beta,m)$

$PT_X(\beta,m) \qquad X$ 3-fold, $H^i(\mathcal{O}_X) = 0 \quad i>0$

Moduli of pairs P parametrize sheaf $\mathcal{F}$ & map $V \to F$ where V is fixed sheaf on X.

1) ——
2) ——

3) There is UNIQUE universal pair

$$q^*V \longrightarrow \mathbb{F} \quad \text{on } X \times P \xrightarrow{q} X$$

4) Has 2-term pot $\quad$ Tan $= Hom([V \to F], F)$
Obs $= Ext^1([V\to F], F)$
$Ext^{>1}([V \to F], F) = 0$

$$[P]^{vir} \in A_\bullet(P)$$

Ex: Grassmannian, Quot, Bundles/pairs on curves/surfaces

Many variations possible:
- Fixing det / traceless Def theory
- CY4
- Quivers
- Reduced virtual fundamental classes (?)

Descendants

Def: $\mathbb{D}^X$ = (super-) commutative algebra generated by $\{ ch_i(\gamma) : i \geq 0,\ \gamma \in H^*(X) \}$

Given a choice of $\mathbb{G}$ we have realization map

$$\xi_{\mathbb{G}} : \mathbb{D}^X \longrightarrow H^*(M)$$

$$ch_i(\gamma) \longmapsto p_*\big(\underbrace{ch_{i+\dim X - s}(\mathbb{G})\, q^*\gamma}_{\in \text{"}H^{i,*}(M)\text{"}}\big)$$

$$\begin{tikzcd}
 & \mathbb{G} \arrow[d, dash] & \\
 & M \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\
M & & X
\end{tikzcd}$$

Invariant descendents

How does $\xi_{\mathbb{G}}(D)$ change if we change $\mathbb{G}$?

Def: Let $R_{-1} : \mathbb{D}^X \to \mathbb{D}^X$ be a derivation defined on generators by

$$R_{-1}(\mathrm{ch}_i(\gamma)) = \mathrm{ch}_{i-1}(\gamma)$$

Suppose that $\mathbb{G}' = \mathbb{G} \otimes p^* L, \quad L \in \mathrm{Pic}(M)$

$$\xi_{\mathbb{G}'}(D) = \sum_{j \geq 0} \xi_{\mathbb{G}}\left(R_{-1}^j(D)\right) \frac{c_1(L)^j}{j!}$$

Def: $\mathbb{D}^X_{\mathrm{inv}} = \{ D \in \mathbb{D}^X : R_{-1}(D) = 0 \}$

Realization map $\mathbb{D}^X_{\mathrm{inv}} \longrightarrow H^*(M)$ doesn't depend on $\mathbb{G}$

Virasoro operators

We define $L_k : \mathbb{D}^X \to \mathbb{D}^X$ as $(k \geq -1)$

$$L_k = R_k + T_k$$

- $R_k : \mathbb{D}^X \to \mathbb{D}^X$ is a derivation defined on generators by

$$R_k(\mathrm{ch}_i(\gamma)) = \frac{(i+k)!}{(i-1)!}\mathrm{ch}_{i+k}(\gamma)$$

- $T_k : \mathbb{D}^X \to \mathbb{D}^X$ is multiplication by

$$\sum_{a+b=k} a!\, b! \sum_i (-1)^{\dim X - p_i^L} \mathrm{ch}_a(\gamma_i^L)\, \mathrm{ch}_b(\gamma_i^R)$$

$$\Delta_*(\mathrm{td}(X)) = \sum_i \gamma_i^L \otimes \gamma_i^R \in H^*(X \times X)$$

$$\gamma_i^L \in H^{p_i^L, q_i^L}(X).$$

Fact: $[L_k, L_\ell] = (\ell - k) L_{k+\ell}$.

For moduli of pairs $\{V \to F\}$ we need small adjustment:

$$L_k^V = R_k + T_k^V = R_k + T_k - k!\, ch_k\left(ch(V^\vee)\, td(X)\right)$$

Remark: $T_k^V = -\sum_{a+b=k} (-1)^{d_i - x - p_i^L} ch_a^{V-F}\, ch_b(td(X))$

$$ch_a^{V-F}(\gamma) = \delta_{a=0} \int_X \gamma \cdot ch(V) - ch_a(\gamma)$$

Related to Def theory of P: $RHom(V \to F, F)$

Virasoro constraints

Conjecture (Pair Virasoro): If P is a moduli of pairs

$$\int_{[P]^{vir}} \xi_F\left(L_k^V(D)\right) = 0 \quad \forall k \geq 0, \; D \in \mathbb{D}^X$$

For M-Virasoro we need to produce relations among invariant descendents.

<u>Def</u>: $\mathcal{L} : \mathbb{D}^X \to \mathbb{D}^X_{inv}$

$$\mathcal{L} = \sum_{j \geq -1} \frac{(-1)^j}{(j+1)!} L_j R_{-1}^j \qquad R_{-1} \circ \mathcal{L} = 0$$

<u>Conjecture (M Virasoro)</u>: M moduli of sheaves

$$\int_{[M]^{vir}} \mathcal{L}(D) = 0 \qquad \forall D \in \mathbb{D}^X$$

<u>Remark</u>: If M parametrizes rank $r>0$ sheaves $\exists!\, G$ s.t. $\xi_G(ch_1(pt)) = 0$.

M Virasoro

$\Updownarrow$ (Dirk's formulation)

$$\forall k \geq -1, \quad 0 = \int_{[M]^{vir}} \xi_G\left(L_k(D) - \frac{1}{r}(k+1)!\, R_{-1}(ch_{k+1}(pt)\, D)\right)$$

Moduli of stable bundles on curves

C genus g curve. F vector bundle on C is $^{(\text{semi})}$stable if

$$G \subsetneq F \Rightarrow \frac{\deg G}{\operatorname{rank}(G)} \underset{(\leq)}{<} \frac{\deg F}{\operatorname{rank} F}$$

$$M = M_C^{ss}(r,d) = \left\{ \text{ss. bundles on } C \text{ w/ } \begin{matrix} \operatorname{rank} = r \\ \deg = d \end{matrix} \right\}$$

If $\gcd(r,d) = 1$ stable = semistable

& M is smooth $\quad T_{[F]} = \operatorname{Ext}^1(F,F)$

$$0 = \operatorname{Ext}^{\geq 2}(F,F)$$

Theorem (Bojko - Lim - M): Virasoro holds for $M_C(r,d)$

Ex: $M_C(1,d) \cong \operatorname{Jac}(C)$

Virasoro can be checked directly (some or less trivial)

Let's focus on $r=2$, d odd

May assume $d \gg 0$ $\quad \left(M_C(2,d) \cong M_C(2,d+2k) \right)$

Thaddeus wall-crossing / rank reduction

Def: $\sigma \in \mathbb{R}_{>0}$

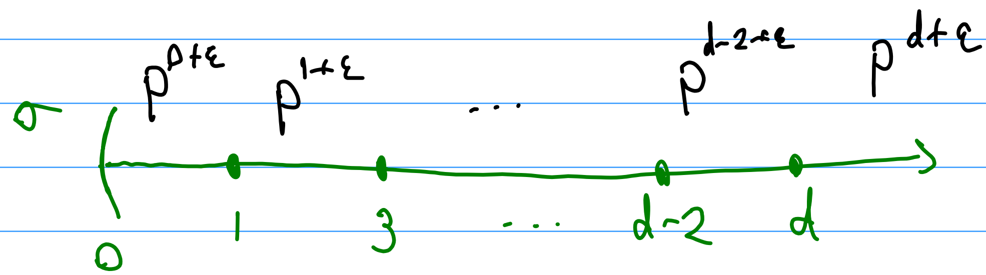
A pair $\mathcal{O}_X \xrightarrow{s} F$ is σ-semistable if $s \neq 0$ and

$$G \subseteq F \implies \frac{\deg G}{\mathrm{rk}(G)} \leq \frac{\deg F + \sigma}{\mathrm{rk}(F)}$$

$$\begin{tikzcd} & \mathcal{O}_X \arrow[dl] \arrow[dr] & \\ G \arrow[rr, phantom, "\subseteq"] & & F \end{tikzcd} \implies \frac{\deg G + \sigma}{\mathrm{rk}(G)} \leq \frac{\deg F + \sigma}{\mathrm{rk}(F)}$$

$\rightsquigarrow P^\sigma(r,d)$ moduli of σ-semistable pairs

$r = 2$



$P^\sigma = P^\sigma(2,d)$ only changes when σ crosses an odd integer

① $P^\sigma = \emptyset$ if $\sigma > d$

$$\begin{tikzcd} & \mathcal{O}_X \arrow[dl] \arrow[dr] & \\ \mathcal{O}_X \arrow[rr] & & F \end{tikzcd}$$

② $P^\varepsilon = \{ \mathcal{O}_X \xrightarrow{s \neq 0} F \mid F \text{ stable} \} \xrightarrow{\pi} M$

Projective bundle w/ fiber $\mathbb{P}(H^0(F))$ over $F \in M$.

$$\begin{tikzcd} P^{0+} \arrow[r, squiggly] \arrow[d] & P^{1+} \arrow[r, squiggly] & \cdots \arrow[r, squiggly] & P^{d-2+\varepsilon} \arrow[r, squiggly] & P^{d+\varepsilon} = \emptyset \\ M \end{tikzcd}$$

Need: • Wall-crossing compatibility
• Projective bundle compatibility

Projective bundle

Prop: M moduli of sheaves w/ G. $H^{>0}(X, G) = 0$ $\forall G \in M$

$P = \mathbb{P}_M(p_* \mathcal{G}) \longrightarrow M$

$\quad = \{ \mathcal{G}_x \to F : F \text{ stable} \}$

P has universal fam $\mathcal{G}_{X \times P} \longrightarrow \pi^* \mathcal{G}(1) = \mathbb{F}$

Virasoro for $P \Rightarrow$ Virasoro for M.

$$\begin{tikzcd} & H^*(P) \\ \mathbb{D}^{\times}_{inv} \arrow[ur] \arrow[dr] & \\ & H^*(M) \arrow[u, "\pi^*"'] \end{tikzcd}$$

Sketch: Want $\int_M \xi_G(\mathcal{L}(D)) = 0$

$\mathcal{L}(D) \in \mathbb{D}^{\times}_{inv} \Rightarrow \xi_{\mathbb{F}}(\mathcal{L}(D)) = \pi^* \xi_G(\mathcal{L}(D))$

Use $\displaystyle\int_M \alpha = \frac{1}{m}\int_P \pi^*\alpha \cdot c_{2m-1}(T_\pi)$ $\qquad m = \chi(G)$, $G \in M$

Write it in terms of descendants and use Virasoro on P

$$ch(T_\pi) = ch(Rp_* \mathbb{F} - G) = p_*\big(ch(\mathbb{F})\, q^* td(X)\big) - 1$$
$$= \xi_{\mathbb{F}}\big(ch_\bullet(td(X))\big) - 1$$

$$c(T_\pi) = \xi_{\mathbb{F}}\Big(\exp\Big(\sum_{\ell \geq 1} (-1)^{\ell-1}(\ell-1)!\, ch_\ell(td(X))\Big)\Big). \square$$

Crossing a wall

σ odd integer. $\sigma - 2 < \sigma_- < \sigma < \sigma_+ < \sigma + 2$

How to relate $\int_{P^{\sigma_-}} D$ and $\int_{P^{\sigma_+}} D$?

Find a "master space" Z w/ $\mathbb{C}^*$ action s.t.

$$Z^{\mathbb{C}^*} = P^{\sigma_-} \sqcup P^{\sigma_+} \sqcup \left\{ \begin{array}{l} S\text{-equivalence classes of} \\ \text{strictly } \sigma\text{-semistable sheaves} \end{array} \right\}$$

$$\cong M\left(1, \tfrac{d+\sigma}{2}\right) \times P\left(1, \tfrac{d-\sigma}{2}\right)$$

$\mathbb{C}^*$ localization $\rightsquigarrow$ $\displaystyle\int_{P^-} D - \int_{P^+} D = \int_{P\left(1, \frac{d-\sigma}{2}\right) \times M\left(1, \frac{d+\sigma}{2}\right)} \cdots$

In Joyce's formalism, $\quad P^{\sigma_-} = P^{\sigma_+} + \left[P\left(1, \tfrac{d-\sigma}{2}\right), M\left(1, \tfrac{d+\sigma}{2}\right) \right]$

Theorem (BLM): "Virasoro is closed under $[\,,\,]$." ← Ankurig's talk

So Virasoro for P^{σ_+}

$$P\left(1, \tfrac{d-\sigma}{2}\right) \cong \mathrm{Sym}^{\frac{d-\sigma}{2}}(C)$$
$$M\left(1, \tfrac{d+\sigma}{2}\right) \cong \mathrm{Jac}(C)$$

Can be checked "by hand"

$\Downarrow$

Virasoro for P^{σ_-}

$C^{an} \to \mathrm{Sym}^{an} C$