Rationality and symmetry of stable pairs generating series for Fano 3-folds

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11 May 2026

VBAC webinar

Counting curves on 3-folds

Counting curves on (smooth projective) 3-folds X is arguably the most important subject in enumerative geometry. Several approaches:

1. Gromov–Witten (GW) theory: counts stable maps $f\colon C \to X$.
2. Donaldson–Thomas (DT) theory: counts 1-dimensional subschemes $C \subseteq X$ or, equivalently, ideal sheaves I_C.
3. Pandharipande–Thomas (PT) theory: counts stable pairs, roughly a 1-dimensional sheaf on X with a section.
4. Maulik–Toda invariants, quasimaps, etc...

Expectation

All these approaches are equivalent.

Counting curves on 3-folds

Conjecture (MNOP '03)

Let X be a Calabi–Yau 3-fold and $\beta \in H_2(X)$ a curve class.

1. *The normalized DT series $Z_\beta^{\mathrm{DT}}(q)/Z_0^{\mathrm{DT}}(q)$ is the Laurent expansion of a rational function.*
2. *After the change of variables $q = -e^{iu}$, the normalized DT series matches the generating series of GW invariants with curve class β.*

Theorem (Toda '08, Bridgeland '10)

The normalized DT series is rational for X Calabi–Yau.

Theorem (Pardon '23)

The GW/PT *correspondence holds for Calabi–Yau 3-folds.*

Stable pairs

The normalization of Z_β^{DT} is meant to "artificially" remove contributions from isolated points. Stable pairs aim to do this geometrically:

Definition (Pandharipande–Thomas '07)

Let X be a projective smooth 3-fold. A stable pair on X is a sheaf F on X with 1-dimensional support together with a section $s\colon \mathcal{O}_X \to F$ such that

- F is pure, i.e. it does not have any 0-dimensional subsheaf.
- The cokernel of s is 0-dimensional.

Heuristically, one can think of a stable pair as the data of a curve and points on that curve:

- The sheaf F is supported on a pure 1-dimensional subscheme $C \subseteq X$.
- $\mathrm{coker}(s)$ is a 0-dimensional sheaf supported on C.

Moduli of stable pairs

Let $P_{\beta,m}$ be the moduli space of stable pairs with

$$[C] = \beta \in H_2(X) \quad \text{and} \quad \mathrm{ch}_3(F) = m \in \frac{1}{2}\mathbb{Z}\,.$$

The moduli of stable pairs is a projective scheme and has a 2-term perfect obstruction theory, and hence a virtual fundamental class

$$[P_{\beta,m}]^{\mathrm{vir}} \in H_{2d_\beta}(P_{\beta,m})$$

where the virtual dimension is given by

$$d_\beta = \int_\beta c_1(X)\,.$$

Remark

1. A stable pair should really be treated as a 2-term complex

$$[\mathcal{O}_X \to F] \in D^b(X)\,.$$

2. The moduli space $P_{\beta,m}$ becomes empty for $m \ll 0$.

PT invariants

If X is Calabi–Yau then the virtual dimension is $d_\beta = 0$, so we get invariants by taking the degree:

$$\mathrm{PT}_{\beta,m} = \deg([P_{\beta,m}]^{\mathrm{vir}}) = \int_{[P_{\beta,m}]^{\mathrm{vir}}} 1 \in \mathbb{Z}\,.$$

If $d_\beta > 0$ (e.g. X is Fano) we should instead integrate natural cohomology classes.

Definition

For $\gamma \in H^*(X)$, $k \geq 0$ define the descendents

$$\mathrm{ch}_k(\gamma) = (\pi_P)_* \left(\mathrm{ch}_k\left(\mathcal{F} - \mathcal{O}\right) \cdot \pi_X^* \gamma\right) \in H^*(P_{\beta,m})$$

where $\mathcal{O} \to \mathcal{F}$ is the universal stable pair on $P_{\beta,m} \times X$ and π_P, π_X are the projections onto $P_{\beta,m}$ and X, respectively.

PT invariants

PT invariants are the integrals of the form

$$\int_{[P_{\beta,m}]^{\mathrm{vir}}} \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_n}(\gamma_n) \in \mathbb{Q}\,.$$

Given $D = \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_n}(\gamma_n)$ we assemble these into a generating series:

$$Z_\beta^{\mathrm{PT}}(q|D) = \sum_m q^m \int_{[P_{\beta,m}]^{\mathrm{vir}}} \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_n}(\gamma_n)\,.$$

Example: lines in $\mathbb{P}^3$

- Let $X = \mathbb{P}^3$ and $\beta = 1$ the class of lines in $\mathbb{P}^3$.
- A point of $P_{1,m}$ corresponds to a line $C \subseteq \mathbb{P}^3$ together with $m+1$ chosen points in $C \simeq \mathbb{P}^1$, so $P_{1,m}$ admits a fibration over $\mathrm{Gr}(2, \mathbb{C}^4)$ with fibers $\mathrm{Sym}^{m+1}(\mathbb{P}^1) \simeq \mathbb{P}^{m+1}$. If $m < -1$ then $P_{1,m}$ is empty.
- The actual dimension of $P_{1,m}$ is $m+5$, but the virtual dimension is 4. The virtual fundamental class can be explicitly identified.
- For example we have

$$
\begin{aligned}
Z_1^{\mathrm{PT}}(q|\mathrm{ch}_7(\mathbf{1})) &= -\frac{1}{9}q^{-1} + \frac{5}{18}q^0 + \frac{11}{9}q - \frac{55}{9}q^2 + \dots \\
&= \frac{(q-1)(2+3q-28q^2+3q^3+2q^4)}{18q(1+q)^3}.
\end{aligned}
$$

Rationality and symmetry

Conjecture (Pandharipande–Thomas '07)

For any smooth projective 3-fold X, β an effective curve class and D as before, $Z_\beta^{\mathrm{PT}}(q|D)$ is the expansion of a rational function satisfying the functional equation

$$Z_\beta^{\mathrm{PT}}(q^{-1}|D) = (-1)^{k_1+\ldots+k_n} Z_\beta^{\mathrm{PT}}(q|D)\,.$$

1. Proved by Toda, Bridgeland when X is Calabi–Yau. They also show that Z_β^{PT} matches the normalized DT generating series.
2. Pandharipande–Pixton proved the rationality part of the conjecture for toric 3-folds and complete intersections on products of projective spaces (with γ_i coming from the ambient space).

Rationality and symmetry

Theorem (Karpov–M, '26)

The rationality and symmetry conjecture holds for X Fano.

1. More generally, we show it under the assumption that $h^{p,0}(X) = 0$ for $p > 1$ (probably not a big deal...) and $\beta' \cdot c_1 > 0$ for every effective curve class $\beta' \leq \beta$ (essential).
2. At the same time and independently, the rationality part was proven by Anderson–Joyce without the constraint on Hodge numbers. They also prove an equivariant version of the conjecture.
3. The proof also controls the poles of $Z_\beta^{\mathsf{PT}}(q|D)$: they can only be $q = 0$ or solutions of

$$(-q)^m = 1 \quad \text{for } m \in \left\{ \beta' \cdot c_1 \,|\, \beta' \leq \beta \right\}.$$

Heuristic and the derived dual

Given a complex $E \in D^b(X)$ we define its derived dual:

$$\delta(E) = \mathrm{RHom}(E, \mathcal{O}_X)[2] \in D^b(X)\,.$$

If we knew that δ sent stable pairs to stable pairs, we would get an isomorphism $P_{\beta,m} \simeq P_{\beta,-m}$ and, in particular, $P_{\beta,m}$ would be empty for $m \gg 0$, so $Z_\beta^{\mathrm{PT}}(q|D)$ would be a Laurent polynomial.
Moreover we would get an equality

$$\int_{[P_{\beta,-m}]^{\mathrm{vir}}} D = (-1)^{k_1+\ldots+k_n} \int_{[P_{\beta,m}]^{\mathrm{vir}}} D\,.$$

While this is not true, Toda's idea is to wall-cross from stable pairs to some other moduli spaces of complexes which are preserved by the derived dual δ.

Moduli of 1-dimensional sheaves

- Fix an ample line bundle H on X. Given a sheaf F with 1-dimensional support we define its slope as

$$\mu(F) = \frac{\mathrm{ch}_3(F)}{\mathrm{ch}_2(F) \cdot H} \in \mathbb{Q}\,.$$

- A 1-dimensional sheaf is semistable if it is pure and for any subsheaf $F' \subsetneq F$ we have $\mu(F') \leq \mu(F)$.
- We let $M_{\beta,m}$ be the moduli space of semistable 1-dimensional sheaves with $\mathrm{ch}_2(F) = \beta, \mathrm{ch}_3(F) = m$.

Moduli of 1-dimensional sheaves

Suppose that:

1. The greatest common divisor of $\beta \cdot H$ and m is 1, so that every semistable sheaf is actually stable.
2. X is Fano, or more generally β is strongly positive.

Then $M_{\beta,m}$ has a virtual fundamental class $[M_{\beta,m}]^{\mathrm{vir}} \in H_2(M_{\beta,m})$ and we can define invariants

$$\int_{[M_{\beta,m}]^{\mathrm{vir}}} D \in \mathbb{Q}\,.$$

"Theorem" (Joyce '21)

If X is Fano we can define generalized intersection numbers even if there are strictly semistable sheaves.

Toda's L-invariants

Toda defines a family of stability conditions μ_s, $s \in \mathbb{R}$, which interpolates between stable pairs (when $s \to +\infty$) and derived duals of stable pairs (when $s \to -\infty$).

Definition (Toda '08)

Let $\mathcal{B}$ be the category of objects $E \in D^b(X)$ which can be obtained by extensions of objects in $\mathrm{Coh}_{\leq 1}(X)$ and $\mathrm{Coh}_{\geq 2}(X)[1]$ and satisfy

$$\mathrm{RHom}(F, E) = 0 = \mathrm{RHom}(G[1], E)$$

for any 0-dimensional sheaf F and any pure 2-dimensional sheaf G.

Worth keeping in mind: $\mathcal{B}$ contains pure 1-dimensional sheaves, $\mathcal{O}_X[1]$, and any stable pair $[\mathcal{O}_X \to F]$.

Toda's L-invariants

Definition (Toda '08)

Let E be an object in $\mathcal{B}$ with $\mathrm{ch}(E) = (-1, 0, \beta, m)$ and let $s \in \mathbb{R}$. We say that E is μ_s-semistable if

1. For any pure 1-dimensional sheaf F and epimorphism $E \twoheadrightarrow F$ in $\mathcal{B}$ we have $\mu(F) \geq s$.
2. For any pure 1-dimensional sheaf F and monomorphism $F \hookrightarrow E$ in $\mathcal{B}$ we have $\mu(F) \leq s$.

Theorem (Toda '08)

1. *When $s \gg 0$, the μ_s-semistable objects are precisely stable pairs.*
2. *The derived dual δ sends a μ_s-semistable object with Chern character $(-1, 0, \beta, m)$ to a μ_{-s}-semistable object with Chern character $(-1, 0, \beta, -m)$*

PT/L wall-crossing formula

"Theorem" (Karpov–M, '26)

There exist generalized intersection numbers $\mathsf{L}^s_{\beta,m}$ "counting" μ_s-semistable objects with Chern character $(-1, 0, \beta, m)$ such that the following wall-crossing formula holds:

$$\mathsf{PT}_{\beta,m} = \sum_{\substack{\beta_0+\ldots+\beta_k=\beta \\ m_0+\ldots+m_k=m \\ 0\leq\frac{m_1}{\beta_1\cdot H}\leq\ldots\leq\frac{m_k}{\beta_k\cdot H}}} \omega\left(\frac{m_1}{\beta_1\cdot H}, \ldots, \frac{m_k}{\beta_k\cdot H}\right) \left[\mathsf{M}_{\beta_k,m_k}, \left[\ldots, \left[\mathsf{M}_{\beta_1,m_1}, \mathsf{L}^{s=0}_{\beta_0,m_0}\right]\ldots\right]\right]$$

1. Here, $\mathsf{PT}_{\beta,m}$, $\mathsf{M}_{\beta,m}$ and $\mathsf{L}^{s=0}_{\beta,m}$ represent all the (generalized) intersection numbers defined on the corresponding moduli spaces.
2. The brackets $[-,-]$ represent some explicit operation. It can be described in terms of a certain vertex algebra defined by Joyce.

PT/L wall-crossing formula

Theorem (Karpov–M, '26)

There exist generalized intersection numbers $\mathsf{L}^s_{\beta,m}$ *"counting"* μ_s*-semistable objects with Chern character* $(-1, 0, \beta, m)$ *such that the following wall-crossing formula holds:*

$$\mathsf{PT}_{\beta,m} = \sum_{\substack{\beta_0+\ldots+\beta_k=\beta \\ m_0+\ldots+m_k=m \\ 0\le\frac{m_1}{\beta_1\cdot H}\le\ldots\le\frac{m_k}{\beta_k\cdot H}}} \omega\left(\frac{m_1}{\beta_1\cdot H},\ldots,\frac{m_k}{\beta_k\cdot H}\right) \left[\mathsf{M}_{\beta_k,m_k},\left[\ldots,\left[\mathsf{M}_{\beta_1,m_1},\mathsf{L}^{s=0}_{\beta_0,m_0}\right]\ldots\right]\right]$$

3. $\omega(\ldots)$ is an explicit combinatorial coefficient. For example if $0=\mu_1=\mu_2<\mu_3<\mu_4=\mu_5$ then

$$\omega(\mu_1,\ldots,\mu_5) = \frac{1}{3!1!2!}\,.$$

PT/L wall-crossing formula

Theorem (Karpov–M, '26)

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$$\mathrm{PT}_{\beta,m} = \sum_{\substack{\beta_0+\ldots+\beta_k=\beta \\ m_0+\ldots+m_k=m \\ 0\leq\frac{m_1}{\beta_1\cdot H}\leq\ldots\leq\frac{m_k}{\beta_k\cdot H}}} \omega\Big(\frac{m_1}{\beta_1 \cdot H}, \ldots, \frac{m_k}{\beta_k \cdot H}\Big) \Big[\mathsf{M}_{\beta_k,m_k}, \Big[\ldots, \Big[\mathsf{M}_{\beta_1,m_1}, \mathsf{L}^{s=0}_{\beta_0,m_0}\Big]\ldots\Big]\Big]$$

4. The result is similar to wall-crossing formulas proved by Joyce ('21) for moduli of sheaves, but his theorems don't apply here. Instead, our proof relies on results of Karpov–M ('25) that use the non-abelian localization theorem of Halpern–Leistner ('25)

Example: lines in $\mathbb{P}^3$

The PT/L wall-crossing $X = \mathbb{P}^3$ and $\beta = 1$ is the class of lines says

$$\begin{aligned}
\mathsf{PT}_{1,-1} &= \mathsf{L}_{1,-1} \\
\mathsf{PT}_{1,0} &= \mathsf{L}_{1,0} + \frac{1}{2}[\mathsf{M}_{1,0}, \mathsf{L}_{0,0}] \\
\mathsf{PT}_{1,1} &= \mathsf{L}_{1,1} + [\mathsf{M}_{1,1}, \mathsf{L}_{0,0}] \\
\mathsf{PT}_{1,m} &= [\mathsf{M}_{1,m}, \mathsf{L}_{0,0}] \quad \text{for } m \geq 2\,.
\end{aligned}$$

We have

$$\begin{aligned}
\int_{[\mathsf{M}_{1,m},\mathsf{L}_{0,0}]} \mathrm{ch}_7(\mathbf{1}) &= \frac{(-1)^m}{6}\int_{\mathsf{M}_{1,m}} \mathrm{ch}_4(\mathbf{1}) - \mathrm{ch}_3(\mathbf{1})\mathrm{ch}_3(H)\,. \\
&= \frac{5}{9}(-1)^m(1-3m^2)\,.
\end{aligned}$$

$$\int_{\mathsf{L}_{1,1}} \mathrm{ch}_7(\mathbf{1}) = \frac{1}{9} = -\int_{\mathsf{L}_{1,-1}} \mathrm{ch}_7(\mathbf{1}) \quad \text{and} \quad \int_{\mathsf{L}_{1,0}} \mathrm{ch}_7(\mathbf{1}) = 0\,.$$

Example: lines in $\mathbb{P}^3$

The PT generating series $Z_1^{\mathrm{PT}}(q|\mathrm{ch}_7(\mathbf{1}))$ is then recovered as

$$-\frac{1}{9}q^{-1} + \frac{1}{9}q + \frac{1}{2} \times \frac{5}{9} + \sum_{m>0} \frac{5}{9}(-q)^m(1-3m^2)$$
$$= \frac{(q-1)(2+3q-28q^2+3q^3+2q^4)}{18q(1+q)^3}\,.$$

Quasi-polynomiality

Definition

A function $f : \mathbb{Z} \to \mathbb{Q}$ is a quasi-polynomial of period N if for every $j \in \mathbb{Z}$ the function $k \mapsto f(kN + j)$ is a polynomial in k.

Proposition

Let $D = \mathrm{ch}_{k_1}(\gamma_1) \ldots \mathrm{ch}_{k_n}(\gamma_n)$*. The function*

$$m \mapsto \int_{\mathrm{M}_{\beta,m}} D \in \mathbb{Q}$$

is a quasi-polynomial of period $H \cdot \beta$*. Moreover, it is even/odd depending on the parity of* $k_1 + \ldots + k_n$*.*

Symmetries

This is a consequence of the following isomorphisms between moduli spaces of 1-dimensional sheaves:

$$- \otimes \mathcal{O}_X(kH) \colon M_{\beta,m} \xrightarrow{\sim} M_{\beta,m+k(\beta\cdot H)}$$
$$\delta \colon M_{\beta,m} \xrightarrow{\sim} M_{\beta,-m}$$

Since μ_0-stability is preserved by the derived dual δ we also have:

Proposition

Given a fixed β, we have $\int_{\mathsf{L}^0_{\beta,m}} D \neq 0$ for only finitely many values of m. Moreover,

$$\int_{\mathsf{L}^0_{\beta,-m}} D = (-1)^{k_1+\ldots+k_n} \int_{\mathsf{L}^0_{\beta,m}} D$$

Combinatorics

Lemma (Karpov–M, '26)

Let $r_1, \ldots, r_k$ be positive integers and let $f_1, \ldots, f_k \colon \mathbb{Z} \to \mathbb{Q}$ be quasi-polynomials. Then the generating series

$$Z(q) = \sum_{0 \leq \frac{n_1}{r_1} \leq \ldots \leq \frac{n_k}{r_k}} \omega\left(\frac{n_1}{r_1}, \ldots, \frac{n_k}{r_k}\right) f_1(n_1) \ldots f_k(n_k) q^{n_1 + \ldots + n_k}$$

is the expansion of a rational function. Moreover, if all f_i are either even or odd, then this rational function satisfies the symmetry

$$Z(q^{-1}) = (-1)^a Z(q)$$

where a is the number of even ones.

The proof uses the beautiful subject of Ehrhart theory, and in particular (a generalization of) Stanley's reciprocity theorem ('74).

Primary insertions

1. Primary insertions are insertions of the form

$$D = \mathrm{ch}_2(\gamma_1) \dots \mathrm{ch}_2(\gamma_n) \, .$$

   We give a new proof of a very strong rationality for primary insertions (that also follows from combining work of Doan–Ionel–Walpuwski '21 and Pardon '23).

2. In particular we show that $Z_\beta^{\mathrm{PT}}(q|D)$ only has poles at $q = 0$ and $q = -1$, but the strongest form of the statement is about a connected version of Z^{PT}.

Primary insertions

Theorem (Karpov–M, '26)

Let X be Fano. If $d_\beta > 1$ then the connected PT generating series $Z_\beta^{\mathrm{PT,conn}}(q|D)$ is a Laurent polynomial in q. If $d_\beta = 1$ then

$$Z_\beta^{\mathrm{PT,conn}}(q|D) = \frac{1}{q^{1/2}+q^{-1/2}}\int_{\mathrm{M}_\beta} D + \text{ Laurent polynomial in } q\,.$$

3. The new aspect of this theorem is that we identify the coefficient of the singular term (=genus 0 Gopakumar–Vafa invariant) as an integral over a moduli of 1-dimensional sheaves.
4. This is analogous (but simpler!) to a conjecture by Katz ('06) for the Calabi–Yau case.

Thank you!